Two hours

THE UNIVERSITY OF MANCHESTER

PREDICATE LOGIC

14th January 2013

9.45 – 11.45

Answer ALL FOUR questions in Section A (56 marks in total).
Answer TWO of the THREE questions in Section B (24 marks in total).

If more than TWO questions from Section B are attempted
then credit will be given for the FIRST TWO answers.

A list of axioms and rules of proof is appended to this examination paper

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The use of calculators is not permitted

SECTION A

Answer all FOUR questions

A1. Do you think the following argument is valid in the sense that the conclusion 'follows' from the premises? You should briefly explain your answer.

Some birds lay eggs
Swans are birds

$\therefore$ *Swans lay eggs*

[5 marks]

A2. Let the language L have a binary relation symbol R and a unary function symbol f.

Which of the following are terms of L? You should briefly justify your answers.

(i) $f(w_1)$

(ii) $f)x_1($

Which of the following are formulae of L? You should briefly justify your answers.

(iii) $\exists w_2(R(w_2, x_1) \to \forall w_1 R(w_1, x_1))$

(iv) $(\neg\exists w_1\, R(x_1, x_1))$

Let M be the structure for L with $|M| = \mathbb{N}^+ = \{1, 2, 3, \ldots\}$, $f^M(n) = n + 1$,

$$R^M = \{\langle n, m\rangle \in |M|^2 \mid n \text{ divides } m\}.$$

Which of the following sentences of L are true in M?

(v) $\forall w_1\, R(w_1, f(w_1))$

(vi) $\forall w_1 \exists w_2\, (R(w_1, w_2) \wedge \neg R(w_2, w_1))$

(vii) $\exists w_1 \forall w_2 \forall w_3\, ((R(w_2, w_1) \wedge R(w_3, w_1)) \to (R(w_2, w_3) \vee R(w_3, w_2)))$

Find formulae $\theta_1(x_1)$, $\theta_2(x_1, x_2)$, $\theta_3(x_1)$, $\theta_4(x_1)$ of L such that for $n, m \in |M|$,

(viii) $M \models \theta_1(n) \iff n = 1$

(ix) $M \models \theta_2(n, m) \iff n = m$

(x) $M \models \theta_3(n) \iff n$ is odd

(xi) $M \models \theta_4(n) \iff n$ is a power of 2

Let K be the structure for L with $|K| = |M| = \mathbb{N}^+$, $f^K(n) = f^M(n) = n + 1$,

$$R^K = \{\langle n, m\rangle \in |K|^2 \mid n \leq m\}.$$

(xii) Find a sentence ϕ of L such that $K \models \phi$ and $M \not\models \phi$.

[27 marks]

A3. Define what is meant by a *formal proof*. Give a formal proof of

$$\forall w_1\, P(w_1) \vdash \forall w_1 \forall w_2\, (P(w_1) \wedge P(w_2))$$

where P is a unary relation symbol.

[11 marks]

A4. State the Completeness Theorem for Relational Languages. Using this theorem or otherwise show that

(a) $\exists w_1 \forall w_2\, (R(w_1, w_2) \vee R(w_2, w_1)) \vdash \exists w_1\, R(w_1, w_1)$

(b) $\forall w_1 \exists w_2\, R(w_1, w_2) \nvdash \exists w_1\, R(w_1, w_1)$

where R is a binary relation symbol.

Is it the case that

$$R(x_1, x_1) \equiv R(x_2, x_2) \quad ?$$

Briefly explain your answer.

[13 marks]

SECTION B

Answer TWO of the THREE questions

If more than TWO questions are attempted then credit will be given for the FIRST TWO answers.

B5. Let L be the language with a unary relation symbol P and a unary function symbol f. Show that no two of the following sentences of L logically imply the third:

(i) $\forall w_1 \, (P(w_1) \to P(f(w_1)))$

(ii) $\forall w_1 \, (P(w_1) \vee \neg P(f(w_1)))$

(iii) $\exists w_1 \, \neg P(f(w_1))$

[12 marks]

B6. Give a formal proof of

$$\forall w_1 \forall w_2 \, (P(w_1) \vee Q(w_2)) \vdash \exists w_1 \, \neg P(w_1) \to \forall w_2 \, Q(w_2)$$

where P, Q are unary relation symbols.

[12 marks]

B7. State the Compactness Theorem for Relational Languages.

Let L be the language with just a ternary relation symbol T. A structure M for L is said to have a *finite separation* if there is a finite subset A of $|M|$ such that for every $b, c \in |M|$, $M \models T(b, a, c)$ for some $a \in A$. Show that there can be no sentence $\theta \in SL$ such that for any structure M for L,

$$M \models \theta \iff M \text{ has a finite separation.}$$

[Hint: Assume that such a θ did exist and consider

$$\Gamma = \{\theta\} \cup \{ \, \neg \exists w_1, \ldots, w_n \forall w_{n+1}, w_{n+2} \bigvee_{i=1}^{n} T(w_{n+1}, w_i, w_{n+2}) \mid n \in \mathbb{N}^+ \, \} \,]$$

[12 marks]

The Rules of Proof and Axiom for the Predicate Calculus

And In (AND)	$\dfrac{\Gamma \mid \theta, \quad \Delta \mid \phi}{\Gamma \cup \Delta \mid \theta \wedge \phi}$	
And Out (AO)	$\dfrac{\Gamma \mid \theta \wedge \phi}{\Gamma \mid \theta}$ $\qquad$ $\dfrac{\Gamma \mid \theta \wedge \phi}{\Gamma \mid \phi}$	
Or In (ORR)	$\dfrac{\Gamma \mid \theta}{\Gamma \mid \theta \vee \phi}$ $\qquad$ $\dfrac{\Gamma \mid \theta}{\Gamma \mid \phi \vee \theta}$	
Disjunction (DIS)	$\dfrac{\Gamma, \theta \mid \psi, \quad \Delta, \phi \mid \psi}{\Gamma \cup \Delta, \theta \vee \phi \mid \psi}$	
Implies In (IMR)	$\dfrac{\Gamma, \theta \mid \phi}{\Gamma \mid \theta \to \phi}$	
Modus Ponens (MP)	$\dfrac{\Gamma \mid \theta, \quad \Delta \mid \theta \to \phi}{\Gamma \cup \Delta \mid \phi}$	
Not In (NIN)	$\dfrac{\Gamma, \theta \mid \phi, \quad \Delta, \theta \mid \neg\phi}{\Gamma \cup \Delta \mid \neg\theta}$	
Not Not Out (NNO)	$\dfrac{\Gamma \mid \neg\neg\theta}{\Gamma \mid \theta}$	
Monotonicity (MON)	$\dfrac{\Gamma \mid \theta}{\Gamma \cup \Delta \mid \theta}$	
All In ($\forall$I)	$\dfrac{\Gamma \mid \theta}{\Gamma \mid \forall w_j\, \theta(w_j/x_i)}$	where x_i does not occur in any formula in Γ and w_j does not occur in θ
All Out ($\forall$O)	$\dfrac{\Gamma \mid \forall w_j\, \theta(w_j, \vec{x})}{\Gamma \mid \theta(t(\vec{x}), \vec{x})}$	for $t(\vec{x}) \in TL$
Exists In ($\exists$I)	$\dfrac{\Gamma \mid \theta}{\Gamma \mid \exists\, w_j\, \theta'}$	where θ' is the result of replacing any number of occurences of the term $t(\vec{x})$ in θ by w_j and w_j does not occur in θ.
Exists Out ($\exists$O)	$\dfrac{\Gamma,\, \phi \mid \theta}{\Gamma, \exists w_j\, \phi(w_j/x_i) \mid \theta}$	where x_i does not occur in θ nor any formula in Γ and w_j does not occur in ϕ.
REF	$\Gamma \mid \theta$ whenever $\theta \in \Gamma$.	

The Equality Axioms, EqL

Eq1 $\forall w_1 \, w_1 = w_1$

Eq2 $\forall w_1, w_2 \, (w_1 = w_2 \to w_2 = w_1)$

Eq3 $\forall w_1, w_2, w_3 \, ((w_1 = w_2 \wedge w_2 = w_3) \to w_1 = w_3)$

Eq4

$$\forall w_1, \ldots, w_{2r} \left(\left(\bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \to (R(w_1, w_2, \ldots, w_r) \leftrightarrow R(w_{r+1}, w_{r+2}, \ldots, w_{2r})) \right)$$

for R an r-ary relation symbol of L.

Eq5

$$\forall w_1, \ldots, w_{2r} \left(\left(\bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \to f(w_1, w_2, \ldots, w_r) = f(w_{r+1}, w_{r+2}, \ldots, w_{2r}) \right)$$

for f an r-ary function symbol of L.

Eq6

$$\forall w_1, \ldots, w_{2r} \left(\left(\bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \to t(w_1, w_2, \ldots, w_r) = t(w_{r+1}, w_{r+2}, \ldots, w_{2r}) \right)$$

for $t(x_1, x_2, \ldots, x_r) \in TL$.

Eq7

$$\forall w_1, \ldots, w_{2r} \left(\left(\bigwedge_{i=1}^{r} w_i = w_{r+i} \right) \to (\theta(w_1, w_2, \ldots, w_r) \leftrightarrow \theta(w_{r+1}, w_{r+2}, \ldots, w_{2r})) \right)$$

for $\theta(x_1, x_2, \ldots, x_r) \in FL$.

END OF EXAMINATION PAPER