Two Hours and Thirty Minutes

UNIVERSITY OF MANCHESTER

NONSTANDARD LOGICS

May 6th 2011
10.00 - 12.30

Answer all six questions in section A (58 marks in all)
and
two of the three questions in section B (11 marks each). If all three questions from Section B are attempted then credit will only be given for the first two answers appearing in the answer book.

The total number of marks on the paper is 80.
A further 20 marks are available from work during the semester making a total of 100.

A list of axioms and rules of proof is appended to this examination paper

Calculators are not allowed
SECTION A

Answer ALL 6 questions

A1. Explain (without proof) how a finite sequence \( s_1, s_2, \ldots, s_m \) of subsets of atoms of a finite language \( L \) determines a rational consequence relation \( \vdash_s \). State the Representation Theorem for Rational Consequence Relations.

In the case where \( L = \{p, q, r\} \), \( \vec{s} = s_1, s_2, s_3 \) and

\[
\begin{align*}
s_1 &= \{\neg p \land q \land \neg r\}, \\
s_2 &= \{p \land \neg q \land r, \ p \land q \land \neg r, \ \neg p \land \neg q \land \neg r\}, \\
s_3 &= \{p \land q \land r\},
\end{align*}
\]

which of the following are true? [You need not justify your answers.]

(i) \( p \lor q \vdash_s p \)
(ii) \( \neg p \land r \vdash_s \neg r \)
(iii) \( r \vdash_s (\neg q \rightarrow p) \)

[10 marks]

A2. (a) Use the Z-algorithm to find the rational closure of \( K = \{\neg(p \land q) \vdash p \lor q, \neg q \vdash \neg p\} \).

[8 marks]

(b) By giving a direct derivation from the GM rules show that the rule

\[
\begin{align*}
\theta \lor \phi &\vdash \theta && \phi \vdash \phi \\
\hline
\phi &\vdash \theta
\end{align*}
\]

is satisfied by all rational consequence relations. [If you wish you may assume any of the derived rules SCL, CON, CC without also proving them.]

[4 marks]

A3. How is the relation \( \models^K \) defined? Show that:

(i) \( \Box(\theta \lor \phi) \models^K \Box \theta \lor \Box \phi \).
(ii) \( \Box p \lor \Box q \not\models^K \Box(p \lor q) \).

[10 marks]

A4. What is meant by a proof in \( D \)? [You need not explicitly state the rules or axioms.]

What does it mean to say that \( \Gamma \vdash^D \theta \)?

Give a formal proof in \( D \) of

\[ \vdash^D \Box(\neg \Box \theta \rightarrow \neg \Box \theta). \]

[10 marks]
A5. State McNaughton’s Theorem in the case when $L = \{p\}$.

Using this theorem, or otherwise, show that if $\theta \in SL$, $L = \{p\}$ and $w$ is a $[0, 1]$-valuation on $L$ with $w(p) = 1/3$ then $w(\theta)$ is one of $0, 1/3, 2/3, 1$.

[6 marks]

A6. State the desirable properties C1-C4 for a function $F_\lambda$.

State the Mostert-Shields Theorem for $F_\lambda$.

Assuming that the function $G : [0, 1]^2 \rightarrow [0, 1]$ defined by

$$G(x, y) = \frac{1}{2} \max \{xy + x + y - 1, 0\}$$

satisfies C1-C4, how does $G$ fit into this classification?

[10 marks]
**SECTION B**

Answer 2 of the 3 questions.

If all three questions from this section are attempted then credit will only be given for the first two answers appearing in the answer book.

**B7.** By using the Representation Theorem for Rational Consequence Relations show that the rule

\[
\frac{\sim \theta \quad \neg \phi \sim \neg \theta}{\sim \phi}
\]

holds for all rational consequence relations, but that the rule

\[
\frac{\psi \sim \theta \quad \neg \phi \sim \neg \theta}{\psi \sim \phi}
\]

fails for some rational consequence relation and choice of \( \theta, \phi, \psi \).

[11 marks]

**B8.** Let \( H \) be \( K \) augmented with the axiom scheme \( \lozenge \theta \vdash \square \theta \). Write \( \Gamma \vdash^H \theta \) if for all thin frames \( (W, E, V) \) and \( i \in W \), if \( i \models \Gamma \) then \( i \models \theta \), where \( (W, E, V) \) is thin if for each \( i \in W \) there is at most one \( j \in W \) such that \( \langle i, j \rangle \in E \).

Outline the key new steps (i.e. beyond those used in proving the corresponding result for \( K \)) of a proof that

\[ \Gamma \vdash^H \theta \iff \Gamma \models^H \theta \]

Hence show that \( \Box p \not\vdash^H \lozenge p \).

[11 marks]

**B9.** Without using the Completeness Theorem for \( L \) show that:

(a) \( \models^L ((p \rightarrow q) \rightarrow p) \rightarrow (q \rightarrow p) \)

(b) \( \not\models^L ((p \rightarrow q) \rightarrow p) \rightarrow p \)

(c) \( \models^L ((p \rightarrow q) \rightarrow p) \rightarrow (q \rightarrow p) \)

[11 marks]
Axioms and Rules of Proof

Sentential, or Propositional, Calculus, $SC$

**Axioms:** REF  \[ \theta \vdash \theta \]

**Rules of Proof:**

\[
\begin{align*}
\text{AND} & : \quad \Gamma \vdash \theta, \quad \Gamma \vdash \phi \\
\quad & \quad \Gamma \vdash \theta \land \phi \\
\text{ANL} & : \quad \Gamma \vdash \theta, \phi \vdash \psi \\
\quad & \quad \Gamma \vdash \theta \land \phi \vdash \psi \\
\text{ORR} & : \quad \Gamma \vdash \theta \land \phi \\
\quad & \quad \Gamma \vdash \theta \lor \phi \\
\text{DIS} & : \quad \Gamma \vdash \theta \lor \phi \\
\quad & \quad \Gamma \vdash \phi \lor \theta \\
\text{IMR} & : \quad \Gamma \vdash \theta \rightarrow \phi \\
\quad & \quad \Gamma \vdash \phi \rightarrow \theta \\
\text{NIN} & : \quad \Gamma \vdash \phi, \quad \Gamma \vdash \neg \phi \\
\quad & \quad \Gamma \vdash \neg \theta, \quad \{\theta\} \vdash \neg \theta \\
\text{NNO} & : \quad \Gamma \vdash \neg \neg \theta \\
\quad & \quad \Gamma \vdash \neg \theta \\
\text{MON} & : \quad \Gamma \vdash \theta \\
\quad & \quad \Gamma \cup \Delta \vdash \theta \\
\text{AO} & : \quad \Gamma \vdash \theta \land \phi \\
\quad & \quad \Gamma \vdash \theta \land \phi \\
\end{align*}
\]

Modal Logics

**Axioms:**

\[
\begin{align*}
K & : \quad \text{REF} + \Box(\theta \rightarrow \phi) \vdash \Box(\theta \rightarrow \Box(\phi) \\
T & : \quad K + \Box \theta \vdash \theta \\
D & : \quad K + \Box \theta \vdash \Box \Box \theta \\
B & : \quad K + \theta \vdash \Box \Box \theta \\
S_4 & : \quad T + \Box \theta \vdash \Box \Box \theta \\
S_5 & : \quad T + \Box \Box \theta \vdash \Box \Box \theta \\
\end{align*}
\]

**Rules of Proof:** All the rules of $SC$ plus NEC  \[ \Gamma \vdash \theta \]

Łukasiewicz Logic, $L$

**Axioms:** REF together with

\[
\begin{align*}
L1 & : \quad \theta \rightarrow (\phi \rightarrow \theta) \\
L2 & : \quad (\theta \rightarrow \phi) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\theta \rightarrow \psi)) \\
L3 & : \quad (\neg \theta \rightarrow \neg \phi) \rightarrow (\phi \rightarrow \theta) \\
L4 & : \quad ((\theta \rightarrow \phi) \rightarrow \phi) \rightarrow ((\phi \rightarrow \theta) \rightarrow \theta) \quad \text{i.e.} \quad (\theta \lor \phi) \rightarrow (\phi \lor \theta) \\
L5 & : \quad \theta \rightarrow (\phi \lor (\phi \rightarrow \theta)) \\
\end{align*}
\]

**Rules of Proof:** Only MP
GM Rules

REF, $\theta \not\vdash \theta$, together with:

- **left logical equivalence**
  \[
  \frac{\theta \equiv \phi, \ \theta \not\vdash \psi}{\phi \not\vdash \psi} \quad \text{LLE}
  \]

- **right weakening**
  \[
  \frac{\theta \not\vdash \phi, \ \phi \models \psi}{\theta \not\vdash \psi} \quad \text{RWE}
  \]

- **cautious monotonicity**
  \[
  \frac{\theta \not\vdash \phi, \ \theta \not\vdash \psi}{\theta \land \phi \not\vdash \psi} \quad \text{CMO}
  \]

- **and on right**
  \[
  \frac{\theta \not\vdash \phi, \ \theta \not\vdash \psi}{\theta \not\vdash \phi \land \psi} \quad \text{AND}
  \]

- **disjunction, or left**
  \[
  \frac{\theta \not\vdash \psi, \ \phi \not\vdash \psi}{\theta \lor \phi \not\vdash \psi} \quad \text{DIS}
  \]

- **rational monotonicity**
  \[
  \frac{\phi \not\models \not\theta, \ \phi \not\vdash \psi}{\theta \land \phi \not\vdash \psi} \quad \text{RMO}
  \]