Two Hours and Thirty Minutes

UNIVERSITY OF MANCHESTER

NONSTANDARD LOGICS

May 6th 2011

10.00 - 12.30

Answer **all** six questions in **section A** (58 marks in all)
and
two of the three questions in **section B** (11 marks each). If all three questions from Section B are attempted then credit will only be given for the first two answers appearing in the answer book.
The total number of marks on the paper is 80.
A further 20 marks are available from work during the semester making a total of 100.

A list of axioms and rules of proof is appended to this examination paper

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Calculators are not allowed

SECTION A

Answer **ALL** 6 questions

A1. Explain (without proof) how a finite sequence $s_1, s_2, ..., s_m$ of subsets of atoms of a finite language L determines a rational consequence relation $\mid\!\sim_{\vec{s}}$. State the *Representation Theorem for Rational Consequence Relations.*

In the case where $L = \{p, q, r\}$, $\vec{s} = s_1, s_2, s_3$ and

$$\begin{aligned} s_1 &= \{\neg p \wedge q \wedge \neg r\}, \\ s_2 &= \{p \wedge \neg q \wedge r,\ p \wedge q \wedge \neg r,\ \neg p \wedge \neg q \wedge \neg r\}, \\ s_3 &= \{p \wedge q \wedge r\}, \end{aligned}$$

which of the following are true? [You need not justify your answers.]

(i) $p \vee q \mid\!\sim_{\vec{s}} p$

(ii) $\neg p \wedge r \mid\!\sim_{\vec{s}} \neg r$

(iii) $r \mid\!\sim_{\vec{s}} (\neg q \rightarrow p)$

[10 marks]

A2. (a) Use the Z-algorithm to find the rational closure of $K = \{\, \neg(p \wedge q) \mid\!\sim p \vee q,\ \neg q \mid\!\sim \neg p \,\}$.

[8 marks]

(b) By giving a direct derivation from the GM rules show that the rule

$$\frac{\theta \vee \phi \mid\!\sim \theta \qquad \theta \mid\!\sim \phi}{\phi \mid\!\sim \theta}$$

is satisfied by all rational consequence relations. [If you wish you may assume any of the derived rules SCL, CON, CC without also proving them.]

[4 marks]

A3. How is the relation $\models^K$ defined? Show that:

(i) $\Box(\theta \vee \phi) \models^K \Box\theta \vee \Diamond\phi$.

(ii) $\Box p \vee \Diamond q \not\models^K \Box(p \vee q)$.

[10 marks]

A4. What is meant by a *proof* in D? [You need not explicitly state the rules or axioms.]

What does it mean to say that $\Gamma \vdash^D \theta$?

Give a formal proof in D of

$$\vdash^D \Box(\neg\Diamond\theta \rightarrow \neg\Box\theta).$$

[10 marks]

A5. State *McNaughton's Theorem* in the case when $L = \{p\}$.

Using this theorem, or otherwise, show that if $\theta \in SL$, $L = \{p\}$ and w is a $[0, 1]$-valuation on L with $w(p) = 1/3$ then $w(\theta)$ is one of $0, \frac{1}{3}, \frac{2}{3}, 1$.

[6 marks]

A6. State the desirable properties C1-C4 for a function $F_\wedge$.

State the Mostert-Shields Theorem for $F_\wedge$.

Assuming that the function $G : [0, 1]^2 \to [0, 1]$ defined by

$$G(x, y) = \frac{1}{2} \max\{xy + x + y - 1, 0\}$$

satisfies C1-C4, how does G fit into this classification?

[10 marks]

SECTION B

Answer **2** of the 3 questions.

If all three questions from this section are attempted then credit will only be given for the first two answers appearing in the answer book.

B7. By using the Representation Theorem for Rational Consequence Relations show that the rule

$$\frac{\mid\!\sim \theta \qquad \neg\phi \mid\!\sim \neg\theta}{\mid\!\sim \phi}$$

holds for all rational consequence relations, but that the rule

$$\frac{\psi \mid\!\sim \theta \qquad \neg\phi \mid\!\sim \neg\theta}{\psi \mid\!\sim \phi}$$

fails for some rational consequence relation and choice of θ, ϕ, ψ.

[11 marks]

B8. Let H be K augmented with the axiom scheme $\Diamond\theta \mid \Box\theta$. Write $\Gamma \models^H \theta$ if for all thin frames $\langle W, E, V\rangle$ and $i \in W$, if $i \models \Gamma$ then $i \models \theta$, where $\langle W, E, V\rangle$ is *thin* if for each $i \in W$ there is at most one $j \in W$ such that $\langle i, j\rangle \in E$.

Outline the key new steps (i.e. beyond those used in proving the corresponding result for K) of a proof that

$$\Gamma \vdash^H \theta \iff \Gamma \models^H \theta$$

Hence show that $\Box p \not\vdash^H \Diamond p$.

[11 marks]

B9. Without using the Completeness Theorem for Ł show that:

(a) $\models^{Ł} ((p \to q) \to p) \to (q \to p)$

(b) $\not\models^{Ł} ((p \to q) \to p) \to p$

(c) $\vdash^{Ł} ((p \to q) \to p) \to (q \to p)$

[11 marks]

Axioms and Rules of Proof

Sentential, or Propositional, Calculus, SC

Axioms: REF $\theta \mid \theta$

Rules of Proof:

AND $\dfrac{\Gamma \mid \theta, \quad \Gamma \mid \phi}{\Gamma \mid \theta \wedge \phi}$ ANL $\dfrac{\Gamma, \theta, \phi \mid \psi}{\Gamma, \theta \wedge \phi \mid \psi}$

ORR $\dfrac{\Gamma \mid \theta}{\Gamma \mid \theta \vee \phi} \quad \dfrac{\Gamma \mid \phi}{\Gamma \mid \theta \vee \phi}$ DIS $\dfrac{\Gamma, \theta \mid \psi, \quad \Gamma, \phi \mid \psi}{\Gamma, \theta \vee \phi \mid \psi}$

IMR $\dfrac{\Gamma, \theta \mid \phi}{\Gamma \mid \theta \to \phi}$ MP $\dfrac{\Gamma \mid \theta \quad \Gamma \mid \theta \to \phi}{\Gamma \mid \phi}$

NIN $\dfrac{\Gamma \mid \phi, \quad \Gamma \mid \neg\phi}{\Gamma - \{\theta\} \mid \neg\theta}$ NNO $\dfrac{\Gamma \mid \neg\neg\theta}{\Gamma \mid \theta}$

MON $\dfrac{\Gamma \mid \theta}{\Gamma \cup \Delta \mid \theta}$ AO $\dfrac{\Gamma \mid \theta \wedge \phi}{\Gamma \mid \theta} \quad \dfrac{\Gamma \mid \theta \wedge \phi}{\Gamma \mid \phi}$

Modal Logics

Axioms:

K := REF + $\Box(\theta \to \phi) \mid \Box\theta \to \Box\phi$
T := $K + \Box\theta \mid \theta$
D := $K + \Box\theta \mid \Diamond\theta$
B := $K + \theta \mid \Box\Diamond\theta$
S_4 := $T + \Box\theta \mid \Box\Box\theta$
S_5 := $T + \Diamond\theta \mid \Box\Diamond\theta$

Rules of Proof: All the rules of SC plus NEC $\dfrac{\mid \theta}{\mid \Box\theta}$

Łukasiewicz Logic, Ł

Axioms: REF together with

Ł1 : $\mid \theta \to (\phi \to \theta)$
Ł2 : $\mid (\theta \to \phi) \to ((\phi \to \psi) \to (\theta \to \psi))$
Ł3 : $\mid (\neg\theta \to \neg\phi) \to (\phi \to \theta)$
Ł4 : $\mid ((\theta \to \phi) \to \phi) \to ((\phi \to \theta) \to \theta)$ i.e. $\mid (\theta \underline{\vee} \phi) \to (\phi \underline{\vee} \theta)$
Ł5 : $\mid (\theta \to \phi) \underline{\vee} (\phi \to \theta)$

Rules of Proof: Only MP

GM Rules

REF, $\theta \mid\!\sim \theta$, together with:

left logical equivalence	$\dfrac{\theta \equiv \phi, \quad \theta \mid\!\sim \psi}{\phi \mid\!\sim \psi}$	LLE
right weakening	$\dfrac{\theta \mid\!\sim \phi, \quad \phi \models \psi}{\theta \mid\!\sim \psi}$	RWE
cautious monotonicity	$\dfrac{\theta \mid\!\sim \phi, \quad \theta \mid\!\sim \psi}{\theta \wedge \phi \mid\!\sim \psi}$	CMO
and on right	$\dfrac{\theta \mid\!\sim \phi, \quad \theta \mid\!\sim \psi}{\theta \mid\!\sim \phi \wedge \psi}$	AND
disjunction, or left	$\dfrac{\theta \mid\!\sim \psi, \quad \phi \mid\!\sim \psi}{\theta \vee \phi \mid\!\sim \psi}$	DIS
rational monotonicity	$\dfrac{\phi \mid\!\not\sim \neg\theta, \quad \phi \mid\!\sim \psi}{\theta \wedge \phi \mid\!\sim \psi}$	RMO

END OF EXAMINATION PAPER