Two Hours and Thirty Minutes

THE UNIVERSITY OF MANCHESTER

NONSTANDARD LOGICS

May 24th 2012

9.45 - 12.15

Answer **all** six questions in **section A** (60 marks in all)

and

two of the three questions in **section B** (10 marks each).

If all three questions from Section B are attempted then credit will only be given for the first two answers appearing in the answer book. The total number of marks on the paper is 80. A further 20 marks are available from work during the semester making a total of 100.

A list of **axioms and rules of proof** is appended to this examination paper

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Calculators are not allowed

SECTION A

Answer **ALL** 6 questions

A1. Explain (without proof) how a finite sequence $s_1, s_2, ..., s_m$ of sets of atoms of a finite language L determines a rational consequence relation $\mid\!\sim_{\vec{s}}$. State the *Representation Theorem for Rational Consequence Relations.*

In the case where $L = \{p, q, r\}$, $\vec{s} = s_1, s_2, s_3$ and

$$\begin{aligned} s_1 &= \{p \wedge q \wedge r,\ p \wedge q \wedge \neg r,\ p \wedge \neg q \wedge r\}, \\ s_2 &= \{\neg p \wedge q \wedge r\}, \\ s_3 &= \{\neg p \wedge q \wedge \neg r\}, \end{aligned}$$

which of the following are true? [You need not justify your answers.]

(i) $\neg p \wedge \neg r \mid\!\sim_{\vec{s}} q$.

(ii) $(p \to q) \mid\!\sim_{\vec{s}} r$.

(iii) $\neg p \wedge \neg q \mid\!\sim_{\vec{s}} p$.

[10 marks]

A2. (a) By giving a direct derivation from the *GM* rules show that the rule

$$\frac{\theta \wedge \neg\phi \mid\!\sim \phi}{\theta \mid\!\sim \phi}$$

is satisfied by all rational consequence relations. [If you wish you may assume any of the derived rules SCL, CON, CC without also proving them.]

[4 marks]

(b) Use the Z-algorithm to find the rational closure of $K = \{\, p \mid\!\sim \neg q,\ \neg q \mid\!\sim \neg p \,\}$.

[8 marks]

A3. State the *Completeness Theorem for B*. Show that:

(i) $\theta \models^B \Box\Box\Diamond\Diamond\theta$.

(ii) $\Box\Box\Diamond\Diamond p \not\models^B p$.

[10 marks]

A4. What is meant by a *proof* in T? [You need not explicitly state the rules or axioms.]

What does it mean to say that $\Gamma \vdash^T \theta$?

Give a formal proof in T of

$$\vdash^T \Box(\Box\theta \to \theta).$$

[8 marks]

 P.T.O.

A5. Show:

(i) If $\models^{Ł} \theta \to \phi$ then $\theta \models^{Ł} \phi$.

(ii) $\models^{Ł} (\theta \to \phi)\underline{\vee}(\neg\theta \to \neg\phi)$.

(iii) If $\vdash^{Ł} \theta \to \phi$ and $\vdash^{Ł} \phi \to \psi$ then $\vdash^{Ł} \theta \to \psi$.

[In (iii) you may assume any result from the course up to, but not including, the Completeness Theorem for Ł.]

[8 marks]

A6. State the desirable properties C1-C4 for a function $F_\wedge : [0,1]^2 \to [0,1]$.

State the Mostert-Shields Theorem for $F_\wedge$.

Show that if $F_\wedge$ satisfies C1-C4 then $F_\wedge(x, 1) = x$ for all $x \in [0, 1]$.

What can be said about the structure $\langle [0,1], F_\wedge, < \rangle$ if in addition to C1-C4 $F_\wedge$ also satisfies that for all $x \in (0, 1)$, $0 < F_\wedge(x, x) < x$?

[12 marks]

SECTION B

Answer **2** of the 3 questions.

If all three questions from this section are attempted then credit will only be given for the first two answers appearing in the answer book.

B7. By using the Representation Theorem for Rational Consequence Relations show that the rule

$$\frac{\theta \wedge \neg\phi \mid\!\sim \phi \vee \psi}{\theta \mid\!\sim \phi \vee \psi}$$

holds for all rational consequence relations, but that the rule

$$\frac{\theta \mid\!\sim \phi \vee \psi}{\theta \wedge \neg\phi \mid\!\sim \phi \vee \psi}$$

fails for some rational consequence relation (and choice of θ, ϕ, ψ).

[10 marks]

B8. Let $\mathcal{R}$ be the modal logic K augmented with the axiom scheme

$$\mid \Diamond(\theta \vee \neg\theta).$$

Give proofs of the key new features (new in the sense that they do not already essentially appear in the corresponding result for K) in the proof of the following completeness theorem for $\mathcal{R}$:

For $\Gamma \subseteq SML$, $\theta \in SML$,

$$\Gamma \vdash^{\mathcal{R}} \theta \iff \text{for all serial frames } \langle W, E, V\rangle \text{ and } i \in W, \text{ if } i \models \Gamma \text{ then } i \models \theta.$$

Hence show that for any $\Gamma \subseteq SML$, $\theta \in SML$,

$$\Gamma \vdash^{\mathcal{R}} \theta \iff \Gamma \vdash^{D} \theta.$$

[10 marks]

B9. Let $L = \{p\}$. State McNaughton's Theorem for L.

Let $\theta \in SML$. Show that there are only finitely many $[0, 1]$-valuations w such that $w(\theta) = 1/3$.

Show that if $w(\theta)$ never takes the value $1/3$ for any $[0, 1]$-valuation then either $\models^{SC} \theta$ or $\models^{SC} \neg\theta$.

[10 marks]

Axioms and Rules of Proof

Sentential, or Propositional, Calculus, SC

Axioms: REF $\theta \mid \theta$

Rules of Proof:

AND $\dfrac{\Gamma \mid \theta, \quad \Gamma \mid \phi}{\Gamma \mid \theta \wedge \phi}$

ANL $\dfrac{\Gamma, \theta, \phi \mid \psi}{\Gamma, \theta \wedge \phi \mid \psi}$

ORR $\dfrac{\Gamma \mid \theta}{\Gamma \mid \theta \vee \phi} \quad \dfrac{\Gamma \mid \phi}{\Gamma \mid \theta \vee \phi}$

DIS $\dfrac{\Gamma, \theta \mid \psi, \quad \Gamma, \phi \mid \psi}{\Gamma, \theta \vee \phi \mid \psi}$

IMR $\dfrac{\Gamma, \theta \mid \phi}{\Gamma \mid \theta \to \phi}$

MP $\dfrac{\Gamma \mid \theta \quad \Gamma \mid \theta \to \phi}{\Gamma \mid \phi}$

NIN $\dfrac{\Gamma \mid \phi, \quad \Gamma \mid \neg\phi}{\Gamma - \{\theta\} \mid \neg\theta}$

NNO $\dfrac{\Gamma \mid \neg\neg\theta}{\Gamma \mid \theta}$

MON $\dfrac{\Gamma \mid \theta}{\Gamma \cup \Delta \mid \theta}$

AO $\dfrac{\Gamma \mid \theta \wedge \phi}{\Gamma \mid \theta} \quad \dfrac{\Gamma \mid \theta \wedge \phi}{\Gamma \mid \phi}$

Modal Logics

Axioms:

$$
\begin{aligned}
\mathrm{K} &:= \mathrm{REF} + \Box(\theta \to \phi) \mid \Box\theta \to \Box\phi \\
\mathrm{T} &:= K + \Box\theta \mid \theta \\
\mathrm{D} &:= K + \Box\theta \mid \Diamond\theta \\
\mathrm{B} &:= K + \theta \mid \Box\Diamond\theta \\
\mathrm{S}_4 &:= T + \Box\theta \mid \Box\Box\theta \\
\mathrm{S}_5 &:= T + \Diamond\theta \mid \Box\Diamond\theta
\end{aligned}
$$

Rules of Proof: All the rules of SC plus NEC $\dfrac{\mid \theta}{\mid \Box\theta}$

Łukasiewicz Logic, Ł

Axioms: REF together with

Ł1 : $\mid \theta \to (\phi \to \theta)$

Ł2 : $\mid (\theta \to \phi) \to ((\phi \to \psi) \to (\theta \to \psi))$

Ł3 : $\mid (\neg\theta \to \neg\phi) \to (\phi \to \theta)$

Ł4 : $\mid ((\theta \to \phi) \to \phi) \to ((\phi \to \theta) \to \theta)$ i.e. $\mid (\theta \underline{\vee} \phi) \to (\phi \underline{\vee} \theta)$

Ł5 : $\mid (\theta \to \phi) \underline{\vee} (\phi \to \theta)$

Rules of Proof: Only MP

GM Rules

REF, $\theta \mid\!\sim \theta$, together with:

left logical equivalence	$\dfrac{\theta \equiv \phi, \quad \theta \mid\!\sim \psi}{\phi \mid\!\sim \psi}$	LLE
right weakening	$\dfrac{\theta \mid\!\sim \phi, \quad \phi \models \psi}{\theta \mid\!\sim \psi}$	RWE
cautious monotonicity	$\dfrac{\theta \mid\!\sim \phi, \quad \theta \mid\!\sim \psi}{\theta \wedge \phi \mid\!\sim \psi}$	CMO
and on right	$\dfrac{\theta \mid\!\sim \phi, \quad \theta \mid\!\sim \psi}{\theta \mid\!\sim \phi \wedge \psi}$	AND
disjunction, or left	$\dfrac{\theta \mid\!\sim \psi, \quad \phi \mid\!\sim \psi}{\theta \vee \phi \mid\!\sim \psi}$	DIS
rational monotonicity	$\dfrac{\phi \mid\!\not\sim \neg\theta, \quad \phi \mid\!\sim \psi}{\theta \wedge \phi \mid\!\sim \psi}$	RMO

END OF EXAMINATION PAPER