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MATH43032/63032 Part 3

Real Valued Logic

A Motivating Example

Pretty young girls are very trustworthy.
Monica is quite pretty and fairly young.

$\therefore$ *Monica is reasonably trustworthy.*

Unlike the previous examples in this course (and the Predicate Calculus in MATH43001/63001) this conclusion does not seem to be a surefire consequence of the premises, in the sense that even if we accepted the premises we might feel the conclusion wasn't *bound* to be true. Nevertheless we would probably feel reasonably happy 'going along' with such arguments in the absence of anything more concrete to go on.

Because such statements are such a common part of everyday communication and (some say) our knowledge of the world, and because reasoning as above seems such a common feature of our world the study of how we make such 'inferences' has received considerable attention by those intent on building 'intelligent computers' – that is computers which can reason as we apparently do with knowledge of this form.

As a first example consider attempting to build an expert system to regulate the environment within a room. Traditionally *Control Theory* would have approached this problem by formulating it in terms of a number of integral-differential equations. The conclusions would (so I'm told) have been computationally demanding, unreliable, and, to anyone except possibly the control engineer responsible, largely incomprehensible in relation to the everydayness of the situation (i.e. room and environment). An alternative, which was first suggested in 1968 by Lotfi Zadeh (The Father of Fuzzy Thinking), was to simply write down the obvious rules you want the system to obey and use 'logic' to get out the answer[1]. This has the obvious advantage that it looks close to the way *we* have been successfully controlling the environment within rooms for the past 3 million years. However to automate this we do need to formulate the rules of 'following' in examples such as 'pretty Monica' above.

For example we might stick into our rule base:

If it is hot and not noisy then open the window.

The problem with this is that 'hot' is not something black and white. The 'degree to which it is hot' is a function of temperature (at least) and doesn't just jump, step function fashion from 'not hot' to 'hot' at one particular temperature. Rather it is a gradual transition from zero degree to which it is hot (say at $0 \deg C$) through to degree of hot one (say at $40 \deg C$). Similarly for the degree to which it is noisy. Now in practice we might expect to know the degree to which it is hot and the degree to which it is noisy. However to apply the rule, and so determine the 'degree to which we should open the window' we need to know the degree to which it is 'hot and not noisy'. How to work that out[2]?

One approach to this problem is to identify the 'degree to which it is hot' with the degree of truth, or truth value, of the statement 'it is hot'. In this way then we are led to accepting also truth values between 0 and 1 and the abovementioned problem can be replaced by the problem of determining the truth value of $p \wedge \neg q$ from the truth values of p and q.

Before we consider the suggestions for how this might be achieved we consider a second example from real life where vague statements (above 'young', 'hot' etc. are vague – Fuzzy Logic is

[1]Dubbed 'Fuzzy Logic' because it applied to fuzzy or vague notions such as 'hot', 'pretty', young' etc..

[2]The general situation is actually even more problematic, since in examples such as with Monica we don't have any sort of objective measure or scale of 'prettiness'.

often refered to as the Logic of Vagueness) apparently need to be confronted. This is the so called Sorites Paradox which goes as follows: We take a guy with a full head of hair, so clearly there's no way he's bald. We now pull out his hairs, one at a time. Now clearly, the paradox argues, pulling out just one hair cannot turn our guy from being 'not bald' into being 'bald'. However, at the end of this painful process he's is only as hairy as a billiard ball, and is now, for sure a 100% baldilocks. The paradox is that if we treat the statement 'he is bald' as having just the two possible truth values, 0,1, then an inductive argument shows that the degree to which it is true that he is 'not bald' must still be 1 even when every last hair has been plucked from his miserable pate. The unsurprising conclusion that some would draw to this paradox is that we must allow degrees of truth, or truth values, between 0 and 1 to capture the truth of 'he is bald' at the intermediate stages of this depapilliatory process.

The introduction of these intermediate truth values is, by itself, hardly too objectionable. It is what comes next which grates. Because, to take the example above of the 'hot and not noisy', in such situations the assumption is now made that the truth value of 'not noisy' is a fixed function, $F_\neg$, of the truth value of 'noisy', and similarly the truth value of 'hot and not noisy' is a fixed function, $F_\wedge$ of the truth values of 'hot' and 'not noisy' (and similarly for disjunctions and implications). We refer to this as the *Truth Functionality Assumption (TFA).*

There are two immediate reasons for making this assumption. The first is that when we only had the two truth values, 0,1 (i.e. as in the Sentential, or Propositional, Calculus) then there were such functions. For example $F_\wedge(x, y) = 1$ if $x = y = 1$ and $F_\wedge(x, y) = 0$ otherwise. Indeed, it is really this $F_\wedge$ that comes first in the sense that this is *the definition of 'and'* in our language, and similarly for negation, disjunction, implication.

The second reason is that with this assumption the truth value of any compound sentence can be calculated as a fixed composition of the functions $F_\neg, F_\wedge$ etc. applied to the truth values of the propositional variables. For example if $w(\theta)$ stands for the truth value of θ then

$$w((p \wedge \neg q) \vee r) = F_\vee(F_\wedge(w(p), F_\neg(w(q))), w(r)).$$

For folk interested in building expert systems such computational simplicity is almost irresistibly attractive.

Interestingly it turned out that several appropriate logics dealing with truth values in the interval $[0, 1]$ had been studied as purely logical objects long before Fuzzy Logic came on the scene. These formed part of *Real Valued Logics*, or more generally *Many Valued Logics*. In this short introduction we shall study in detail one such logic (Łukasiewicz Logic) and, briefly, consider it in relation to Fuzzy Logic. The reason for so doing is very largely because of the wide influence of Fuzzy Logic, which some of you will surely come into contact with in your future careers.

In order to study these logics we need introduce a formal framework. As usual let L be a language (i.e. non-empty set of propositional variables) and SL the sentences of L. The TFA is the assumption:

The Truth Functionality Assumption: There are fixed functions $F_\wedge, F_\vee, F_\rightarrow : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ and $F_\neg : [0, 1] \longrightarrow [0, 1]$ such that if $w(\theta) \in [0, 1]$ is the *truth value* of θ, for $\theta \in SL$, then

$$w(\neg\theta) = F_\neg(w(\theta)),$$

$$w(\theta \wedge \phi) = F_\wedge(w(\theta), w(\phi)),$$

$$w(\theta \vee \phi) = F_\vee(w(\theta), w(\phi)),$$

$$w(\theta \to \phi) = F_{\to}(w(\theta), w(\phi)).$$

Each such choice of $F_{\neg}, F_{\wedge}, F_{\vee}, F_{\to}$ yields a logic in an analogous fashion to the way the functions on $\{0, 1\}$ given by the conventional (2-valued) truth tables

θ	ϕ	$\neg\theta$	$\theta \wedge \phi$	$\theta \vee \phi$	$\theta \to \phi$
1	1	0	1	1	1
1	0	0	0	1	0
0	1	1	0	1	1
0	0	1	0	0	1

yielded the Sentential Calculus SC. Namely we analogously define a *valuation* (or a $[0, 1]$-valuation if there is ever any danger of confusing it with a standard $\{0, 1\}$-valuation) w on L to be a function from L to $[0, 1]$ and extend w by induction on the levels SL_n (uniquely, by unique readability) to all of SL using the schema in the TFA above. We now define a semantic version of 'follows' by saying that θ follows from Γ in this logic if for all valuations w, if $w(\phi) = 1$ for all $\phi \in \Gamma$ then $w(\theta) = 1$[3]. Given this notion of follows we can ask if there is a corresponding proof theory, completeness theorem etc.

Of course we could spend the rest of our lives looking at such logics, but it only really makes sense to consider logics which have some other, independent reason for being interesting (although, as mentioned earlier, some such logics which were previously considered were, apparently, only later found to be of independent interest). So, since our motivation has come via Fuzzy Logic, what are the choices of $F_{\neg}, F_{\wedge}, F_{\vee}, F_{\to}$ which are appropriate there? Or, more pointedly, what justifications does Fuzzy Logic profer for making any particular choice of these functions?

In practical applications of 'Fuzzy Logic' this choice is almost always between one of:-

$$\begin{array}{lrcl}
\underline{\mathbb{F}^1 :} & F_{\neg}(x) & = & 1 - x \\
 & F_{\wedge}(x, y) & = & min\{x, y\} \\
 & F_{\vee}(x, y) & = & max\{x, y\} \\
 & F_{\to}(x, y) & = & min\{1, 1 - x + y\}
\end{array}$$

$$\begin{array}{lrcl}
\underline{\mathbb{F}^2 :} & F_{\neg}(x) & = & 1 - x \\
 & F_{\wedge}(x, y) & = & xy \\
 & F_{\vee}(x, y) & = & x + y - xy \\
 & F_{\to}(x, y) & = & 1 \text{ if } x \leq y \\
 & & = & y/x \text{ if } y < x
\end{array}$$

$$\begin{array}{lrcl}
\underline{\mathbb{F}^3 :} & F_{\neg}(x) & = & 1 - x \\
 & F_{\wedge}(x, y) & = & max\{0, x + y - 1\} \\
 & F_{\vee}(x, y) & = & min\{1, x + y\} \\
 & F_{\to}(x, y) & = & min\{1, 1 - x + y\}
\end{array}$$

The choice $\mathbb{F}^3$ yields what is nowadays frequently referred to as Łukasiewicz Logic (although Łukasiewicz actually studied the logic based on $\mathbb{F}^1$). $\mathbb{F}^2$ yields what is called Product Logic

[3]Clearly this is not the only notion of 'semantic follows' we could have chosen here, although from our point of view it seems to be the best one to consider.

and $\mathbb{F}^1$ with the alternative

$$F_{\rightarrow}(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{if } y < x \end{cases}$$

gives Gödel Logic.

As far as justifying the TFA and any particular choice of $F_{\neg}, F_{\wedge}, F_{\vee}, F_{\rightarrow}$ there are three possible approaches. One is to look for some sort of semantics for 'degrees of truth' – what does it mean to say that 'Monica is young' is true to degree 0.64? – and (hopefully) show that these semantics lead us inevitably to some particular choice for $F_{\neg}$ etc. In fact a number of different semantics have been proposed. However the most obvious of these do not even justify the TFA, let alone direct us towards a particular choice of $F_{\neg}$ etc. It is true that there are some possible semantics which do yield the TFA, and lead to particular $F_{\neg}$ etc. However these are, frankly, rather hard to swallow, except in some special circumstances. [It is easy to see why one might encounter difficulties in attempting to provide a justifying semantics. For suppose, as a simple example, both θ and $\neg\theta$ had the same truth value (as is allowed in each of $\mathbb{F}^1, \mathbb{F}^2, \mathbb{F}^3$). Then we would have

$$w(\theta \wedge \theta) = F_{\wedge}(w(\theta), w(\theta)) = F_{\wedge}(w(\theta), w(\neg\theta)) = w(\theta \wedge \neg\theta),$$

$$w(\theta \vee \theta) = F_{\vee}(w(\theta), w(\theta)) = F_{\vee}(w(\theta), w(\neg\theta)) = w(\theta \vee \neg\theta),$$

so any suitable semantics must be able to accommodate, or explain, this situation. However, I imagine that most of us would feel that under any reasonable interpretation $w(\theta \wedge \theta)$ and $w(\theta \vee \theta)$ should both equal $w(\theta)$ whilst $w(\theta \wedge \neg\theta)$ should be 0 and and $w(\theta \vee \neg\theta)$ should be 1.]

A second justification, which seems to be held by some of the big cheeses in the subject is that the TFA holds because of the socially accepted *meaning* of 'and','or','not','implies'. In other words, in our language, the functions $F_{\neg}, F_{\wedge}, F_{\vee}$, $F_{\rightarrow}$ *define* the meaning of these connectives when dealing with degrees of truth in $[0,1]$, just as the usual truth tables mentioned earlier define these connectives for truth values on $\{0,1\}$. This would seem to nicely avoid the problem, if only it was pretty clear that we were all using the same choice of $F_{\neg}, F_{\wedge}$, etc. Unfortunately, this doesn't seem so self evident, indeed even the big cheeses seem to concede that in practice we might be using different choices in different contexts. [There is an interesting variation on this idea which says that if the TFA is made in some expert system for some a particular choice of $F_{\neg}, F_{\wedge}$, etc then all this means is that *within this system this is how 'not', 'and' etc are being defined*. One can scarcely argue with this, on the other hand it is hard to imagine that anyone, even the inventor of the system, won't unconsciously slip back into his/her old interpretations of the connectives when actually working with the system.]

The third justification is in terms of 'desirable properties'. Namely, the idea is simply make the TFA and then impose certain conditions on $F_{\neg}, F_{\wedge}$, etc which you think are natural and desirable given their intended interpretation. For example, introducing the shorthand notation $\neg x$ for $F_{\neg}(x)$, $x \wedge y$ for $F_{\wedge}(x,y)$ etc. (so $\neg$ stands now both for a connective and a particular function from SL into $[0,1]$ etc.), in the case of $F_{\neg}$ such desirable properties might be:-

(N1) $\neg 0 = 1,\ \neg 1 = 0$,
(N2) $\neg$ is decreasing,
(N3) $\neg\neg x = x$ for $x \in [0, 1]$.

Thus (N1) says that the negation of a certainly false statement is certainly true and the negation of a certainly true statement is certainly false. (N2) says that as the truth value of a statement increases so the truth value of its negation decreases. (N3) says that two negations cancel each other out, as in classical propositional logic.

Assuming (N1-3) we conclude, by (N3), that $\neg$ is 1-1 onto $[0, 1]$ since for $x, y \in [0, 1]$, $x = \neg(\neg x)$ and

$$\neg x = \neg y \ \Rightarrow\ \neg\neg x = \neg\neg y \ \Rightarrow x = y.$$

It follows then that by (N2) $\neg$ must be strictly decreasing, and hence by elementary analysis continuous.

Theorem 1 (Trillas) *If $\neg$ satisfies (N1-3) then $\langle\, [0, 1], \neg, <\rangle$ is isomorphic to $\langle\, [0, 1], 1-x, <\rangle$ (and conversely since $\neg x = 1 - x$ satisfies (N1-3)).*

Proof First notice that since $\neg 0 = 1 > 0$ and $\neg 1 = 0 < 1$ the continuity of $\neg$ ensures by the Intermediate Value Theorem (IVT) the existence of $0 < c < 1$ such that $\neg c = c$. Define for $x \in [0, 1]$,

$$f(x) = \begin{cases} x/2c & \text{if } x \le c, \\ \\ 1 - \neg x/2c & \text{if } x \ge c. \end{cases}$$

Then $f(0) = 0$, $f(1) = 1$, $f(c) = 1/2$ (either way!) and it is easy to check that f is strictly increasing and continuous, hence onto $[0, 1]$. Finally for $x \le c$, $\neg x \ge \neg c = c$ so

$$f(\neg x) = 1 - \neg\neg x/2c = 1 - x/2c = 1 - f(x),$$

as required, and similarly for $x \ge c$. Hence f is the required isomorphism. ■

Notice that this f is not unique, for example the above proof also works with $2(x/2c)^2$ in place of $x/2c$ etc.

Justification for the choice of $F_\wedge$

The function $\wedge; [0, 1]^2 \to [0, 1]$ might intuitively be argued to satisfy the following *desiderata*:

(C1) $0 \wedge 1 = 1 \wedge 0 = 0,\ 1 \wedge 1 = 1$,
(C2) $\wedge$ is continuous,
(C3) $\wedge$ is increasing (not necessarily strictly) in each coordinate,
(C4) $\wedge$ is associative, i.e. $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ for $x, y, z \in [0, 1]$.

A function satisfying (C1-4) is known as a *continuous T-norm*, or just a T-norm if we drop (C2).

(C1) might be justified on the grounds that for the categorical (or certain) truth values 0,1 $\wedge$ should act like classical conjunction. (C2) is justified on the grounds that microscopic changes in the truth values of θ and ϕ should not cause macroscopic changes in the truth value of $\theta \wedge \phi$. (C3) is justified on the grounds that increasing the truth value of θ (with ϕ's truth value fixed) should not decrease the truth value of $\theta \wedge \phi$, and similar with θ, ϕ reversed. Finally (C4) could be defended on the grounds that in natural language we do not differentiate between $\theta \wedge (\phi \wedge \psi)$ and $(\theta \wedge \phi) \wedge \psi$ so their truth values should not differ either. [A similar argument to justify $\wedge$

being commutative seems open to question although in fact as we shall see later commutativity actually follows from (C1-4).] In view of (C4) we can unambiguously drop parentheses from multiple applications of $\wedge$.

Notice that the $F_\wedge$ in $\mathbb{F}^1, \mathbb{F}^2, \mathbb{F}^3$ satisfy (C1-4). Indeed as we shall eventually show *any* $\wedge$ satisfying (C1-4) must in a sense be a hybrid or *chimera* of the $F_\wedge$ of $\mathbb{F}^1, \mathbb{F}^2, \mathbb{F}^3$. This result will be a consequence of the next two theorems and discussion.

In what follows assume that $\wedge$ satisfies (C1-4). We first show that (C1) can be strengthened to

$$x \wedge 0 = 0 \wedge x = 0, \quad x \wedge 1 = 1 \wedge x = x, \quad \text{for } x \in [0,1].$$

The first of these identities follows directly from (C1) and (C3). For the second notice that by (C1), for $x \in [0,1]$,

$$0 \wedge 1 = 0 \leq x \leq 1 = 1 \wedge 1,$$

so by continuity, (C3), and the Intermediate Value Theorem (IVT), $x = t \wedge 1$ for some $0 \leq t \leq 1$. Then

$$x \wedge 1 = (t \wedge 1) \wedge 1 = t \wedge (1 \wedge 1) = t \wedge 1 = x$$

by (C1) and (C4). The identity $1 \wedge x = x$ follows similarly.

Using these identities and (C3) we now have that

$$x \wedge y \leq x \wedge 1 = x$$

and similarly $x \wedge y \leq y$. We shall use these identities and inequalities frequently and without further mention in what follows.

Now suppose that $x \wedge x = x$. Then for $y \geq x \geq z$,

$$x = x \wedge 1 \geq z \geq x \wedge 0 = 0,$$

so by the IVT and continuity, (C2), $z = x \wedge t$ for some $t \in [0,1]$ and

$$z \geq y \wedge z = y \wedge x \wedge t \geq x \wedge x \wedge t = x \wedge t = z.$$

We conclude that for $y \geq x \geq z$ and $x \wedge x = x$,

$$y \wedge z = z = \min\{y, z\} \quad (= z \wedge y \ \text{ similarly}).$$

In particular we now have the following theorem.

Theorem 2 *If in addition to (C1-4) $\wedge$ also satisfies $x \wedge x = x$ for all $x \in [0,1]$ then $\wedge = \min$ (as in $\mathbb{F}^1$).* ■

Notice that this result gives that under these conditions $\wedge$ is commutative. We shall shortly see that in fact this follows just from (C1-4).

Now suppose that $x \wedge x \neq x$. Then since $0 \wedge 0 = 0 < x < 1 \wedge 1 = 1$, by continuity there must be a largest $a \leq x$ (in fact $a < x$) such that $a \wedge a = a$ and a smallest $b \geq x$ (in fact $b > x$) such that $b \wedge b = b$. Notice that for $x_1, y_1 \in [a,b]$, $x_1 \wedge y_1 \in [a,b]$ since

$$a = a \wedge a \leq x_1 \wedge y_1 \leq b \wedge b = b.$$

Also, of course, for $a < z < b$, $z \wedge z < z$. For these a, b we now have the following theorem.

Theorem 3 *$\langle [a,b], \wedge, < \rangle$ is either isomorphic to $\langle [0,1], \times, < \rangle$ (as in $\mathbb{F}^2$) or to $\langle [0,1], \max\{0, x+y-1\}, < \rangle$ (as in $\mathbb{F}^3$). The latter holds just if for some $a < z < b$, $z \wedge z = a$.*

Proof First suppose that for no $a < z < b$ do we have $z \wedge z = a$. We shall show that $\wedge$ is *strictly* increasing on $(a,b]^2$. To see this suppose $a < c \leq b$, $a < x < y \leq b$ but $c \wedge x = c \wedge y$. Then since, by the discussion above,

$$y \wedge a = \min\{a, y\} = a < x < y = \min\{b, y\} = y \wedge b$$

there exists $a < t < b$ such that $x = y \wedge t$. Hence

$$c \wedge x \wedge t = c \wedge y \wedge t = c \wedge x.$$

Let $s \geq a$ be minimal such that $c \wedge x \wedge s = c \wedge x$. Clearly s exists by continuity and also $s \leq t < b$. Furthermore $a < s$ since if $a = s$ then

$$c \wedge y = c \wedge x = c \wedge x \wedge a = \min\{c \wedge x, a\} = a,$$

so for $z = \min\{c, y\} > a$,

$$a = c \wedge y \geq z \wedge z > a,$$

since $a < z \leq b$, contradiction. It follows that $a < s < b$. But then

$$c \wedge x \wedge s \wedge s = c \wedge x \wedge s = c \wedge x$$

and $a < s \wedge s$ (since we are assuming here that $a < z \wedge z$ whenever $a < z < b$) so by minimality of s we must have that $a < s = s \wedge s < b$, contradiction! We conclude then that in this case $\wedge$ must be strictly increasing on $(a,b]^2$.

We are now ready to construct the required isomorphism. For notational simplicity(!) we shall, for $x \in [a,b]$, $n \in \mathbb{N}^+$, henceforth write $x^{\wedge n}$ for

$$\underbrace{x \wedge x \wedge x \wedge \ldots \wedge x}_{n \text{ times}}.$$

Fix $a < \alpha < b$. For $0 < m \in \mathbb{N}$ notice that $a = a^{\wedge m} < \alpha < b = b^{\wedge m}$ so by continuity and the now proven *strict monotonicity* there is a unique $a < \beta < b$ such that $\beta^{\wedge m} = \alpha$. Denote by $\alpha^{\wedge \frac{n}{m}}$ the number $\beta^{\wedge n}$. By this device we now have 'fractional powers' of α. Furthermore this interpretation is justified in that we have cancellation for 'vulgar fractions'. Precisely, suppose also $\delta^{\wedge rm} = \alpha$, so $\alpha^{\wedge \frac{rn}{rm}} = \delta^{\wedge rn}$. Then $\beta^{\wedge m} = \delta^{\wedge rm} = (\delta^{\wedge r})^{\wedge m}$ so by strict monotonicity $\beta = \delta^{\wedge r}$ and

$$\alpha^{\wedge \frac{n}{m}} = \beta^{\wedge n} = (\delta^{\wedge r})^{\wedge n} = \delta^{\wedge rn} = \alpha^{\wedge \frac{rn}{rm}}.$$

Using this 'cancellation rule' we see that

$$\alpha^{\wedge \frac{n}{m}} \wedge \alpha^{\wedge \frac{r}{s}} = \alpha^{\wedge \frac{ns}{ms}} \wedge \alpha^{\wedge \frac{mr}{ms}} = \alpha^{\wedge \frac{ns+mr}{ms}} = \alpha^{\wedge (\frac{n}{m} + \frac{r}{s})}. \tag{1}$$

Furthermore, if $\frac{n}{m} < \frac{p}{q}$ then $\alpha^{\wedge (\frac{p}{q} - \frac{n}{m})} < b$ so

$$\alpha^{\wedge \frac{p}{q}} = \alpha^{\wedge \frac{n}{m}} \wedge \alpha^{\wedge (\frac{p}{q} - \frac{n}{m})} < \alpha^{\wedge \frac{n}{m}} \wedge b = \alpha^{\wedge \frac{n}{m}}. \tag{2}$$

Now notice that the sequence $\alpha^{\wedge n}$ is decreasing and bounded below by a so must reach a limit, $\gamma \geq a$. If $\gamma > a$ then just as for α we could find $\gamma^{\wedge \frac{1}{2}}$ and

$$\gamma^{\wedge \frac{1}{2}} = \gamma^{\wedge \frac{1}{2}} \wedge b > \gamma^{\wedge \frac{1}{2}} \wedge \gamma^{\wedge \frac{1}{2}} = \gamma.$$

Hence for some n, $\alpha^{\wedge n} < \gamma^{\wedge\frac{1}{2}}$. But then

$$\alpha^{\wedge 2n} < \gamma^{\wedge\frac{1}{2}} \wedge \gamma^{\wedge\frac{1}{2}} = \gamma,$$

contradiction! We conclude that

$$\lim_{n\to\infty} \alpha^{\wedge n} = a. \tag{3}$$

By an exactly similar proof we obtain that

$$\lim_{n\to\infty} \alpha^{\wedge\frac{1}{2^n}} = b. \tag{4}$$

At this point we generalize the relationships given in (1),(2) from positive rational powers of α to all positive real powers. We do this as follows. Suppose $a < \beta < b$ and let

$$r_\beta = \sup\{\, p/q \in \mathbb{Q} \,|\, \alpha^{\wedge\frac{p}{q}} \geq \beta \,\}. \tag{5}$$

Notice that $r_\beta \in \mathbb{R}^+$ exists alright because by (3) and (4) for large enough k, $\alpha^{\wedge k} < \beta < \alpha^{\wedge\frac{1}{2^k}}$ so this set over which we are taking the sup is non-empty and bounded above.

Here we have defined $r_\beta \in \mathbb{R}^+$ from $\beta \in (a, b)$. In fact for any $r \in \mathbb{R}^+$ there is a $\beta \in (a, b)$ such that $r = r_\beta$. Indeed let

$$\beta = \inf\{\alpha^{\wedge\frac{p}{q}} \,|\, p/q \leq r\,\}. \tag{6}$$

Then from the definitions (5), (6),

$$p/q \leq r \;\Rightarrow\; \alpha^{\wedge\frac{p}{q}} \geq \beta \;\Rightarrow\; p/q \leq r_\beta. \tag{7}$$

Conversely suppose that $p/q > r$. Pick $p/q > s/t > r$. Then

$$\alpha^{\wedge\frac{p}{q}} < \alpha^{\wedge\frac{s}{t}} \leq \alpha^{\wedge\frac{p'}{q'}} \quad \text{for all } p'/q' \leq r$$

so

$$\alpha^{\wedge\frac{p}{q}} < \alpha^{\wedge\frac{s}{t}} \leq \beta.$$

Hence $s/t \geq p'/q'$ whenever $\alpha^{\wedge\frac{p'}{q'}} \geq \beta$, so $p/q > s/t \geq r_\beta$ and the arrows in (7) reverse to give $r = r_\beta$ and in turn

$$\beta = \inf\{\alpha^{\wedge\frac{p}{q}} \,|\, p/q \leq r_\beta\,\} \tag{8}$$

From this it follows that $\beta \mapsto r_\beta$ provides a 1-1 onto correspondence between (a, b) and $(0, \infty)$ $(= \mathbb{R}^+)$. In view of this let $\alpha^{\wedge r}$ denote β when $r = r_\beta$. Notice that if $r = p/q$ then these 'two' meanings to $\alpha^{\wedge r}$ agree so this new notation merely extends to all positive reals the old notation.

From (1), (2), (5), (8) and continuity it now follows that for $r_1, r_2 \in \mathbb{R}^+$

$$r_1 < r_2 \;\Rightarrow\; \alpha^{\wedge r_1} > \alpha^{\wedge r_2}, \tag{9}$$

$$\alpha^{\wedge r_1} \wedge \alpha^{\wedge r_2} = \alpha^{\wedge(r_1+r_2)}. \tag{10}$$

Define $g : [a, b] \to [0, 1]$ by $g(a) = 0$, $g(b) = 1$ and $g(\alpha^{\wedge r}) = (1/2)^r$ for $r \in \mathbb{R}^+$. Then since every $\delta \in (0, 1)$ is of the form $(1/2)^r$ for some unique $r \in \mathbb{R}^+$, g is 1-1 onto. Also by (9) g is order preserving and by (10)

$g(\alpha^{\wedge r_1} \wedge \alpha^{\wedge r_2}) = g(\alpha^{\wedge(r_1+r_2)}) = (1/2)^{r_1+r_2}$

$$= (1/2)^{r_1} \times (1/2)^{r_2} = g(\alpha^{\wedge r_1}) \times g(\alpha^{\wedge r_2})$$

so, together with the properties of g and $\wedge$ at the edge values a, b,

$$g : \langle [a, b], \wedge, < \rangle \cong \langle [0, 1], \times, < \rangle,$$

as required.

The investigation of the second case, that is when there is $a < x < b$ such that $x \wedge x = a$, goes along similar lines except that now our choice of α is forced to be the largest $x \in (a, b]$ such that $x \wedge x = a$. Again we prove a suitable monotonicity result which enables us to define an exponentiation such that any $c \in [a, b]$ is of the from $\alpha^{\wedge r}$ for some $r \in [0, 2]$ (notice the 2 here rather than ∞ since already $\alpha^{\wedge 2} = \alpha \wedge \alpha = a$) and from this, using just the same ideas as before, we now obtain that

$$\langle [a, b], \wedge, < \rangle \;\cong\; \langle [0, 2], \min\{x + y, 2\}, > \rangle \;\cong\; \langle [0, 1], \max\{x + y - 1, 0\}, < \rangle,$$

as required. ■

Summing up then we have the following result which has been reproved by numerous people over the years but seems first to have originated with Mostert-Shields in 1957:

Theorem 4 (The Mostert-Shields Theorem (for $F_\wedge$)) *Let $F_\wedge$ satisfy (C1-4) and let $A = \{ x \in [0, 1] \mid F_\wedge(x, x) = x \}$. Then for $x \in A$ and $0 \le z \le x \le y \le 1$,*

$$F_\wedge(z, y) = F_\wedge(y, z) = z = \min\{y, z\},$$

and if $a < b$, $a, b \in A$ and $(a, b) \cap A = \emptyset$ then on $[a, b]$ either

$$\langle [a, b], F_\wedge, < \rangle \;\cong\; \langle [0, 1], \times, < \rangle$$

or

$$\langle [a, b], F_\wedge, < \rangle \;\cong\; \langle [0, 1], \max\{0, x + y - 1\}, < \rangle^4,$$

with the latter of these holding just if $F_\wedge(c, c) = a$ for some $a < c < b$.

The Mostert-Shields Theorem provides a strongish argument for the special status afforded to at least the $F_\wedge$ of $\mathbb{F}^1, \mathbb{F}^2, \mathbb{F}^3$.

Notice that from this theorem it must be the case that for $\wedge$ satisfying (C1-4) $x \wedge y = y \wedge x$. i.e. $\wedge$ is commutative. This would seem a pleasing and far from obvious consequence of these assumptions.

Justification for the choice of $F_\vee$

By direct analogy with the case of $F_\wedge$ the following would seem to be natural requirements on $F_\vee$ (abbreviated to just $\vee$ as usual):

- (D1) $0 \vee 0 = 0, \quad 1 \vee 0 = 0 \vee 1 = 1$,
- (D2) $\vee$ is continuous,
- (D3) $\vee$ is increasing (not necessarily strictly) in each coordinate.

[4] In other words, on $[a, b]$ $F_\wedge$ either looks like a copy of the $F_\wedge$ of Product Logic, $\mathbb{F}^2$ (on [0,1]) or looks like a copy of the $F_\wedge$ of Łukasiewicz Logic, $\mathbb{F}^3$.

(D4) $\quad \vee$ is associative, i.e. $(x \vee y) \vee z = x \vee (y \vee z)$ for all $x, y, z \in [0, 1]$.

A function satisfying (D1-4) is known as a *continuous T-conorm*, or just a T-conorm if we drop (D2).

Fortunately (in terms of the work involved) results on continuous T-conorms follow directly from results on continuous T-norms (and conversely). This is because if $\vee$ satisfies (D1-4) then $\wedge$ defined by

$$x \wedge y = 1 - (1 - x) \vee (1 - y)$$

satisfies (C1-4) and conversely if $\wedge$ satisfies (C1-4) then $\vee$ defined by

$$x \vee y = 1 - (1 - x) \wedge (1 - y)$$

satisfies (D1-4). What's more if we start with $\vee$, produce $\wedge$ as above and from this produce a $\vee$ as above then we get back to the one we started with (and similarly with $\vee$, $\wedge$ transposed).

In particular from the results on conjunction we get the 'disjunction version' of the Mostert-Shields Theorem

Theorem 5 (Mostert-Shields Theorem for $F_\vee$) *Let $F_\vee$ satisfy (D1-4) and let $A = \{ x \in [0,1] \mid F_\vee(x,x) = x \}$. Then for $x \in A$ and $0 \leq z \leq x \leq y \leq 1$,*

$$F_\vee(z, y) = F_\vee(y, z) = z = \max\{y, z\},$$

and if $a < b$, $a, b \in A$ and $(a, b) \cap A = \emptyset$ then on $[a, b]$ either

$$\langle [a, b], F_\vee, < \rangle \;\cong\; \langle [0, 1], x + y - xy, < \rangle$$

or

$$\langle [a, b], F_\vee, < \rangle \;\cong\; \langle [0, 1], \min\{1, x + y\}, < \rangle^5,$$

with the latter of these holding just if $F_\vee(c, c) = b$ for some $a < c < b$.

Again the Mostert-Shields Theorem provides a strongish argument for the special status afforded to at least the $F_\vee$ of $\mathbb{F}^1, \mathbb{F}^2, \mathbb{F}^3$. Notice that we again get commutativity for free.

Remark The 'desirable properties' of $F_\rightarrow$ are, maybe, not quite so obvious as they were for $F_\neg, F_\wedge, F_\vee$ (although some such characterising results have certainly been obtained).
One condition that has been suggested for $F_\rightarrow$ is that it satisfy that for $x, y, z \in [0, 1]$,

$$x \leq F_\rightarrow(y, z) \iff F_\wedge(x, y) \leq z. \tag{11}$$

This corresponds the requirement, for sentences θ, ϕ, ψ, that

> *The truth degree of θ is less or equal the truth degree to which ϕ implies ψ iff the truth degree of θ and ϕ is less or equal the truth degree of ψ.*

In fact this relation (11) does hold for $\mathbb{F}^2, \mathbb{F}^3$ and Gödel Logic, but not $\mathbb{F}^1$. [For those of you familiar with lattice theory a lattice with an operation $\rightarrow$ satisfying this is known as a *residuated lattice*.]

[5] In other words, on $[a, b]$ $F_\vee$ either looks like a copy of the $F_\vee$ of Product Logic, $\mathbb{F}^2$ (on [0,1]) or looks like a copy of the $F_\vee$ of Łukasiewicz Logic, $\mathbb{F}^3$.

Despite the attraction of the Mostert-Shields Theorem the justifications for any one choice of $\mathbb{F}^1, \mathbb{F}^2, \mathbb{F}^3$ as the 'logic of vagueness' seems to me not entirely convincing. Perhaps overall $\mathbb{F}^3$ just edges it, and since this logic has been studied quite extensively within mathematical logic for it's own sake (and as we shall see in some sense covers also $\mathbb{F}^1$) it is this one that we now go on to look at in more detail.

Łukasiewicz Logic

We shall now turn our attention to the logic Ł (standing for Łukasiewicz Logic) given by $\mathbb{F}^3$, that is:

$$\begin{array}{rcl} F_\neg(x) & = & 1-x \\ F_\wedge(x,y) & = & max\{0, x+y-1\} \\ F_\vee(x,y) & = & min\{1, x+y\} \\ F_\rightarrow(x,y) & = & min\{1, 1-x+y\} \end{array}$$

For the rest of this section we shall fix $F_\neg, F_\wedge, F_\vee, F_\rightarrow$ to be these specific functions as immediately above. Notice that

$$F_\rightarrow(x,y) = 1 \iff x \leq y,$$

a fact we shall use frequently in what follows without further mention.

One reason for choosing to concentrate on this logic Ł is that it simple seems to have attracted more interest in than the others amongst mathematical logicians (although $\mathbb{F}^1$ appears the more favoured in AI and Control Theory – indeed amongst some practitioners the term 'Fuzzy Logic' is sometimes identified with using this logic). In fact, as the following proposition shows, $\mathbb{F}^1$ and $\mathbb{F}^3$ are in a sense the same thing in that $\mathbb{F}^3$ actually 'contains', as a definable connectives, the $F_\wedge, F_\vee$ of $\mathbb{F}^1$, and conversely.

Proposition 6 *For $x, y \in [0,1]$,*

$$max\{x,y\} = F_\rightarrow(F_\rightarrow(x,y),y), \qquad min\{x,y\} = F_\neg(F_\rightarrow(F_\rightarrow(y,x),F_\neg(y))),$$

$$F_\wedge(x,y) = F_\neg F_\vee(F_\neg(x),F_\neg(y)), \qquad F_\vee(x,y) = F_\neg F_\wedge(F_\neg(x),F_\neg(y)), \quad \text{(de Morgan's Laws)},$$

$$F_\wedge(x,y) = F_\neg F_\rightarrow(x,F_\neg(y)), \qquad F_\vee(x,y) = F_\rightarrow(F_\neg(x),y).$$

In particular it follows from this proposition that $F_\wedge, F_\vee$ are actually redundant in Ł, they can be defined from $F_\neg$ and $F_\rightarrow$ alone. Indeed, from now on, and in the modern spirit of this subject, we shall assume that we only have the two connectives, $\neg, \rightarrow$ and that for the other connectives $\theta \wedge \phi, \theta \vee \phi, \theta \underline{\wedge} \phi, \theta \underline{\vee} \phi$ are abbreviations standing for

$$\neg(\theta \rightarrow \neg\phi), \ \neg\theta \rightarrow \phi, \ \neg((\phi \rightarrow \theta) \rightarrow \neg\phi), \ (\theta \rightarrow \phi) \rightarrow \phi,$$

respectively. [So $F_{\underline{\wedge}} = min, F_{\underline{\vee}} = max$.] Note that we shall continue to use SL for the set of sentences of Ł despite the fact that, as basic connectives, we have now cut out $\wedge$ and $\vee$.

We define a valuation for Ł (or a $[0,1]$-valuation if we wish to distinguish it from a standard $\{0,1\}$-valuation) to be a function $w : L \rightarrow [0,1]$. So w gives a truth value in $[0,1]$ to each propositional variable in L. As expected we extend w uniquely to the whole of SL by induction on the $|\theta|$ by:

$w(\neg\theta) \;=\; F_\neg(w(\theta)) \;=\; 1 - w(\theta),$

$w(\theta \to \phi) \;=\; F_\to(w(\theta), w(\phi)) \;=\; min\{1, 1 - w(\theta) + w(\phi)\}.$

From this it follows, of course, that

$w(\theta \wedge \phi) \;=\; F_\wedge(w(\theta), w(\phi)) \;=\; max\{0, w(\theta) + w(\phi) - 1\},$

$w(\theta \vee \phi) \;=\; F_\vee(w(\theta), w(\phi)) \;=\; min\{1, w(\theta) + w(\phi)\},$

$w(\theta \underline{\wedge} \phi) \;=\; F_{\underline{\wedge}}(w(\theta), w(\phi)) \;=\; min\{w(\theta), w(\phi)\},$

$w(\theta \underline{\vee} \phi) \;=\; F_{\underline{\vee}}(w(\theta), w(\phi)) \;=\; max\{w(\theta), w(\phi)\}.$

Notice that since $F_\neg, F_\wedge, F_\vee, F_\to$ agree with the standard truth table definitions of $\neg, \wedge$, etc. in the $\{0,1\}$-valued Sentential Calculus (indeed this is true for each of $\mathbb{F}^1, \mathbb{F}^2, \mathbb{F}^3$) any $\{0,1\}$-valuation on SL is also an $[0,1]$-valuation on SL. In more detail, if $V : SL \to \{0,1\}$ is a $\{0,1\}$-valuation in the sense of SC then V is also a $[0,1]$-valuation in the sense of Ł (or $\mathbb{F}^1, \mathbb{F}^2$) and, by induction on the level, for $\theta \in SL$ $V(\theta)$ defined as in SC (with V viewed as a $\{0,1\}$-valuation) equals $V(\theta)$ defined as in Ł (with V viewed as a $[0,1]$-valuation). In particular then this latter truth value is in $\{0,1\}$.

Clearly for $\theta \in SL$ mentioning at most the propositional variables $p_1, p_2, ..., p_n$ there is a fixed function, $F_\theta : [0,1]^n \longrightarrow [0,1]$, such that for any valuation w on L,

$$F_\theta(w(p_1), w(p_2), ..., w(p_n)) = w(\theta).$$

This raises the question as to exactly what functions $F : [0,1]^n \longrightarrow [0,1]$ can arise in this way, or putting it in another way, 'How expressive is Ł?' In the case of the SC it is well know that for *any* function $F : \{0,1\}^n \longrightarrow \{0,1\}$ there is a $\theta \in SL$ such that

$$F(V(p_1), V(p_2), ..., V(p_n)) = V(\theta)$$

for any $\{0,1\}$-valuation V on L ($L \supseteq \{p_1, p_2, ...p_n\}$) (see Examples 3). In the case of Ł it is easy to see that we cannot hope to get such a strong result (with $[0,1]$ in place of $\{0,1\}$). Precisely which functions can arise in this way is given by McNaughton's Theorem, which we shall state and sketch prove in the case $n = 1$, the full theorem being an obvious, yet rather messy to state and prove, generalization.

Theorem 7 (McNaughton's Theorem (for $L = \{p\}$)) *A function $F : [0,1] \longrightarrow [0,1]$ is of the form F_θ for some $\theta \in SL$, with $L = \{p\}$, iff there exist some $0 = \gamma_1 < \gamma_2 < \gamma_3 < ... < \gamma_{n-1} < \gamma_n = 1$ and $n_i, m_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, n-1$, such that on each $[\gamma_i, \gamma_{i+1}]$ $F(x) = m_i + n_i x$ ($\in [0,1]$).*

Notes

(1) Since F is a 'single valued function' this last condition forces that at the endpoints $\gamma_2, \gamma_3, ..., \gamma_{n-1}$ these linear segments join up. Furthermore, unless the two functions on each side of γ_i are the same (in which case we can just drop γ_i) γ_i must be rational, because it is the simultaneous solution to the two (now distinct) linear forms giving $F(x)$ on each side of γ_i.

(2) The general case for $L = \{p_1, ..., p_n\}$ is similar (and is proved in a similar way). Namely, in place of intervals $[\gamma_i, \gamma_{i+1}]$ we have convex polytopes in the unit cube $[0,1]^n$ on each of which $F(\vec{x}) = m_0 + m_1 x_1 + m_2 x_2 + ... + m_n x_n \in [0,1]$ for some $m_1, m_2, ..., m_n \in \mathbb{Z}$, and again these 'planes' meet up at the edges of the polytopes.

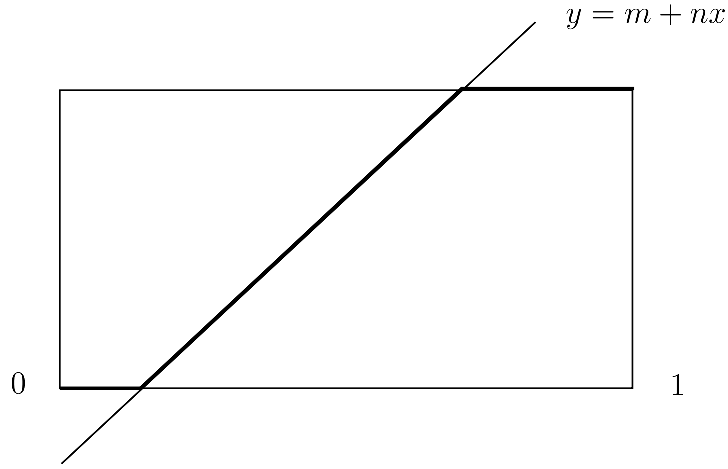
Proof $\Rightarrow$: We show the existence of the γ_i and n_i, m_i by induction on the n such that $\theta \in SL_n$. If $n = 0$ (so $\theta = p$) then we can take $\gamma_1 = 0, \gamma_2 = 1$ and on here $F(x) = 1 \cdot x + 0$. Now assume the result for $n-1$. It is now enough to notice that if F_1, F_2 satisfy the given conditions (and notice that we may assume the $\vec{\gamma}$'s are the same in both cases) then $F_\neg F_1$ and $F_\to(F_1, F_2)$ again satisfy them. This is checked by cases, the only non-trivial case being for $F_\to$ when $F_1(x) = m^{(1)} + n^{(1)}x$, $F_2(x) = m^{(2)} + n^{(2)}x$ on $[\gamma_{i-1}, \gamma_i]$ and $F_1(\gamma) = F_2(\gamma)$ for some $\gamma_{i-1} < \gamma < \gamma_i$. In this case either $F_1 \leq F_2$ on $[\gamma_{i-1}, \gamma]$, in which case

$$F_\to(F_1(x), F_2(x)) = \begin{cases} 1 & \text{on } [\gamma_{i-1}, \gamma] \\ 1 - F_1(x) + F_2(x) \ (\in [0,1]) & \text{on } [\gamma, \gamma_i] \end{cases}$$

or, $F_1 \geq F_2$ on $[\gamma_{i-1}, \gamma]$, in which case

$$F_\to(F_1(x), F_2(x)) = \begin{cases} 1 & \text{on } [\gamma, \gamma_i] \\ 1 - F_1(x) + F_2(x) \ (\in [0,1]) & \text{on } [\gamma_{i-1}, \gamma]. \end{cases}$$

We now prove the significantly more involved opposite direction. For $n, m \in \mathbb{Z}$ let $]m + nx[$ be the function $\min\{1, \max\{0, m + nx\}\}$, so the graph of $]m + nx[$ looks like the dark line in



We first show that for any $n, m \in \mathbb{Z}$ there is a $\theta \in SL$ ($L = \{p\}$) such that for any $[0,1]$-valuation w,

$$w(\theta) =]m + nw(p)[,$$

equivalently,

$$F_\theta(x) =]m + nx[.$$

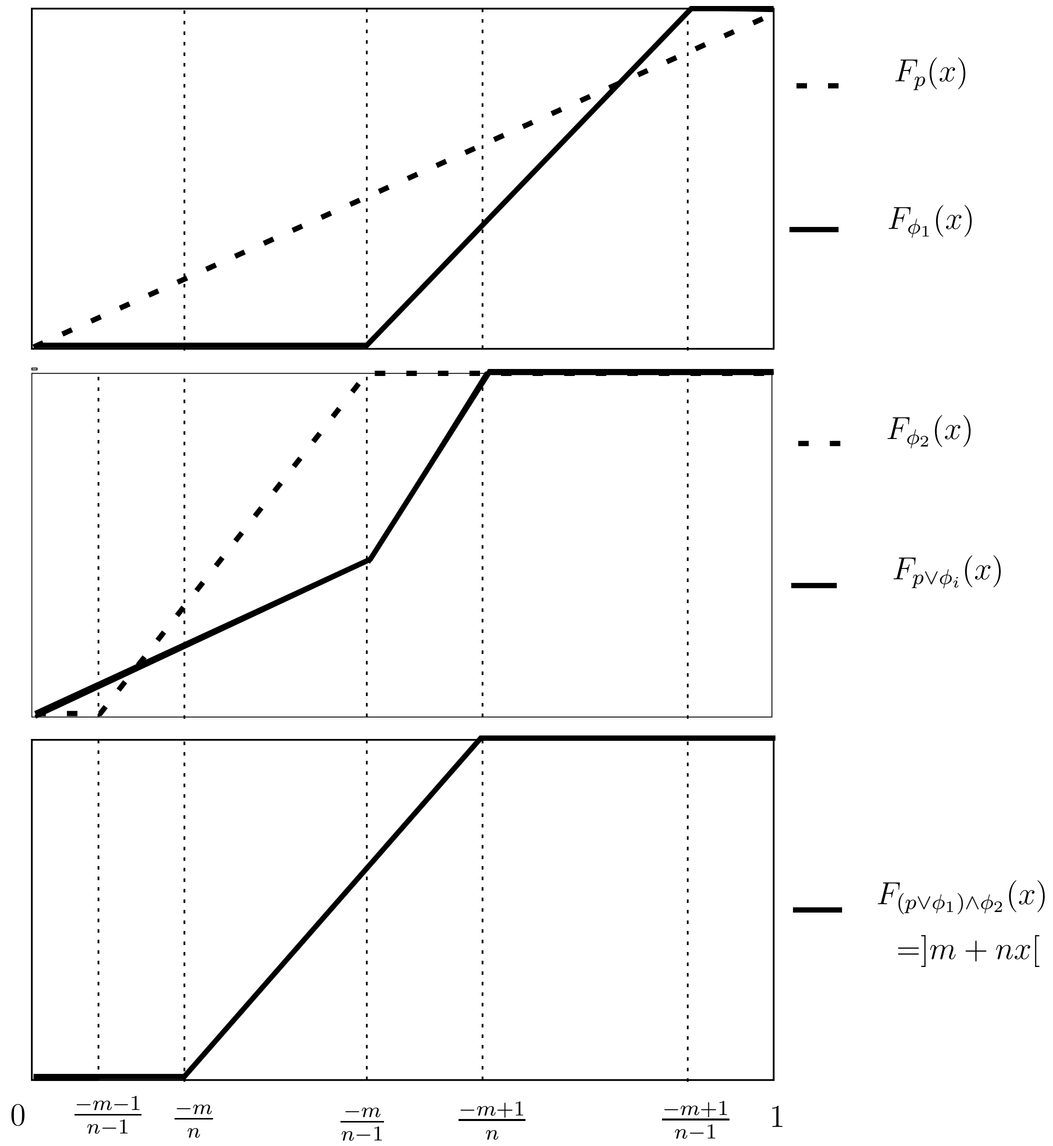
The proof is by induction on $|n|$ (for all m). In case $n = 0$ we can take

$$\theta = \begin{cases} p \to p & \text{if } m > 0, \\ \neg(p \to p) & \text{if } m \leq 0. \end{cases}$$

Now assume the result for $|n| - 1 (\geq 0)$ and all $m \in \mathbb{Z}$. In the first case suppose $n > 0$, so $|n| - 1 = n - 1$. By I.H. let $\phi_1, \phi_2 \in SL$ be such that

$$F_{\phi_1}(x) =]m + (n-1)x[, \qquad F_{\phi_2}(x) =]m + 1 + (n-1)x[.$$

Then $(\phi_1 \vee p) \wedge \phi_2$ is the required θ as can be seen(!) from the following graphs:



In the other case $n < 0$, so $|n| - 1 = |n+1|$ and a similar diagram shows that if we take $\psi_1, \psi_2 \in SL$ such that

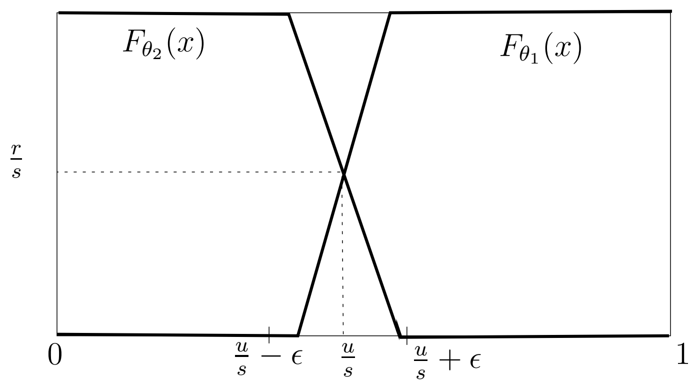
$$F_{\psi_1}(x) =](m-1) + (n+1)x[, \qquad F_{\psi_2}(x) =]m + (n+1)x[$$

then the required θ is $(\psi_1 \vee \neg p) \wedge \psi_2$.

The next step in the proof is to notice that if we are given $0 < r/s, u/s < 1$ and $0 < \epsilon$, with $s, r, u \in \mathbb{N}$ and u, s relatively prime, then there are $\theta_1, \theta_2 \in SL$ such that the graphs of $F_{\theta_1}(x), F_{\theta_2}(x)$ look like the dark lines in figure 2.
Precisely, by the previous result pick θ_1 such that $F_{\theta_1}(x) =]b + ax[$ where $au + bs = r$ and $a \in \mathbb{N}$ is very large, so

$$F_{\theta_1}(u/s) =]b + au/s[= r/s$$

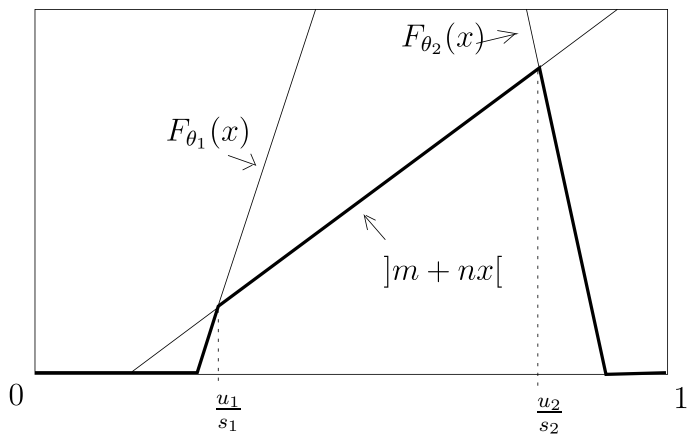
and $F_{\theta_1}(x)$ cuts the x-axis between $u/s - \epsilon$ and u/s etc.. It is possible to find such a, b since because u, s are relatively prime, by the Euclidean Algorithm, there are a', b' such that $a'u + b's = 1$ and we can then take $a = ra' + ks$, $b = rb' - ku$ for a sufficiently large $k \in \mathbb{N}$.



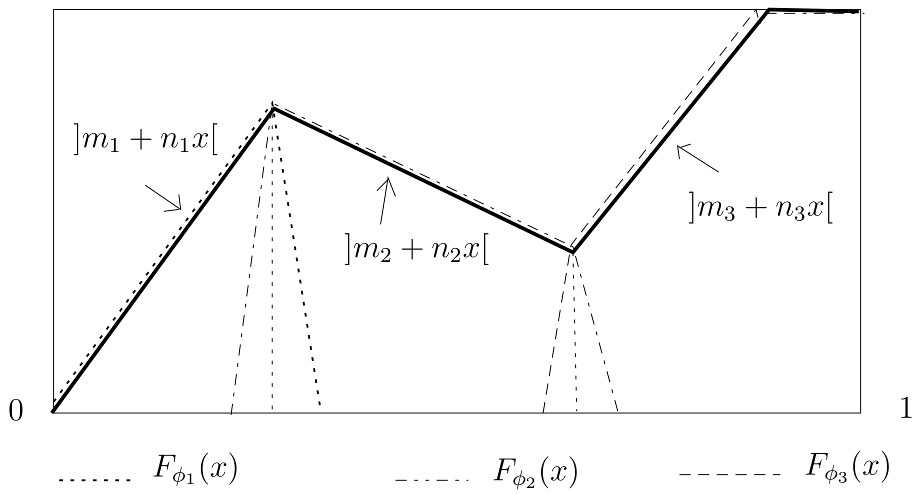
Now suppose $\phi \in SL$, $F_\phi(x) =]m+nx[$ and $0 \leq u_1/s_1 < u_2/s_2 \leq 1$, with both fractions proper, i.e. u_1, s_1 and u_2, s_2 relatively prime. Notice that

$$F_\phi(u_1/s_1) =]m + nu_1/s_1[= r_1/s_1, \qquad F_\phi(u_2/s_2) =]m + nu_2/s_2[= r_2/s_2$$

for some r_1, r_2. It now follows that we can, as above, find $\theta_1, \theta_2 \in SL$ with arbitrarily steep slopes (up at u_1/s_1 for θ_1, down at u_2/s_2 for θ_2) such that $F_{\theta_1}(x), F_{\theta_2}(x)$ and $F_\theta(x)$, given by the dark line, for $\theta = (\theta_1 \underline{\triangle} \theta_2) \underline{\triangle} \phi$, look as in:



Finally by taking maximums of functions of this form we can fabricate any function of the type described in McNaughton's Theorem (provided we take the γ_i to be rational, which we've already seen we can do). This construction is indicated in:



Here the required θ, where $F_\theta(x)$ is the dark line, is given by $(\phi_1 \underline{\vee} \phi_2) \underline{\vee} \phi_3$. ■

We now turn our attention to proving a completeness theorem[6] for Ł. That is, we would like to produce a set of rules and axioms for Ł such that for $\Gamma \subseteq SL, \theta \in SL$,

$$\Gamma \models^{\text{Ł}} \theta \ \Leftrightarrow \ \Gamma \vdash^{\text{Ł}} \theta,$$

where, as usual, $\Gamma \vdash^{\text{Ł}} \theta$ means that there is a (proof in Ł) $\Gamma_1|\theta_1, \Gamma_2|\theta_2, ..., \Gamma_k|\theta_k$ where each Γ_i is a finite subset of SL, each $\theta_i \in SL$, $\Gamma_k \subseteq \Gamma$, $\theta_k = \theta$ and each sequent in this proof is 'justified' either by being an axiom (of the proof system) or by following from some earlier sequents by one of the rules, and $\Gamma \models^{\text{Ł}} \theta$ means that for any valuation w (for Ł), if $w(\phi) = 1$ for all $\phi \in \Gamma$ then $w(\theta) = 1$.

[6]As it turns out this will only be possible for *finite* Γ.

As usual when $\Gamma = \emptyset$ we write $\models^{Ł} \theta$ instead of $\emptyset \models^{Ł} \theta$. In this case the definition reduces to,

$$\models^{Ł} \theta \iff w(\theta) = 1 \text{ for all } [0,1]-\text{valuations } w.$$

which we might on occasions shorten to just 'θ is a tautology of Ł'.

Notice that (unlike the case for modal and non-monotonic logic(s)) $\models^{Ł}$ does not extend $\models^{SC}$. For example

$$\not\models^{Ł} (p \to \neg p) \to \neg p$$

whereas

$$\models^{SC} (p \to \neg p) \to \neg p.$$

Instead however the converse does hold (so, in this sense, Ł is weaker than SC) since because any $\{0,1\}$-valuation on SL is also an $[0,1]$-valuation on SL, if $\Gamma \models^{Ł} \theta$ then $\Gamma \models^{SC} \theta$.

We shall now give a system of rules and axioms due to Łukasiewicz (together also with REF which we add in in order to allow 'assumptions' - more on that later). It is, to my mind, rather surprising that these suffice given how far away, in terms of how much work has to be done, they still are from 'the Completeness Theorem' (which Łukasiewicz himself did not obtain, it was proved later by Rose & Rosser and, independently Wajsberg). In fact it turned out much later that one of these axioms, Ł5, is derivable from the others, although we shall keep it in because that simplifies things slightly. In (one) of these axioms, and in what follows, we shall use the 'defined connectives' $\vee, \wedge, \underline{\wedge}, \underline{\vee}$ etc., This will make rules etc easier to appreciate and work with, although we shouldn't forget what they actually stand for.

Unfortunately proving this completeness result is considerably more difficult than those for SC or the modal logics K, T, D etc, although the proof again uses the idea of forming some sort of 'maximal consistent set of sentences'. In part the reason for this is that in the step from a maximal consistent subset of SL to a valuation 'satisfying it' we somehow need to conjure up real truth values for our propostional variables and, unfortunately these don't simply fall into our lap in the way the truth values 0,1 did in the corresponding proof for SC. The proof we shall briefly sketch (there isn't time to do all the messy checking) is based on some simplifications due to Dana Scott.

A proof theory for Ł

Axioms

REF: For $\theta \in \Gamma$, $\Gamma|\theta$ (notice that this is an inessential generalisation of some earlier versions of this axiom)

Ł1 : $|\theta \to (\phi \to \theta)$
Ł2 : $|(\theta \to \phi) \to ((\phi \to \psi) \to (\theta \to \psi))$
Ł3 : $|(\neg\theta \to \neg\phi) \to (\phi \to \theta)$
Ł4 : $|((\theta \to \phi) \to \phi) \to ((\phi \to \theta) \to \theta)$ i.e. $|(\theta \underline{\vee} \phi) \to (\phi \underline{\vee} \theta)$
Ł5 : $|(\theta \to \phi)\underline{\vee}(\phi \to \theta)$

Rules

$$\text{MP}: \quad \frac{\Gamma|\theta \to \phi, \quad \Delta|\theta}{\Gamma, \Delta|\phi}$$

As usual we define:

Definition. A *proof* (in Ł) is a finite sequence of sequents

$$\Gamma_1|\,\theta_1, \ldots, \Gamma_m|\,\theta_m,$$

where $\theta_i \in SL$ and $\Gamma_i \subseteq SL$ are finite, such that for each $i = 1, \ldots, m$, either $\Gamma_i|\,\theta_i$ is an instance of an axiom of Ł or for some $j, k < i$

$$\frac{\Gamma_j|\,\theta_j \qquad \Gamma_k|\,\theta_k}{\Gamma_i|\,\theta_i}$$

is an instance of MP.

The natural number m here is called the *length* of the proof.
For $\theta \in SL$ and $\Gamma \subseteq SL$, we define

$$\Gamma \vdash^{\text{Ł}} \theta \iff \text{there is a proof } \Gamma_1|\,\theta_1, \ldots, \Gamma_m|\,\theta_m \text{ in Ł such that } \Gamma_m \subseteq \Gamma \text{ and } \theta_m = \theta.$$

In this case $\Gamma_1|\,\theta_1, \ldots, \Gamma_m|\,\theta_m$ is called a *proof*[7] *of* θ *from* Γ *in* Ł.

Proposition 8 *If* $\Gamma|\theta$ *is any instance of an axiom of Ł then* $\Gamma \vdash^{Ł} \theta$. *If*

$$\frac{\Gamma|\theta \to \phi, \quad \Delta|\theta}{\Gamma, \Delta|\phi}$$

is any instance of the rule MP of Ł and $\Gamma \vdash^{Ł} \theta \to \phi$, $\Delta \vdash^{Ł} \theta$, *then* $\Gamma, \Delta \vdash^{Ł} \phi$.

We leave the proof of Proposition 8 as an exercise because it is completely parallel to the proof of proposition 1.1 in the case of modal logic. Again, as for 1.1, we will use Proposition 8 extensively, and often without explicit mention, in what follows.

[7]We insert the adjective 'formal' when we want to distinguish between a proof in this sense and an 'informal proof' such as we would give to justify a theorem.

Notice that Ł has no rules for moving sentences across the $|$. So the left hand side 'Γ' in a proof need simply consist of the 'assumptions' which find their way to the righthand side of $|$ by the rule REF. In the standard Łukasiewicz Logic, where the interest is purely in showing a completness theorem when the lefthand sides are empty, i.e.

$$\models^{Ł} \theta \iff \vdash^{Ł} \theta,$$

the rule REF does not appear.

Notice then that if $\Gamma \vdash^{Ł} \theta$ then also $\Gamma, \Delta \vdash^{Ł} \theta$, simply from the definition of a proof in Ł, so in what follows we may always unhesitatingly use the usual rule MON when required.

Proposition 9 *If $\Gamma|\theta$ is any instance of an axiom of L then it is valid in L i.e. $\Gamma \models^{L} \theta$. If*

$$\frac{\Gamma|\theta \to \phi, \quad , \Delta|\theta}{\Gamma, \Delta|\phi}$$

is an instance of the rule MP of L and $\Gamma \models^{L} \theta \to \phi$, $\Delta \models^{L} \theta$ then $\Gamma, \Delta \models^{L} \phi$.

Proof We just have to check through each of them in turn. We'll just do the axioms Ł2 and Ł5 and leave the rest as exercises.

Ł2: Suppose w is a valuation. If $w(\phi \to \psi) \leq w(\theta \to \psi))$ then $w((\phi \to \psi) \to (\theta \to \psi)) = 1$ so clearly

$$w(\theta \to \phi) \leq w((\phi \to \psi) \to (\theta \to \psi)),$$

and hence,

$$w((\theta \to \phi) \to ((\phi \to \psi) \to (\theta \to \psi))) = 1,$$

as required. Otherwise $w(\theta \to \psi) < w(\phi \to \psi) \leq 1$ so

$$w(\theta \to \psi) = 1 - w(\theta) + w(\psi) \leq w(\phi \to \psi) \leq 1 - w(\phi) + w(\psi).$$

Hence $w(\phi) \leq w(\theta)$ and

$$\begin{aligned} w(\theta \to \phi) &= 1 - w(\theta) + w(\phi) \\ &= 1 + (1 - w(\theta) + w(\psi)) - (1 - w(\phi) + w(\psi)) \\ &= 1 + w(\theta \to \psi) - (1 - w(\phi) + w(\psi)) \\ &\leq 1 + w(\theta \to \psi) - w(\phi \to \psi) \\ &= w((\phi \to \psi) \to (\theta \to \psi)) \end{aligned}$$

and so again in this case,

$$w((\theta \to \phi) \to ((\phi \to \psi) \to (\theta \to \psi))) = 1,$$

as required.

Ł5: If $w(\theta) \leq w(\phi)$ then $w(\theta \to \phi) = 1$ so

$$w((\theta \to \phi)\underline{\vee}(\phi \to \theta)) = max\{w(\theta \to \phi), w(\phi \to \theta)\} = 1.$$

Otherwise $w(\theta) > w(\phi)$ and so $w(\phi \to \theta) = 1$ and again

$$w((\theta \to \phi)\underline{\vee}(\phi \to \theta)) = 1,$$

as required. ■

Corollary 10 (The Correctness Theorem for Ł)) *For* $\Gamma \subseteq SL,\ \theta \in SL,$

$$\Gamma \vdash^{L} \theta \implies \Gamma \models^{L} \theta.$$

Proof As usual, by induction on the length of proof of $\Gamma \vdash^{Ł} \theta$ using Proposition 9. ■

At this point one would usually aim to extend the Correctness Theorem to the:

Theorem 11 (The Completeness Theorem for Ł) *For* $\xi \in SL$ *and finite* $\Gamma \subseteq SL,$

$$\Gamma \models^{L} \xi \iff \Gamma \vdash^{L} \xi.$$

Unfortunately we do not have the time in this course to go through the proof and will merely sketch some of its key features.

Notes

(1) We cannot improve the Completeness Theorem to *infinite* Γ as the following example shows:- Let

$$\Gamma = \{\neg p \to p^n \mid n > 0\},$$

where $p^n = p \wedge (p \wedge (p \wedge ... \wedge (p \wedge p)...))$ with n conjunctions $\wedge$. Then $\Gamma \models^{Ł} p$, since suppose w was a valuation such that $w(p) < 1$, say $w(p) < 1 - 1/n$. Then

$$w(p^n) = \max\{0, 1 - nw(\neg p)\} = \max\{0, nw(p) - (n-1)\} = 0 \quad \text{(see Examples 2 Q13)},$$

so

$$w(\neg p \to p^n) \leq 1 - w(\neg p) + 0 = w(p) < 1,$$

and hence such a w cannot satisfy Γ.

However $\Gamma \nvdash^{Ł} p$, since suppose $\Gamma \vdash^{Ł} p$. Then, straight from the definition of a proof, there would be a finite subset

$$\Gamma_0 = \{\neg p \to p^{n_1}, \neg p \to p^{n_2}, ..., \neg p \to p^{n_k}\}$$

of Γ such that $\Gamma_0 \vdash^{Ł} p$. But this is not possible by the Correctness Theorem since if w is the valuation which gives p value $1 - (1 + max\{n_1, n_2, ..., n_k\})^{-1}$ then $w(\neg p) \leq w(p^{n_i})$. Therefore $w(\neg p \to p^{n_i}) = 1$, for $i = 1, 2, ..., k$, so w satisfies Γ_0 whilst $w(p) < 1$.

[Clearly the 'problem' here is that our valuations are taking values in the standard real interval $[0, 1]$. If instead we allowed our valuations to take values in arbitrary non-standard extensions of $[0, 1]$ then we could allow Γ to be infinite in the Completeness Theorem – exercise!!]

(2) Notice that if we define the relation $\mid\!\sim$ by, $\Gamma \mid\!\sim \theta$ if for all valuations w

$$\inf\{\, w(\phi) \mid \phi \in \Gamma\,\} \leq w(\theta)$$

then for finite Γ,

$$\Gamma \mid\!\sim \theta \iff \vdash^{Ł} \underline{\bigwedge}\Gamma \to \theta.$$

In preparation for proving the Completeness Theorem for Ł one needs to give proofs in Ł of a rather long list of derived rules and sentences of Ł. Some of these we shall prove, largely to give you some insight into how such proofs go[8] others will be in the examples sheets, but some I'm afraid, like most of what follows now, you'll just have you will have to just accept or prove for yourself.

[8] They are often fiendishly difficult to produce first time round, as one might have already guessed from the fact that it was quite late on that Ł5 was shown to be derivable from the other axioms.

Proposition 12 .

(r1)	If $\Gamma \vdash^{Ł} \theta \to \phi$, $\Gamma \vdash^{Ł} \phi \to \psi$ then $\Gamma \vdash^{Ł} \theta \to \psi$
(a11)	$\vdash^{Ł} \phi \to ((\phi \to \theta) \to \theta)$
$Ł4'$	$\vdash^{Ł} (\theta \to (\phi \to \psi)) \to (\phi \to (\theta \to \psi))$
(a1)	$\vdash^{Ł} \theta \to \theta$
(a2)	$\vdash^{Ł} (\psi \to \eta) \to ((\theta \to \psi) \to (\theta \to \eta))$
(r2)	If $\Gamma \vdash^{Ł} \psi \to \eta$ then $\Gamma \vdash^{Ł} (\theta \to \psi) \to (\theta \to \eta)$
(r3)	If $\Gamma \vdash^{Ł} \psi \to \eta$ then $\Gamma \vdash^{Ł} (\eta \to \theta) \to (\psi \to \theta)$
(r4)	If $\Gamma \vdash^{Ł} \psi$ then $\Gamma \vdash^{Ł} \theta \to \psi$
(a3)	$\vdash^{Ł} \neg\neg\theta \to \theta$
(a4)	$\vdash^{Ł} (\theta \to \neg\phi) \to (\phi \to \neg\theta)$
(a5)	$\vdash^{Ł} \theta \to \neg\neg\theta$
(a6)	$\vdash^{Ł} (\neg\theta \to \phi) \to (\neg\phi \to \theta)$
(a7)	$\vdash^{Ł} \neg\theta \to (\theta \to \neg(p \to p))$
(a8)	$\vdash^{Ł} (\theta \to \neg(p \to p)) \to \neg\theta$
(a13)	$p \vdash^{Ł} \neg(p \to \neg p)$
(r5)	If $\Gamma \vdash^{Ł} \phi \to \psi$, then $\Gamma \vdash^{Ł} ((\psi \to \phi) \to (\psi \to \theta)) \to (\phi \to \theta)$
(r6)	If $\Gamma \vdash^{Ł} \theta \to \phi$ then $\Gamma \vdash^{Ł} ((\phi \to \theta) \to \theta) \to \phi$
(r7)	If $\Gamma \vdash^{Ł} \theta \to \phi$, $\Gamma \vdash^{Ł} (\phi \to \theta) \to \psi$ then $\Gamma \vdash^{Ł} (\psi \to \theta) \to \phi$
(a14)	$\vdash^{Ł} (\theta \to \phi) \to ((\phi \to \psi) \to ((\theta \underline{\vee} \phi) \to \psi))$
(a15)	$\vdash^{Ł} (\theta \to \psi) \to ((\phi \to \psi) \to ((\theta \underline{\vee} \phi) \to \psi))$

Proof

(r1): By Ł2, $\vdash^{Ł} (\theta \to \phi) \to ((\phi \to \psi) \to (\theta \to \psi))$ and the rule follows by using MP twice.

(a11): By Ł1 $\vdash^{Ł} \phi \to ((\theta \to \phi) \to \phi)$ and the conclusion follows by Ł4 and (r1).

Ł4′: By Ł2,

$$\vdash^{Ł} ((\theta \to (\phi \to \psi)) \to ((\phi \to \psi) \to \psi) \to (\theta \to \psi)),$$

and

$$\vdash^{Ł} (\phi \to ((\phi \to \psi) \to \psi)) \to ((((\phi \to \psi) \to \psi) \to (\theta \to \psi)) \to (\phi \to (\theta \to \psi))).$$

Hence by this last expression with MP and (a11),

$$\vdash^{Ł} (((\phi \to \psi) \to \psi) \to (\theta \to \psi)) \to (\phi \to (\theta \to \psi)),$$

and using (r1) and the first expression Ł4′ follows.

(a1): $\vdash^{Ł} \theta \to (\phi \to \theta)$ by Ł1, so by Ł4′ and MP, $\vdash^{Ł} \phi \to (\theta \to \theta)$. Taking ϕ to be any axiom, so $\vdash^{Ł} \phi$, gives via MP that $\vdash^{Ł} \theta \to \theta$.

(a14): From (r2),

$$\vdash^{Ł} (\phi \to \psi) \to ((\theta \to \phi) \to \phi) \to ((\theta \to \phi) \to \psi)).$$

By Ł4′ and (r2),

$$\vdash^{Ł} ((\phi \to \psi) \to ((\theta \to \phi) \to \phi) \to ((\theta \to \phi) \to \psi)))$$
$$\to ((\phi \to \psi) \to ((\theta \to \phi) \to (((\theta \to \phi) \to \phi) \to \psi))),$$

so by MP
$$\vdash^{Ł} ((\phi \to \psi) \to ((\theta \to \phi) \to (((\theta \to \phi) \to \phi) \to \psi))),$$
and again by MP and Ł4′,

$$\vdash^{Ł} ((\theta \to \phi) \to ((\phi \to \psi) \to (((\theta \to \phi) \to \phi) \to \psi))),$$

as required.

(a15): By Ł2,

$$\vdash^{Ł} ((\psi \to \phi) \to (\theta \to \phi)) \to (((\theta \to \phi) \to \phi) \to ((\psi \to \phi) \to \phi)).$$

Also by Ł2,
$$\vdash^{Ł} (\theta \to \psi) \to ((\psi \to \phi) \to (\theta \to \phi)),$$
so by (r1),
$$\vdash^{Ł} (\theta \to \psi) \to (((\theta \to \phi) \to \phi) \to ((\psi \to \phi) \to \phi)).$$
Since by Ł4
$$\vdash^{Ł} ((\psi \to \phi) \to \phi) \to ((\phi \to \psi) \to \psi),$$
using (r2) twice gives

$$\vdash^{Ł} ((\theta \to \psi) \to (((\theta \to \phi) \to \phi) \to ((\psi \to \phi) \to \phi)))$$
$$\to ((\theta \to \psi) \to (((\theta \to \phi) \to \phi) \to ((\phi \to \psi) \to \psi))).$$

and an application of MP gives

$$\vdash^{Ł} (\theta \to \psi) \to (((\theta \to \phi) \to \phi) \to ((\phi \to \psi) \to \psi)).$$

Finally Ł4′, (r1) and MP yield (a15).

The remainder of the proof either appears on the example sheets, or the take home test, or is left as an easy exercise. ■

You might wonder at this point what has happened to the so called 'Deduction Theorem', that is, the rule
$$\frac{\Gamma, \theta \mid \phi}{\Gamma \mid \theta \to \phi},$$
which we did have in SC (as IMR) and Non-monotonic Logic (as CON). Unfortunately it fails in Ł, since $p \vdash^{Ł} \neg(p \to \neg p)$ (see (a13)) whilst if $w(p) = 1/2$ then

$$w(p \to \neg(p \to \neg p)) = 1/2 \neq 1$$

so $\nvdash^{Ł} (p \to \neg(p \to \neg p))$ by Proposition 10.

However we do have a version of this 'theorem' which in fact plays a vital role in the proof of the Completeness Theorem.

Theorem 13

$$\Gamma, \theta \vdash^{Ł} \phi \iff \text{for some } n \in \mathbb{N} \;\; \Gamma \vdash^{Ł} \theta \to_n \phi,$$

where $\theta \to_n \phi$ *is defined recursively by*

$$(\theta \to_0 \phi) = \phi$$

$$(\theta \to_{(n+1)} \phi) = (\theta \to (\theta \to_n \phi)).$$

THE MATERIAL WHICH FOLLOWS WILL NOT BE EXAMINED, IT IS INCLUDED HERE JUST FOR INTEREST

Before giving the proof of Theorem 13 it will be useful to have a key lemma.

Lemma 14 *If*

$$\Delta \vdash^{Ł} \theta \to_n \phi,$$

$$\Delta \vdash^{Ł} \theta \to_n (\phi \to \psi),$$

then

$$\Delta \vdash^{Ł} \theta \to_{2n} \psi.$$

Proof By induction on n. If $n = 0$ the result is immediate by MP. So assume that $n > 0$ and the result holds for $n - 1$. From the assumptions of the lemma we have that

$$\Delta \vdash^{Ł} \theta \to_{(n-1)} (\theta \to \phi), \tag{12}$$

$$\Delta \vdash^{Ł} \theta \to_{(n-1)} (\theta \to (\phi \to \psi)). \tag{13}$$

By Ł4′

$$\Delta \vdash^{Ł} (\theta \to (\phi \to \psi)) \to (\phi \to (\theta \to \psi))$$

so by (r1) and (r2) repeatedly

$$\Delta \vdash^{Ł} (\theta \to_{(n-1)} (\theta \to (\phi \to \psi))) \to (\theta \to_{(n-1)} (\phi \to (\theta \to \psi))).$$

With (13) then and MP,

$$\Delta \vdash^{Ł} \theta \to_{(n-1)} (\phi \to (\theta \to \psi)). \tag{14}$$

From (r2) we have

$$\Delta \vdash^{Ł} (\phi \to (\theta \to \psi)) \to ((\theta \to \phi) \to (\theta \to (\theta \to \psi))$$

and as above we can now obtain that

$$\Delta \vdash^{Ł} (\theta \to_{(n-1)} (\phi \to (\theta \to \psi))) \to (\theta \to_{(n-1)} ((\theta \to \phi) \to (\theta \to (\theta \to \psi)))).$$

Hence with (14) and MP,

$$\Delta \vdash^{Ł} \theta \to_{(n-1)} ((\theta \to \phi) \to (\theta \to (\theta \to \psi)).$$

Now by inductive hypothesis with (12),

$$\Delta \vdash^{\text{Ł}} \theta \to_{2(n-1)} (\theta \to (\theta \to \psi)),$$

as required. ■

We are now ready to give the proof of Theorem 13.

Proof
$\Leftarrow$: Proof by induction on n. Assume $\Gamma \vdash^{\text{Ł}} \theta \to_n \phi$. If $n = 0$ then $\Gamma, \theta \vdash^{\text{Ł}} \theta \to_n \phi$ by MON. Assume the result for $n-1$. Then since $\Gamma, \theta \vdash^{\text{Ł}} \theta$ by REF (and Proposition 8), by MP (and Proposition 8), $\Gamma, \theta \vdash^{\text{Ł}} \theta \to_{(n-1)} \phi$ and the result follows by the inductive hypothesis.

$\Rightarrow$: Assume $\Gamma, \theta \vdash^{\text{Ł}} \phi$, say $\Gamma_1|\phi_1, \;\; \Gamma_2|\phi_2, ..., \Gamma_m|\phi_m$ is a proof of this (so Γ_m is a finite subset of $\Gamma \cup \{\theta\}$ and $\phi_m = \phi$). We show by induction that for each $i = 1, 2, ..., m$ there exists $n_i \in \mathbb{N}$, such that $\Gamma_i - \{\theta\} \vdash^{\text{Ł}} \theta \to_{n_i} \phi_i$. Clearly this suffices.

So suppose the result holds for $j < i$. If $|\phi_i$ is one of the axioms Ł1-5 then $\Gamma_i \vdash^{\text{Ł}} \phi_i$ so the conclusion holds with $n_i = 0$. If $\Gamma_i, \theta \vdash^{\text{Ł}} \phi_i$ is an instance of REF then either $\phi_i \neq \theta$, in which case again $\Gamma_i - \{\theta\} \vdash^{\text{Ł}} \phi_i$ and we can take $n_i = 0$, or $\phi_i = \theta$ in which case $\Gamma_i - \{\theta\} \vdash^{\text{Ł}} \theta \to \phi_i$ by Proposition 12(a1) and we can take $n_i = 1$.

The last case is where $\Gamma_i \vdash^{\text{Ł}} \phi_i$ is justified by the rule MP, in which case there are $j, k < i$ with $\phi_k = (\phi_j \to \phi_i)$ and $\Gamma_i = \Gamma_k \cup \Gamma_j$. By IH

$$\Gamma_j - \{\theta\} \vdash^{\text{Ł}} \theta \to_{n_j} \phi_j,$$

$$\Gamma_k - \{\theta\} \vdash^{\text{Ł}} \theta \to_{n_k} (\phi_j \to \phi_i),$$

for some $n_j, n_k \in \mathbb{N}$. By (r4) repeatedly and MON both these also hold if we replace n_j, n_k by $r = max\{n_j, n_k\}$ and $\Gamma_j - \{\theta\}$, $\Gamma_k - \{\theta\}$, by $\Gamma_i - \{\theta\} = (\Gamma_j - \{\theta\}) \cup (\Gamma_k - \{\theta\})$ to give

$$\Gamma_i - \{\theta\} \vdash^{\text{Ł}} \theta \to_r \phi_j,$$

$$\Gamma_i - \{\theta\} \vdash^{\text{Ł}} \theta \to_r (\phi_j \to \phi_i).$$

But then by Lemma 14

$$\Gamma_i - \{\theta\} \vdash^{\text{Ł}} \theta \to_{2r} \phi_i,$$

as required to complete the proof of the result. ■

Corollary 15 *If* $\Gamma, \theta \vdash^{L} \psi$ *and* $\Gamma, \phi \vdash^{L} \psi$ *then* $\Gamma, \theta \underline{\vee} \phi \vdash^{L} \psi$

Proof
By Theorem 13 and (r4) repeatedly the hypotheses $\Gamma, \theta \vdash^{\text{Ł}} \psi$ and $\Gamma, \phi \vdash^{\text{Ł}} \psi$ give us that $\Gamma \vdash^{\text{Ł}} \theta \to_n \psi$ and $\Gamma \vdash^{\text{Ł}} \phi \to_n \psi$ for some n. We prove by induction on n that this gives that $\Gamma, \theta \underline{\vee} \phi \vdash^{\text{Ł}} \psi$, equivalently, by Theorem 13, that

$$\Gamma \vdash^{\text{Ł}} (\theta \underline{\vee} \phi) \to_m \psi$$

for some m. If $n = 0$ the result is immediate, whilst if $n = 1$ it follows easily using MP with (a15). So assume $n > 1$ and the result is true, for all θ, ϕ, ψ, for $n - 1$. From (a15) and (r2) repeatedly we obtain

$$\Gamma \vdash^{\text{Ł}} (\theta \to_{n-1} (\theta \to \psi)) \to (\theta \to_{n-1} ((\phi \to \psi) \to ((\theta \underline{\vee} \phi) \to \psi)),$$

which with the assumption $\Gamma \vdash^{\text{Ł}} \theta \to_n \psi$ and MP gives

$$\Gamma \vdash^{\text{Ł}} (\theta \to_{n-1} ((\phi \to \psi) \to ((\theta \underline{\vee} \phi) \to \psi)).$$

Again by a similar trick with (a14), and the fact that $\Gamma \vdash^{\text{Ł}} \phi \to_{n-1} (\theta \to \phi)$ follows from Ł1 and (r4) (for $n > 2$), we obtain that

$$\Gamma \vdash^{\text{Ł}} (\phi \to_{n-1} ((\phi \to \psi) \to ((\theta \underline{\vee} \phi) \to \psi)).$$

By inductive hypothesis then

$$\Gamma \vdash^{\text{Ł}} ((\theta \underline{\vee} \phi) \to_m ((\phi \to \psi) \to ((\theta \underline{\vee} \phi) \to \psi)),$$

for some m and similarly,

$$\Gamma \vdash^{\text{Ł}} ((\theta \underline{\vee} \phi) \to_m ((\theta \to \psi) \to ((\theta \underline{\vee} \phi) \to \psi)).$$

Again since by Ł1,

$$\vdash^{\text{Ł}} \psi \to (\theta \to \psi),$$

so by (r2),

$$\vdash^{\text{Ł}} (\phi \to_n \psi) \to (\phi \to_{n-1} (\phi \to (\theta \to \psi)).$$

Hence using our assumption that $\Gamma \vdash^{\text{Ł}} \phi \to_n \psi$,

$$\Gamma \vdash^{\text{Ł}} \phi \to_{n-1} (\phi \to (\theta \to \psi)).$$

By symmetry, together with using Ł4′, (r2) repeatedly and MP,

$$\Gamma \vdash^{\text{Ł}} (\theta \to_{n-1} (\phi \to (\theta \to \psi)).$$

By inductive hypothesis then,

$$\Gamma \vdash^{\text{Ł}} (\theta \underline{\vee} \phi) \to_m (\phi \to (\theta \to \psi)),$$

for some m. By Theorem 13 we have now shown that

$$\Gamma, \theta \underline{\vee} \phi \vdash^{\text{Ł}} (\theta \to \psi) \to ((\theta \underline{\vee} \phi) \to \psi),$$

$$\Gamma, \theta \underline{\vee} \phi \vdash^{\text{Ł}} \phi \to (\theta \to \psi), \quad \text{etc.}$$

From these and (r1) we now obtain

$$\Gamma, \theta \underline{\vee} \phi \vdash^{\text{Ł}} \phi \to ((\theta \underline{\vee} \phi) \to \psi),$$

and symmetrically (with Ł4),

$$\Gamma, \theta \underline{\vee} \phi \vdash^{\text{Ł}} \theta \to ((\theta \underline{\vee} \phi) \to \psi).$$

By (a15) and MP twice we now obtain

$$\Gamma, \theta \underline{\vee} \phi \vdash^{\text{Ł}} (\theta \underline{\vee} \phi) \to ((\theta \underline{\vee} \phi) \to \psi),$$

from which $\Gamma, \theta \underline{\vee} \phi \vdash^{\text{Ł}} \psi$ follows by REF and MP. [Phew, what a struggle, there's a silver shilling on offer to the first person who can significantly simplify this proof.] ■

We are now ready to prove 'the big one'.

The Completeness Theorem for Ł
For $\xi \in SL$ and finite $\Gamma \subseteq SL$,

$$\Gamma \models^{\text{Ł}} \xi \iff \Gamma \vdash^{\text{Ł}} \xi.$$

Proof
$\Leftarrow$: This direction is immediate by the Correctness Theorem 10.

$\Rightarrow$: Assume $\Gamma \nvdash^{\text{Ł}} \xi$. We need to construct a valuation w such that $w(\phi) = 1$ for all $\phi \in \Gamma$ whilst $w(\xi) < 1$. Clearly since Γ is finite we may assume the overlying language L is finite, so each of the SL_j are also finite. Let Ω be a maximal subset of SL such that $\Omega \nvdash^{\text{Ł}} \xi$. Such a subset exists since if $\theta_n, n = 0, 1, 2, ...$ enumerates SL and we set $\Omega_0 = \Gamma$,

$$\Omega_{n+1} = \begin{cases} \Omega_n \cup \{\theta_n\} & \text{if } \Omega_n \cup \{\theta_n\} \nvdash^{\text{Ł}} \xi \\ \Omega_n & \text{otherwise,} \end{cases}$$

then $\Omega = \bigcup_n \Omega_n$ suffices.

For $\theta, \phi \in SL$ define $\theta \preceq \phi$ if $\Omega \vdash^{\text{Ł}} \theta \to \phi$, and $\theta \approx \phi$ if $\theta \preceq \phi$ and $\phi \preceq \theta$. By Proposition 12(a1),(r1) $\preceq$ is transitive and reflexive and $\approx$ is an equivalence relation.

Indeed $\preceq$ is connected. To see this suppose on the contrary that $\theta, \phi \in SL$ and neither $\theta \preceq \phi$ nor $\phi \preceq \theta$ hold. Then certainly neither $\theta \to \phi$ nor $\phi \to \theta$ can be in Ω. Therefore, by the maximality of Ω, it must be the case that

$$\Omega, (\theta \to \phi) \vdash^{\text{Ł}} \xi \quad \text{and} \quad \Omega, (\phi \to \theta) \vdash^{\text{Ł}} \xi.$$

By Corollary 15 then,

$$\Omega, (\theta \to \phi) \underline{\vee} (\phi \to \theta) \vdash^{\text{Ł}} \xi,$$

so by Theorem 13,

$$\Omega \vdash^{\text{Ł}} ((\theta \to \phi) \underline{\vee} (\phi \to \theta)) \to_n \xi,$$

for some n. But since, by Ł5,

$$\Omega \vdash^{\text{Ł}} (\theta \to \phi) \underline{\vee} (\phi \to \theta),$$

repeated use of MP gives $\Omega \vdash^{\text{Ł}} \xi$, contradiction.

Let

$$\mathcal{P} = \{\theta \in SL | \Omega \nvdash^{\text{Ł}} \theta\},$$

so $\xi \in \mathcal{P}$.

We now do something totally unexpected. Namely we drop our 'preconceptions' about what the $\theta \in SL$ 'really are' and temporarily treat them as simply orthonormal vectors from which we can form the inner product vector space $\mathbb{V}$ over the field of rationals $\mathbb{Q}$ of *expressions*

$$a_{\theta_1}\theta_1 + a_{\theta_2}\theta_2 + ... + a_{\theta_k}\theta_k,$$

where the $a_{\theta_i} \in \mathbb{Q}$ and the $\theta_i \in SL$ (with the obvious addition and scalar multiplication) and inner product determined by

$$\theta \cdot \phi = \begin{cases} 1 & \text{if } \theta = \phi, \\ 0 & \text{if } \theta \neq \phi. \end{cases}$$

Let $\mathcal{N}$ be the following subset of $\mathbb{V}$ (where $\theta = 1\theta$ etc.),

$$\mathcal{N} = \{\pm(\psi + \theta - \phi) | \phi \preceq \theta \text{ and } \psi \approx (\theta \to \phi)\}.$$

The key lemma relating $\mathcal{P}, \mathcal{N}$ is:-

Lemma 16 *Let* $\gamma_1, ..., \gamma_n \in \mathcal{N}$ *and* $\rho_1, ..., \rho_r \in \mathcal{P}$ *with* $r > 0$. *Then, in* $\mathbb{V}$,

$$\gamma_1 + \gamma_2 + ... + \gamma_n \neq \rho_1 + \rho_2 + ... + \rho_r.$$

Proof Suppose on the contrary that we had

$$\gamma_1 + \gamma_2 + ... + \gamma_n = \rho_1 + \rho_2 + ... + \rho_r.$$

for some $\gamma_1, ..., \gamma_n \in \mathcal{N}$ and $\rho_1, ..., \rho_r \in \mathcal{P}$ with $r > 0$. Then since ρ_1 appears on the righthand side of this equation there must be some $\gamma_i = \lambda_1 - \lambda_2 \pm \nu_1$, where $\lambda_1, \lambda_2, \nu_1 \in SL$, such that $\lambda_1 = \rho_1$. Now because of the form of the elements of $\mathcal{N}$ we can assume that the $\lambda_2, \pm\nu_1$ have been chosen so that one of the following four possibilities holds:-

(1) $\lambda_1 \approx (\nu_1 \to \lambda_2)$, $\pm = +$, $\lambda_2 \preceq \nu_1$,

(2) $\nu_1 \approx (\lambda_1 \to \lambda_2)$, $\pm = +$, $\lambda_2 \preceq \lambda_1$,

(3)$\lambda_2 \approx (\nu_1 \to \lambda_1)$, $\pm = -$, $\lambda_1 \preceq \nu_1$,

(4) $\nu_1 \approx (\lambda_2 \to \lambda_1)$, $\pm = -$, $\lambda_1 \preceq \lambda_2$,,

and $\lambda_2 \in \mathcal{P}$. That one of these forms (1)-(4) is possible is clear, so it only remains to show that we can also assume $\lambda_2 \in \mathcal{P}$, i.e. $\Omega \nvdash^{\text{Ł}} \lambda_2$. In cases (1) and (2) it is easy to show that if $\Omega \vdash^{\text{Ł}} \lambda_2$ then $\Omega \vdash^{\text{Ł}} \lambda_1$, contradiction, so it is enough to consider cases (3) and (4).

In case (3) suppose on the contrary that $\Omega \vdash^{\text{Ł}} \lambda_2$. If $\Omega \nvdash^{\text{Ł}} \nu_1$ then transposing ν_1, λ_2 gives (4) holding (with the required condition on λ_2), so we may also assume that $\Omega \vdash^{\text{Ł}} \nu_1$. But then since also $\lambda_2 \approx (\nu_1 \to \lambda_1)$, we have that

$\Omega \vdash^{\text{Ł}} \lambda_2 \to (\nu_1 \to \lambda_1)$,
$\Omega \vdash^{\text{Ł}} \lambda_2$,
$\Omega \vdash^{\text{Ł}} \nu_1$,

from which, by a couple of applications of MP, $\Omega \vdash^{\text{Ł}} \lambda_1$, contradiction. Case (4), of course, follows similarly.

Now since $-\lambda_2$ and appears in the lefthandside of the equation

$$\gamma_1 + \gamma_2 + ... + \gamma_n = \rho_1 + \rho_2 + ... + \rho_r.$$

in γ_i it must cancel with a positive copy of λ_2 apearing in some γ_j. Again we may assume

$$\gamma_j = \lambda_2 - \lambda_3 \pm \nu_2$$

with $\lambda_3 \in \mathcal{P}$. Now since there are only a finite number of these γ's they can't continue canceling out λ's indefinitely in this way. The reason it stops is because we must eventually repeat a γ, say the s-th γ equals the t-th γ, $s < t$ and that happens because it is ν_s which cancels with the $-\lambda_{t+1}$.

What this means is then that a subset of the γ's can be written as

$\lambda_s - \lambda_{s+1} + \lambda_{t+1}, \qquad (\pm\nu_s = \lambda_{t+1})$
$\lambda_{s+1} - \lambda_{s+2} \pm \nu_{s+1},$
$\lambda_{s+2} - \lambda_{s+3} \pm \nu_{s+2},$
....
$\lambda_t - \lambda_{t+1} \pm \nu_{t+1},$

where the $\lambda_s, \lambda_{s+1}, ..., \lambda_{t+1} \in \mathcal{P}$.

Since the ordering $\preceq$ is connected there must be some $s < k \leq t+1$ such that $\lambda_i \preceq \lambda_k$ for all $s < i \leq t+1$. Now replace each of the γ's in the above list by the corresponding expression in the list:-

$\lambda_s - (\lambda_k \to \lambda_{s+1}) + (\lambda_k \to \lambda_{t+1}),$
$(\lambda_k \to \lambda_{s+1}) - (\lambda_k \to \lambda_{s+2}) \pm \nu_{s+1},$
$(\lambda_k \to \lambda_{s+2}) - (\lambda_k \to \lambda_{s+3}) \pm \nu_{s+2},$
....
$(\lambda_k \to \lambda_t) - (\lambda_k \to \lambda_{t+1}) \pm \nu_{t+1}.$

Our claim now is that all of these are also in $\mathcal{N}$. Without loss of generality consider the second expression (vector),

$$(\lambda_k \to \lambda_{s+1}) - (\lambda_k \to \lambda_{s+2}) \pm \nu_{s+1}.$$

There are now a number of cases that need checking according to why the original vector

$$\lambda_{s+1} - \lambda_{s+2} \pm \nu_{s+1},$$

was in $\mathcal{N}$ in the first place.

Case 1: $\lambda_{s+1} \approx (\nu_{s+1} \to \lambda_{s+2}),\ \pm = +,\ \lambda_{s+2} \preceq \nu_{s+1}.$

In this case we need to show that

$$(\lambda_k \to \lambda_{s+1}) \approx (\nu_{s+1} \to (\lambda_k \to \lambda_{s+2})),$$

$$(\lambda_k \to \lambda_{s+2}) \preceq \nu_{s+1}.$$

The first of these follows since if $\lambda_{s+1} \approx (\nu_{s+1} \to \lambda_{s+2})$ then

$$(\lambda_k \to \lambda_{s+1}) \approx (\lambda_k \to (\nu_{s+1} \to \lambda_{s+2})) \approx (\nu_{s+1} \to (\lambda_k \to \lambda_{s+2})),$$

by (a2), MP, Ł4′, (r2), whilst the second follows directly from (r7).

The remaining cases are left as exercises.

Now we are ready to derive our contradiction. For $\gamma = \pm(\psi + \theta - \phi) \in \mathcal{N}$ let $\sharp\gamma$ be the number of ψ, θ, ϕ which are in $\mathcal{P}$. Now we may suppose that the equation

$$\gamma_1 + \gamma_2 + ... + \gamma_n = \rho_1 + \rho_2 + ... + \rho_r$$

has been chosen so that $\sum_i \sharp\gamma_i$ is as small as possible, subject to $r > 0$. The above discussion has shown that we can replace those γ's on the lefthandside which appear in

$\lambda_s - \lambda_{s+1} + \lambda_{t+1}, \qquad (\pm\nu_s = \lambda_{t+1})$
$\lambda_{s+1} - \lambda_{s+2} \pm \nu_{s+1},$
$\lambda_{s+2} - \lambda_{s+3} \pm \nu_{s+2},$
....
$\lambda_t - \lambda_{t+1} \pm \nu_{t+1},$

by the vectors

$\lambda_s - (\lambda_k \to \lambda_{s+1}) + (\lambda_k \to \lambda_{t+1}),$
$(\lambda_k \to \lambda_{s+1}) - (\lambda_k \to \lambda_{s+2}) \pm \nu_{s+1},$
$(\lambda_k \to \lambda_{s+2}) - (\lambda_k \to \lambda_{s+3}) \pm \nu_{s+2},$
....
$(\lambda_k \to \lambda_t) - (\lambda_k \to \lambda_{t+1}) \pm \nu_{t+1}.$

These vectors are still in $\mathcal{N}$ and the equality with the sum of the ρ_j is still maintained. However, whereas all the $\lambda_s, \lambda_{s+1}, ..., \lambda_{t+1}$ were in $\mathcal{P}$ at least one of their replacements

$$(\lambda_k \to \lambda_s), (\lambda_k \to \lambda_{s+1}), ..., (\lambda_k \to \lambda_{t+1}),$$

namely $(\lambda_k \to \lambda_k)$, is not. This then contradicts the minimality of the sum $\sum_i \sharp\gamma_i$, and we have our result. ■

In order to use this we need a corollary of the following (widely useful) result from the theory of vector spaces:-

Lemma 17 *Given a set of vectors $Q = \{\vec{q_1}, \vec{q_2}, ..., \vec{q_r}\}$ in $\mathbb{Q}^n$ let $Q^\angle$ be the cone*

$$\{e_1\vec{q_1} + e_2\vec{q_2} + ... + e_r\vec{q_r} | 0 \leq e_1, e_2, ..., e_r \in \mathbb{Q}\}.$$

Suppose $\vec{a} \notin Q^\angle$. Then there is a $\vec{b}$ such that $\vec{b} \cdot \vec{a} < 0$ whilst $0 \leq \vec{b} \cdot \vec{x}$ for all $\vec{x} \in Q^\angle$.

Proof
Let $\vec{c}$ be the nearest point to $\vec{a}$ in $Q^\angle$. [To see that such a point exists first notice that by a convexity argument there is a unique

$$\langle s_1, \ldots, s_r \rangle \in \{s \in \mathbb{R} \mid s \geq 0\}^r$$

for which

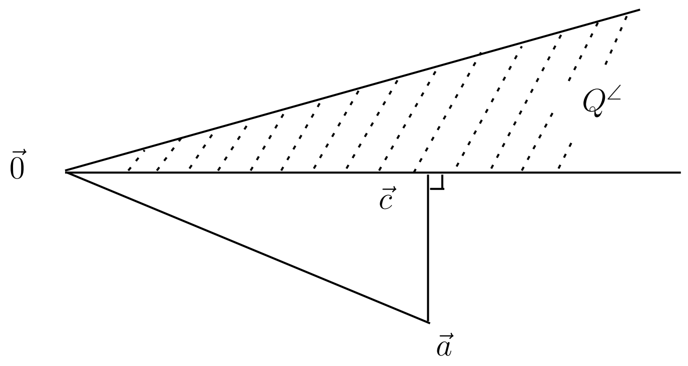
$$|\,\vec{a} - \sum_{i=1}^{r} s_i\vec{q_i}\,| \tag{15}$$

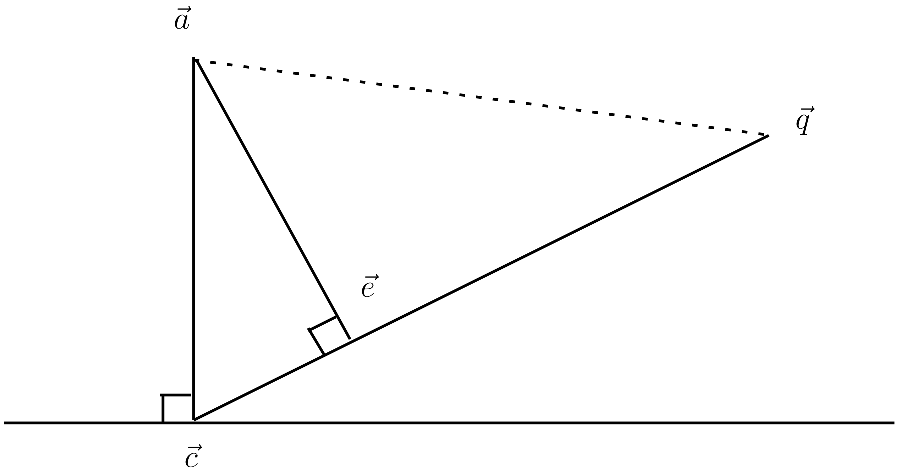
is minimal. But then by considering the usual Lagrange Multiplier method of solving for the non-zero s_i minimizing (15) we see that these s_i must be rational.]
Then if $\vec{c} \neq \vec{0}$, $\vec{b}$ must be perpendicular to $\vec{c}$ so $\vec{b} \cdot \vec{c} = 0$. Therefore,

$$\vec{a} \cdot \vec{b} = -(\vec{c} - \vec{a}) \cdot \vec{b} = -\vec{b} \cdot \vec{b} < 0.$$

On the other hand if $\vec{c} = \vec{0}$ then again $\vec{b} \cdot \vec{c} = 0$ so similarly $\vec{a} \cdot \vec{b} < 0$.

We claim that $\vec{b} \cdot \vec{q} \geq 0$ for all $\vec{q} \in Q^\angle$. For suppose on the contrary that $\vec{b} \cdot \vec{q} < 0$ for some $\vec{q} \in Q^\angle$. Then in the plane determined by the points $\vec{a}, \vec{c}, \vec{q}$ we have,





(since on the line segment from $\vec{q}$ to $\vec{a}$ we will always have $\vec{b} \cdot \vec{x} < 0$ so this line cannot cut the line through $\vec{c}$ in this plane which is perpendicular to $\vec{b}$). But then the indicated point $\vec{e}$ is in $Q^{\angle}$ and is nearer to $\vec{a}$ than $\vec{c}$, contradiction. ■

Corollary 18 *Let V be the vector subspace*

$$\{e_1\vec{v_1} + e_2\vec{v_2} + ... + e_m\vec{v_m} | e_1, e_2, ..., e_m \in \mathbb{Q}\}$$

of some vector space V' over $\mathbb{Q}$ and let $\vec{q_1}, \vec{q_2}, ..., \vec{q_r}$ be linearly independent vectors in V' such that $V \cap Q^{\angle} = \{\vec{0}\}$ where $Q = \{\vec{q_1}, \vec{q_2}, ..., \vec{q_r}\}$. Then there exists $\vec{b} \in V'$ such that $\vec{b} \cdot \vec{x} = 0$ for all $\vec{x} \in V$ and $\vec{b} \cdot \vec{y} \geq 0$ for all $\vec{y} \in Q^{\angle}$, with equality just if $\vec{y} = \vec{0}$.

Proof
First notice that for each $\vec{q_i}$,

$$\vec{q_i} \notin \{\vec{v_1}, \vec{v_2}, ..., \vec{v_m}, -\vec{v_1}, -\vec{v_2}, ..., -\vec{v_m}, -\vec{q_1}, -\vec{q_2}, ..., -\vec{q_r}\}^{\angle} \text{ } (= H, \text{ say}),$$

since otherwise, for some $u_1, u_2, ..., u_{2m+r} \geq 0$,

$$u_{2m+1}\vec{q_1} + ... + u_{2m+r}\vec{q_r} + \vec{q_i} = u_1\vec{v_1} + ... + u_{2m}(-\vec{v_m}) \in V,$$

with the coefficient of $\vec{q_i}$ on the left hand side being $u_{2m+i} + 1 > 0$, contradiction.

Therefore, by Lemma 16 we can pick $\vec{b_i}$ such that $\vec{b_i} \cdot \vec{q_i} > 0$ and $\vec{b_i} \cdot \vec{x} \leq 0$ for all $\vec{x} \in H$. In particular $\vec{b_i} \cdot \vec{v_j}$, $\vec{b_i} \cdot (-\vec{v_j}) \leq 0$, so $\vec{b_i} \cdot \vec{v_j} = 0$ for $j = 1, 2, ..., m$, and $\vec{b_i} \cdot \vec{q_k} \geq 0$ for $k = 1, 2, ..., r$. Thus for

$$\vec{b} = \sum_{i=1}^{r} \vec{b_i}$$

we have that $\vec{b} \cdot \vec{x} = 0$ for all $\vec{x} \in H$ and $\vec{b} \cdot \vec{q_k} > 0$ for $k = 1, 2, ..., r$. Hence $\vec{b} \cdot (\sum u_k \vec{q_k}) \geq 0$ for $\vec{u} \geq 0$ with equality just if $\vec{u} = \vec{0}$, as required. ■

Proof of Theorem 11 cont.
Let h be such that $h > 2$, $\xi \in SL_{h-1}$ and $\Gamma \subset SL_{h-1}$ (this is where we use that Γ is finite). By Lemma 16 the vector subspace of $\mathbb{V}$ generated by the

$$\{(\psi + \theta - \phi) \in \mathcal{N} \mid \psi, \theta, \phi \in SL_h\},$$

has only the zero vector in common with the cone $\{\rho \in \mathcal{P} | \rho \in SL_h\}^{\angle}$. Notice that the $\rho \in \mathcal{P}$ are linearly independent. By Corollary 18 there is a vector

$$\vec{b} = \sum_{\theta \in SL_h} b_\theta \theta$$

such that for $(\psi + \theta - \phi) \in \mathcal{N}$ with $\psi, \theta, \phi \in SL_h$, $\vec{b} \cdot (\psi + \theta - \phi) = 0$, i.e.

$$b_\psi + b_\theta - b_\phi = 0.$$

and for $\rho \in \mathcal{P}$, $\vec{b} \cdot \rho > 0$, i.e. $b_\rho > 0$. The following claim now just about gives us the valuation we have been seeking.

Claim

(1) For $\theta \in SL_h$, if $\Omega \vdash^{\text{Ł}} \theta$ then $b_\theta = 0$.

(2) For $\theta \in SL_h$, $b_\theta \geq 0$.

(3) For $\theta, \phi \in SL_{h-1}$, $\theta \preceq \phi \iff b_\theta \geq b_\phi$.

(4) If $\theta \in SL_{h-1}$, $p \in L$ then $b_\theta \leq b_{\neg(p \to p)}$.

Proof of Claim.

(1): If $\Omega \vdash^{\text{Ł}} \theta$ then $\Omega \vdash^{\text{Ł}} (p \to p) \to \theta$ by (r4) and similarly $\Omega \vdash^{\text{Ł}} \theta \to (p \to p)$, since $\Omega \vdash^{\text{Ł}} p \to p$, so $\theta \approx (p \to p)$. Therefore $(\theta + p - p) \in \mathcal{N}$ so $b_\theta = b_\theta + b_p - b_p = 0$.

(2): This follows from (1) and the fact that if $\Omega \nvdash^{\text{Ł}} \theta$, i.e. $\theta \in \mathcal{P}$, then $b_\theta > 0$.

(3): Since $\theta, \phi \in SL_{h-1}$, $(\theta \to \phi) \in SL_h$. If $\theta \preceq \phi$ then $((\phi \to \theta) + \phi - \theta) \in \mathcal{N}$ so

$$b_{\phi \to \theta} + b_\phi - b_\theta = 0.$$

Therefore, since $b_{\phi \to \theta} \geq 0$ by (2), $b_\theta \geq b_\phi$.

For the converse, if not $\theta \preceq \phi$ then $(\theta \to \phi) \in \mathcal{P} \cap SL_h$, so $b_{\theta \to \phi} > 0$, and $\phi \preceq \theta$, (recall $\preceq$ is connected) so $((\theta \to \phi) + \theta - \phi) \in \mathcal{N} \cap SL_h$ and

$$b_{\theta \to \phi} + b_\theta - b_\phi = 0.$$

Combining these gives that $b_\theta < b_\phi$.

(4): Since, by (a1), $\Omega \vdash^{\text{Ł}} p \to p$, by (r1) $\Omega \vdash^{\text{Ł}} \neg\theta \to (p \to p)$ and by (a4) and MP, $\Omega \vdash^{\text{Ł}} \neg(p \to p) \to \theta$. Hence $\neg(p \to p) \preceq \theta$ and the result follows by (3).

Now fix $p \in L$ and define the $[0,1]$-valuation $w : L \longrightarrow [0,1]$ by

$$w(q) = 1 - \frac{b_q}{b_{\neg(p\to p)}},$$

for $q \in L$. Notice that by the claims (2),(4) $w(q) \in [0,1]$ and by (4) the value of denominator is actually independent of p.

We now claim that for $\psi \in SL_{(h-1)}$,

$$w(\psi) = 1 - \frac{b_\psi}{b_{\neg(p\to p)}}.$$

This will complete the proof of the Completeness Theorem, since if $\psi \in \Gamma(\subseteq \Omega)$ then $\psi \in SL_{(h-1)}$ and, by (1), $b_\psi = 0$, so $w(\psi) = 1$, whilst, since $\xi \notin \Omega$, $\xi \in \mathcal{P}$, and $\xi \in SL_{(h-1)}$, $b_\xi > 0$ so $w(\xi) < 1$.

We prove this claim by induction on $|\psi|$, the length of ψ. Clearly we have it by definition of w for $|\psi| = 1$. To prove the remaining cases it is enough to show that if $\theta \to \phi, \neg\theta \in SL_{(h-1)}$ and

$$w(\theta) = 1 - b_\theta / b_{\neg(p\to p)}, \quad w(\phi) = 1 - b_\phi / b_{\neg(p\to p)}$$

then

(5) $w(\theta \to \phi) = \min\{1, 1 - w(\theta) + w(\phi)\} = 1 - b_{\theta\to\phi}/b_{\neg(p\to p)}$,

(6) $w(\neg\theta) = 1 - w(\theta) = 1 - b_{\neg\theta}/b_{\neg(p\to p)}$.

To show (5) first suppose $w(\theta) \geq w(\phi)$ (so $w(\theta \to \phi) = 1 - w(\theta) + w(\phi)$). Then $b_\theta \leq b_\phi$ so $\phi \preceq \theta$ by (3) and $((\theta \to \phi) + \theta - \phi) \in \mathcal{N}$, $(\theta \to \phi), \theta, \phi \in SL_h$. Hence

$$b_{(\theta\to\phi)} + b_\theta - b_\phi = 0,$$

giving

$$1 - b_{(\theta\to\phi)}/b_{\neg(p\to p)} = 1 - (1 - b_\theta/b_{\neg(p\to p)}) + (1 - b_\phi/b_{\neg(p\to p)}),$$

as required.

On the other hand, if $w(\theta) \leq w(\phi)$ (so $w(\theta \to \phi) = 1$), then $b_\theta \geq b_\phi$, $\theta \preceq \phi$ by (3). Hence $\Omega \vdash^{\text{Ł}} (\theta \to \phi)$ and

$$1 - b_{(\theta\to\phi)}/b_{\neg(p\to p)} = 1,$$

again as required.

Turning now to (6), from (a7),(a8), $\neg\theta \approx \theta \to \neg(p \to p)$ and $\neg(p \to p) \preceq \theta$. Hence $(\neg\theta + \theta - \neg(p \to p)) \in \mathcal{N}$ and since $\neg\theta \in SL_h$, this gives

$$b_{\neg\theta} + b_\theta - b_{\neg(p\to p)} = 0,$$

i.e.

$$1 - b_{\neg\theta}/b_{\neg(p\to p)} = 1 - (1 - b_\theta/b_{\neg(p\to p)}),$$

as required. The Completeness Theorem is proved! ■

MATH43032/63032 Examples 3

1. [Revision] Let $F : \{0,1\}^n \longrightarrow \{0,1\}$. [Such a function is called a *Boolean Function.*] Show that there is a $\theta \in SL$, where $L = \{p_1, p_2, ..., p_n\}$, such that for any $\{0,1\}$-valuation V,

$$V(\theta) = F(V(p_1), V(p_2), ..., V(p_n)).$$

2. Show that if $w : SL \longrightarrow [0,1]$ is a probability function then w is not necessarily truth functional (for any $F_\neg, F_\wedge, F_\vee, F_\rightarrow$).
Show that, however, for any $\theta, \phi \in SL$,

$$F^3_\wedge(w(\theta), w(\phi)) \leq w(\theta \wedge \phi) \leq F^1_\wedge(w(\theta), w(\phi)),$$

$$F^1_\vee(w(\theta), w(\phi)) \leq w(\theta \vee \phi) \leq F^3_\vee(w(\theta), w(\phi)),$$

where $F^1_\wedge$ is $F_\wedge$ of $\mathbb{F}^1$ (i.e. *min*) etc.

3. In Ł sketch the graphs of the functions $F_{p\underline{\wedge}\neg p}, F_{p\underline{\vee}\neg p}, F_{p\wedge\neg p}, F_{p\vee\neg p}$.

4. Show:

(1) $\not\models^{Ł} (p_1 \underline{\wedge} \neg p_1) \rightarrow p_2$, (2) $\models^{Ł} (\theta \underline{\wedge} \neg\theta) \rightarrow \theta$,

(3) $\not\models^{Ł} p\underline{\vee}\neg p$, (4) $\models^{Ł} (\theta \rightarrow \phi)\underline{\vee}(\phi \rightarrow \theta)$.

5. Let w be a $[0,1]$-valuation on L. Define a $\{0,1\}$-valuation $\overline{w}$ by

$$\overline{w}(p) = 1 \iff w(p) > 1/2.$$

Show that in Gödel Logic, if $\theta \in SL$, does not mention $\rightarrow$ then

$$w(\theta) > 1/2 \implies \overline{w}(\theta) = 1,$$

$$w(\theta) < 1/2 \implies \overline{w}(\theta) = 0.$$

Hence show that for such θ,

$$\models^{SC} \theta \text{ iff for any } [0,1]\text{-valuation } w',\ w'(\theta) \geq 1/2 \qquad \dagger$$

Show that we cannot replace $\geq$ in † by $>$.
Does † still hold (for such θ) if we work instead in Łukasiewicz Logic, Ł?

6. Show that if $F_\wedge$ satisfies (C1-4) and we define $F_\vee$ by

$$F_\vee(x,y) = 1 - F_\wedge(1-x, 1-y)$$

then $F_\vee$ satisfies (D1-4) and

$$F_\wedge(x,y) = 1 - F_\vee(1-x, 1-y).$$

7. [1997 Exam Question] Let $F_\wedge$ satisfy (C1-4). Show (directly) that $F_\wedge$ (which as usual we abbreviate to just $\wedge$) must be commutative as follows: Assume that $0 \leq x, y \leq 1$ and $x \wedge y < y \wedge x$ and derive a contradiction by showing that

(i) There exists $s \in [0,1]$ such that $y \wedge x > (y \wedge x) \wedge \dot{s}^2 > x \wedge y$, where as usual $\dot{s}^0 = 1, \dot{s}^1 = s, \dot{s}^2 = s \wedge s$ etc.

For s as in (i),
(ii)$\lim_{n\to\infty} \dot{s}^n$ ($= c$ say) exists and $c \wedge c = c, c < y \wedge x$.
(iii) There are n, m such that

$$\dot{s}^{n+1} < x \leq \dot{s}^n, \quad \dot{s}^{m+1} < y \leq \dot{s}^m, \quad \dot{s}^{n+m+2} \leq x \wedge y, y \wedge x \leq \dot{s}^{n+m}.$$

8. Define a function $\curlywedge : [0,1]^2 \longrightarrow [0,1]$ by

$$x \curlywedge y = \frac{xy}{\max\{x, y, 1/2\}}.$$

Show that $\curlywedge$ satisfies (C1-4).
What is $\curlywedge$ according to the classification of such functions as chimera of $min\{x,y\}, xy$, $max\{0, x+y-1\}$?

9. By using McNaughton's Theorem, or otherwise, show that if $L = \{p\}$ and $w(\theta) = 1$ for some $[0,1]$-valuation w then there is a valuation w' such that $w'(p) \in \mathbb{Q}$ and $w'(\theta) = 1$.

10. Show that if w_1, w_2 are valuations and

$$|w_1(p) - w_2(p)| < \epsilon$$

for all $p \in L$ then for $\theta \in SL$,

$$|w_1(\theta) - w_2(\theta)| < 2^k \epsilon,$$

where k equals the number of connectives in θ.

11. Let $Ł_n = \{0, 1/(n-1), 2/(n-1), ..., (n-2)/(n-1), 1\}$ and call a $[0,1]$-valuation w a $Ł_n$-valuation if $w(p) \in Ł_n$ for all $p \in L$.
Show that if w is an $Ł_n$-valuation then $w(\theta) \in Ł_n$ for all $\theta \in SL$.
Define $\Gamma \vDash_n^{Ł} \theta$ if whenever w is an $Ł_n$-valuation such that $w(\phi) = 1$ for all $\phi \in \Gamma$ then $w(\theta) = 1$.
Show that if $\Gamma \models^{Ł} \theta$ then $\Gamma \vDash_n^{Ł} \theta$, but that the converse is false.
Show, however, that if Γ is finite and $\Gamma \nvDash^{Ł} \theta$ then for some n, $\Gamma \nvDash_n^{Ł} \theta$. [Hint: Use Question 5 above and recall the following weak version of Kronecker's Theorem: If $a_1, a_2, ..., a_r \in \mathbb{R}$ and $K > 0$ then there are $n, n_1, n_2, ..., n_r \in \mathbb{Z}$, $n > 0$, such that for each $i = 1, 2, ..., r$, $|a_i - n_i/n| < 1/nK$.]

12. Assuming the axioms Ł1-5 and the already proven (a1),Ł4′,(r1) show:-
(r2) If $\Gamma \vdash^{Ł} \psi \to \eta$ then $\Gamma \vdash^{Ł} (\theta \to \psi) \to (\theta \to \eta)$.
(r3) If $\Gamma \vdash^{Ł} \psi \to \eta$ then $\Gamma \vdash^{Ł} (\eta \to \theta) \to (\psi \to \theta)$.
(r4) If $\Gamma \vdash^{Ł} \psi$ then $\Gamma \vdash^{Ł} \theta \to \psi$.
(a3) $\vdash^{Ł} \neg\neg\theta \to \theta$.
[Hint: Consider starting with the instance $\neg\neg\theta \to (\neg\neg\phi \to \neg\neg\theta)$ of Ł1
where $\vdash^{Ł} \phi$.]
(a4) $\vdash^{Ł} (\theta \to \neg\phi) \to (\phi \to \neg\theta)$.
(a5) $\vdash^{Ł} \theta \to \neg\neg\theta$.
(a6) $\vdash^{Ł} (\neg\theta \to \phi) \to (\neg\phi \to \theta)$.
(a7) $\vdash^{Ł} \neg\theta \to (\theta \to \neg(p \to p))$.
(a8) $\vdash^{Ł} (\theta \to \neg(p \to p)) \to \neg\theta$.

13. For $\theta \in SL$ define θ^n for $n > 0$ inductively by

$$\theta^1 = \theta, \qquad \theta^{(n+1)} = \theta \wedge \theta^n.$$

For a valuation w what is $w(\theta^n)$ as a function of $w(\theta)$?
What is $w(\theta \to_n \phi)$ as a function of $w(\theta), w(\phi)$?
By using the Completeness Theorem show that for Γ finite,

$$\Gamma, \theta \vdash^{Ł} \phi \iff \Gamma \vdash^{Ł} \theta^n \to \phi \text{ for some } n > 0.$$

14. By using the Completeness Theorem, or otherwise, show that

$$\Gamma, \theta, \phi \vdash^{Ł} \psi \iff \Gamma, \theta \underline{\triangle} \phi \vdash^{Ł} \psi.$$

[Note that we do not need to assume Γ finite here.]