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# Pure Inductive Logic

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Jeff Paris,  
School of Mathematics,  
University of Manchester,  
Manchester M13 9PL, UK.

`jeff.paris@manchester.ac.uk`

# Tutorial I

## Introduction

I have commonly heard philosophers say that Goodman's GRUE Paradox, [12], [13], spells the end of Carnap's Inductive Logic Programme, see [1], [2], [3], [4], [7], [8].<sup>1</sup> That may indeed be so if one intends it to be an applied subject, to be applicable to the problem of our assigning probabilities in the real world, as I suppose was Carnap's primary aim. However as Carnap himself pointed out one can also treat Inductive Logic as a pure subject, just in the way the same way that mathematics splits into pure and applied mathematics, where one studies logical or rational principles of probability assignment for their own sake. Indeed Carnap used the very phrase 'Pure Inductive Logic'. From this perspective the GRUE Paradox simply isn't a paradox.

For much of it's history the Pure and Applied sides of Inductive Logic have been tangled together, just as with Pure and Applied Mathematics. I have come to realize however that my own interest are very firmly located in the Pure, and in that sense I see it as part of Mathematical Logic, more specifically a branch of Uncertain Reasoning. That's not to say (I hope!) that it is of no interest to anyone except a pure mathematician. The topic is inspired by issues that concern 'applied inductive logicians'. The arguably rational principles that we formulate and study derive from our perceptions of the real world and I would hope that the conclusions we draw from them by the agency of mathematics might still say something of wider interest than just being purely technical mathematical theorems. Not that I would want at all to say that  $A$ ,  $B$  and  $C$  are clearly rational and therefore you should believe them. Instead the most I would wish to point out is that  $D$  is a mathematical consequence of  $A$ ,  $B$ ,  $C$  so *if* you accept  $A$ ,  $B$ ,  $C$  *then* perforce you should accept  $D$ .

My aim in the first of these two tutorials is to describe a result, de Finetti's Representation Theorem, which has really been the cornerstone of Pure Inductive Logic up to the millenium. In the second tutorial I shall describe two principles, Language Invariance and Spectrum Exchangeability, which seem to me to be the major players in where Pure Inductive Logic, PIL for short, is right now. Before we can do any of that however we need to set the scene, explain what the problem is.

## Context and Notation

For the mathematical setting we need to make the formalism completely clear. Whilst there are various possible choices here the language which seems best for our study, and corresponds to most of the literature, including Carnap's, is where we work with a first order language  $L$  with relation symbols  $R_1, R_2, \dots, R_q$ , say of arities  $r_1, r_2, \dots, r_q$  respectively, and constants  $a_n$  for  $n \in \mathbb{N}^+ = \{1, 2, 3, \dots\}$ , and no function symbols nor (in general) equality. The intention here is that the  $a_i$  name all the individuals in some population though there is no prior assumption that they necessarily name different individuals. Let  $SL$  denote the set of first order sentences of this language  $L$  and  $QFSL$  the quantifier free sentences of this language.

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<sup>1</sup>And also W.E.Johnson's earlier [17].

Let  $\mathcal{T}$  denote the set of structures for  $L$  with universe  $\{a_1, a_2, a_3, \dots\}$ , with the obvious interpretation of  $a_i$  as  $a_i$  itself.

To capture the underlying problem that PIL aims to address imagine an agent who inhabits some structure  $M \in \mathcal{T}$  but knows nothing about what is true in  $M$ . Then the problem is,

*Q: In this situation of zero knowledge, logically, or rationally, what belief should our agent give to a sentence  $\theta \in SL$  being true in  $M$ ?*

There are several terms in this question which need explaining. Firstly ‘zero knowledge’ means that the agent has no intended interpretation of the  $a_i$  nor the  $R_j$ . To mathematicians this seems a perfectly easy idea to accept, we already do it effortlessly when proving results about, say, an arbitrary group. In these cases all you can assume is the axioms and you are not permitted to bring in new facts because they happen to hold in some particular group you have in mind. Unfortunately outside of Mathematics this sometimes seems to be a particularly difficult idea to embrace and much confusion has found its way into the folklore as a result.

In a way this is at the heart of the difference between the ‘Pure Inductive Logic’ proposed here as Mathematics and the ‘Applied Inductive Logic’ of Philosophy. For many philosophers would (I think) argue that in this latter the language is intended to carry with it an interpretation, that without it one is doing Pure Mathematics not Philosophy. It is the reason why GRUE is a paradox in Philosophy and simply an invalid argument in Mathematics. Nevertheless, mathematicians or not, we all need to be on our guard when it comes to allowing interpretations to slip in subconsciously. Carnap himself was very well aware of this division, and the dangers presented by ignoring it, and spent some effort explaining it in [5]. Indeed in that paper he describes Inductive Logic, IL, as the study of just such a zero knowledge agent, a ‘robot’ as he termed it.

A second unexplained term is ‘logical’ and its synonym (as far as this text is concerned) ‘rational’. In this case, as already indicated, we shall offer no definition, it is to be taken as intuitive, something we recognize when we see it without actually being able to give it a definition. This will not be a great problem for our purpose is to propose and mathematically investigate principles for which it is enough that we may simply *entertain* the idea that they are logical or rational. The situation parallels that of the intuitive notion of an ‘effective process’ in recursion theory, and similarly we may hope that our investigations will ultimately lead to a clearer understanding.

The third unexplained term above is ‘belief’. In PIL we identify belief, or more precisely degree of belief, with (subjective) probability. To my mind the *Dutch Book Argument* provides a strong justification for this identification. This of course now requires us to make precise what we mean by ‘probability’, or more precisely a ‘probability function’:

## Probability Functions

A function  $w : SL \rightarrow [0, 1]$  is a *probability function* on  $SL$  if for all  $\theta, \phi, \exists x \psi(x) \in SL$ ,

$$(P1) \quad \models \theta \Rightarrow w(\theta) = 1.$$

$$(P2) \quad \theta \models \neg\phi \Rightarrow w(\theta \vee \phi) = w(\theta) + w(\phi).$$

$$(P3) \quad w(\exists x \psi(x)) = \lim_{n \rightarrow \infty} w(\psi(a_1) \vee \psi(a_2) \vee \dots \vee \psi(a_n)).$$

Condition (P3) is often referred to as *Gaifman's Condition*, see [9], and is a special addition to the conventional conditions (P1), (P2) appropriate to this context. It intends to capture the idea that the  $a_1, a_2, a_3, \dots$  exhaust the universe.

All the standard, simple, properties you'd expect of a probability function follow from these (P1-3):

**Proposition 1** *Let  $w$  be a probability function on  $SL$ . Then for  $\theta, \phi \in SL$ ,*

- (a)  $w(\neg\theta) = 1 - w(\theta)$ .
- (b)  $\models \neg\theta \Rightarrow w(\theta) = 0$ .
- (c)  $\theta \models \phi \Rightarrow w(\theta) \leq w(\phi)$ .
- (d)  $\theta \equiv \phi \Rightarrow w(\theta) = w(\phi)$ .
- (e)  $w(\theta \vee \phi) = w(\theta) + w(\phi) - w(\theta \wedge \phi)$ .

Proofs may be found in [26].

On the face of it it might appear that because of the great diversity of sentences in  $SL$  probability functions would be very complicated objects and not easily described. In fact this is not the case as we shall now explain. The first step in this direction is the following theorem of Gaifman, [9]:<sup>2</sup>

**Theorem 2** *Suppose that  $w : QFSL \rightarrow [0, 1]$  satisfies (P1) and (P2) for  $\theta, \phi \in QFSL$ . Then  $w$  has a unique extension to a probability function on  $SL$  satisfying (P1), (P2), (P3) for any  $\theta, \phi, \exists x \psi(x) \in SL$ .  $SL$ .*

In view of this theorem then the 'game' of picking a rational probability function is really being played at the level of quantifier free sentences. In fact, it's even simpler than that:

## State Descriptions

As usual let  $L$  be our default language with relation symbols  $R_1, R_2, \dots, R_q$  of arities  $r_1, r_2, \dots, r_q$  respectively. For distinct constants  $a_{i_1}, a_{i_2}, \dots, a_{i_m}$  coming from  $a_1, a_2, \dots$ , a *State Description* for  $a_{i_1}, a_{i_2}, \dots, a_{i_m}$  is a sentence of  $L$  of the form

$$\bigwedge_{k=1}^q \bigwedge_{c_1, c_2, \dots, c_{r_k}} R_k^{\epsilon_k}(c_1, c_2, \dots, c_{r_k})$$

where the  $c_1, c_2, \dots, c_{r_k}$  range over all (not necessarily distinct) choices from  $a_{i_1}, a_{i_2}, \dots, a_{i_m}$ , the  $\epsilon_k \in \{0, 1\}$  and  $R_k^{\epsilon_k}$  stands for  $R_k$  if  $\epsilon_k = 1$  and  $\neg R_k$  if  $\epsilon_k = 0$ .

In other words, a state description for  $a_{i_1}, a_{i_2}, \dots, a_{i_m}$  tells us exactly which of the  $R_k(c_1, c_2, \dots, c_{r_k})$  hold and which do not hold for  $R_k$  a relation symbol from our language and any arguments  $c_1, c_2, \dots, c_{r_k}$  from  $a_{i_1}, a_{i_2}, \dots, a_{i_m}$ . Hence any two distinct<sup>3</sup> (inequivalent) state descriptions for  $a_{i_1}, a_{i_2}, \dots, a_{i_m}$  are exclusive in the sense that their conjunction is inconsistent. We allow

<sup>2</sup>For a proof in the notation of these tutorials see Theorem 7 of [26].

<sup>3</sup>Following standard practice we shall identify two state descriptions if they are the same *up to the ordering of their conjuncts*. Since throughout we will only really be concerned with sentences up to logical equivalence this abuse should not cause any distress.

here the possibility that  $m = 0$  in which case the sole state description for these constants is a tautology, for which we use the symbol  $\top$ .

As an example, if  $L$  had just the binary relation symbol  $R$  and the unary relation (or predicate) symbol  $P$  then

$$P(a_1) \wedge \neg P(a_2) \wedge \neg R(a_1, a_1) \wedge R(a_1, a_2) \wedge R(a_2, a_1) \wedge R(a_2, a_2)$$

would be a state description for  $a_1, a_2$ .

We shall use upper case  $\Theta, \Phi, \Psi$  etc. for state descriptions. By using Theorem 2 and the Disjunctive Normal Form Theorem it is straightforward to see that:

**Proposition 3** *A probability function is determined by its values on the state descriptions.*

So, actually, the ‘game’ is being played here at the level of state descriptions.

Returning to our central question  $\mathcal{Q}$ , it now amounts to:

*$\mathcal{Q}$ : In this situation of zero knowledge, logically, or rationally, what probability function  $w : SL \rightarrow [0, 1]$  should our agent adopt when  $w(\theta)$  is to represent the agent’s probability that a sentence  $\theta \in SL$  is true in the ambient structure  $M$ ?*

So how is the agent supposed to make this choice? As far as PIL is concerned, by the application of:

## Rational Principles

That is, the agent formulates rational or logical (I’m using the two words synonymously) principles or rules of probability assignment and then adopts a probability function which satisfies those principles. One might object that one first needs to know what ‘rational’ means in this context and that appears to be a major, if not downright impossible, task.

However as far as PIL is concerned that isn’t actually the case. As already mentioned we have some sort of intuitive ideas of what it means to behave rationally, or more especially perhaps we recognize irrational behavior, and it is enough for PIL that we try to capture these feelings as formal principles and then then go on to investigate their consequences and relationships to other such putatively rational principles. Thus at this stage of its development I see PIL as an experiment where we are free to investigate any principles provided they bear some sort of relevance to the situation, i.e. could be argued to encapsulate some feature of rationality. For this we don’t need to know what we mean by rational, just as those mathematicians who first tried to formulate the idea of an ‘effective process’ did not actually have to know what this was before they started – the notion crystalized out of their investigations. And possibly a notion(s) of ‘rational’ just might crystalize out of this investigation.

In a way I see this as similar to the situation in Set Theory where we propose axioms based on some intuitions we have about the ‘universe of sets’, some of which are almost universally accepted whilst others others are highly contentious, and then investigate their consequences. Sometimes these axioms are inconsistent with each other but nevertheless the feeling is that by investigating these choices we are getting closer to understanding and perhaps even closing in on the set theoretic universe. So it is too with PIL, we again find that intuitions can conflict.

As far as such ‘rational principles’ (from now on I will drop any quotes) are concerned there appear to date to be three main sources from which they spring: Symmetry, Relevance and Irrelevance. Of these principles based on symmetry, the requirement that the adopted probability function  $w$  should preserve existing symmetries, seems to be the most convincing, and easiest to formulate. We start with one such that is so widely accepted that we will henceforth *assume it throughout without further explicit mention*.

### The Constant Exchangeability Principle, Ex<sup>4</sup>

For  $\phi(a_1, a_2, \dots, a_m) \in SL^5$  and (distinct)  $i_1, i_2, \dots, i_m \in \mathbb{N}^+$ ,

$$w(\phi(a_1, a_2, \dots, a_m)) = w(\phi(a_{i_1}, a_{i_2}, \dots, a_{i_m})).$$

The rational justification here is that the agent has no knowledge about any of the  $a_i$  so it would be irrational to treat them differently when assigning probabilities.<sup>6</sup>

In an exactly similar way we could justify the Principles of *Predicate Exchangeability* (where we require  $w$  to be fixed when we transpose relation symbols of the same arity) and *Strong Negation* where we replace a relation symbol throughout by its negation.

Now that we have introduced this Constant Exchangeability Principle, Ex, we would like to investigate what it entails, what follows from the assumption that  $w$  satisfies Ex. Often a major step in PIL after one has formulated a principle is to prove a representation theorem for the probability functions satisfying that principle by showing that they must look like a combination of certain ‘simple building block functions’. There are such results for probability functions satisfying Ex, the first of these, and historically the most important, being the so call *de Finetti’s Representation Theorem* in the case where the language  $L$  is unary.

## Unary Inductive Logic

In the initial investigations of Johnson and Carnap (and indeed almost exclusively up to the millennium) the language of Inductive Logic was taken to be *unary*. That is, in the notation we are adopting here the relation symbols  $R_1, R_2, \dots, R_q$  of the language  $L$  all had arity 1, and so would more commonly be referred to in Philosophy as ‘predicate’ symbols. Assume for the present that  $L$  is the unary language as above.

Now a state description for  $a_{i_1}, a_{i_2}, \dots, a_{i_m}$  is of the form

$$\bigwedge_{j=1}^m \bigwedge_{k=1}^q R_k^{\epsilon_{jk}}(a_{i_j}),$$

equivalently of the form

$$\bigwedge_{j=1}^m \alpha_{h_j}(a_{i_j}), \tag{1}$$

where the  $\alpha_{h_j}(x)$  ( $1 \leq h_j \leq 2^q$ ) are *atoms* of  $L$ , that is come from amongst the  $2^q$  formulae of the form

$$R_1^{\epsilon_1}(x) \wedge R_2^{\epsilon_2}(x) \wedge \dots \wedge R_q^{\epsilon_q}(x),$$

<sup>4</sup>Johnson’s *Permutation Postulate* and Carnap’s *Axiom of Symmetry*.

<sup>5</sup>The convention is that when a sentence  $\phi$  is written in this form it is assumed (unless otherwise stated) that the displayed constants are distinct and include all the constants actually occurring in  $\phi$ .

<sup>6</sup>The agent is not supposed to ‘know’ that  $a_1$  comes before  $a_2$  which comes before . . . in our list.

where the  $\epsilon_1, \epsilon_2, \dots, \epsilon_q \in \{0, 1\}$ . [There are  $2^q$  of them because there are two choices of  $\epsilon_i$  for  $i = 1, 2, \dots, q$ .]

For example if  $q = 3$  the conjunction of

$$\begin{array}{ccccccc} R_1(a_1) & R_1(a_2) & \neg R_1(a_3) & R_1(a_4) & \neg R_1(a_5) & \neg R_1(a_6) & \neg R_1(a_7) \\ R_2(a_1) & R_2(a_2) & \neg R_2(a_3) & R_2(a_4) & \neg R_2(a_5) & \neg R_2(a_6) & R_2(a_7) \\ \neg R_3(a_1) & \neg R_3(a_2) & \neg R_3(a_3) & \neg R_3(a_4) & R_3(a_5) & R_3(a_6) & \neg R_3(a_7) \end{array}$$

is a state description for  $a_1, a_2, \dots, a_7$ , it tells us everything there is to know about  $a_1, a_2, \dots, a_7$ . Indeed in this simple unary language the first column, equivalently the atom  $R_1(x) \wedge R_2(x) \wedge \neg R_3(x)$  which  $a_1$  satisfies, already tells us everything there is to know about  $a_1$  etc..

A useful way of thinking about atoms is as *colours* in a situation where the  $a_i$  are balls and that's their only distinguishing feature. Thus to know the colour of a ball is to know everything there is to know about that ball.

Now let  $0 \leq x_1, x_2, \dots, x_{2^q} \leq 1$  with  $\sum_{i=1}^{2^q} x_i = 1$  and define  $w_{\vec{x}}$ , where  $\vec{x} = \langle x_1, x_2, \dots, x_{2^q} \rangle$ , on the state description (1) by

$$w_{\vec{x}} \left( \bigwedge_{j=1}^m \alpha_{h_j}(a_{i_j}) \right) = \prod_{j=1}^m x_{h_j},$$

equivalently

$$w_{\vec{x}}(\alpha_{h_1}(a_{i_1}) \wedge \alpha_{h_2}(a_{i_2}) \wedge \dots \wedge \alpha_{h_m}(a_{i_m})) = x_{h_1} x_{h_2} \dots x_{h_m}. \quad (2)$$

It is straightforward to show that  $w_{\vec{x}}$ , defined as here on state descriptions, in fact extends to a probability function on  $SL$ . Indeed we can see already from (2) that  $w_{\vec{x}}$  looks like it is acting as follows: Given  $a_{i_j}$  it is picking an atom for  $a_{i_j}$  to satisfy, the probability of picking  $\alpha_{h_j}$  being  $x_{h_j}$ . Thus the probability of (independently) picking  $\alpha_{h_1}, \alpha_{h_2}, \dots, \alpha_{h_m}$  for  $a_{i_1}, a_{i_2}, \dots, a_{i_m}$  to satisfy is  $x_{h_1} x_{h_2} \dots x_{h_m}$ .

Notice that these  $w_{\vec{x}}$  satisfy Ex on state descriptions, since the right hand side of (2) does not depend on the actual (distinct) numbers  $i_1, i_2, \dots, i_m$  and in fact tracking through the proof of Proposition 3 shows that this is enough to ensure that  $w_{\vec{x}}$  satisfies Ex on *all* sentences. It turns out that these  $w_{\vec{x}}$  are precisely the 'simple building blocks' referred to above from which to generate all probability functions satisfying Ex.

## de Finetti's Representation Theorem

The following theorem due to Bruno de Finetti may be found in [6] (for a proof of this result in the notation being used here see Theorem 10 of [26]).

**de Finetti's Representation Theorem 4** *A probability function  $w$  on a unary language  $L$  satisfies Ex just if it is a mixture of the  $w_{\vec{x}}$ .*

*More precisely, just if*

$$w = \int w_{\vec{x}} d\mu(\vec{x})$$

where  $\mu$  is a countably additive measure on the Borel subsets of

$$\{\langle x_1, x_2, \dots, x_{2^q} \rangle \mid 0 \leq x_1, x_2, \dots, x_{2^q}, \sum_i x_i = 1\}. \quad (3)$$

On the face of it this theorem may seem to only be of interest to mathematicians, it doesn't seem to be saying much about induction or rationality which would be of interest to a philosopher. However it yields consequences and observations which surely are of interest in this regard. The mathematical power of this theorem lies in the fact that it often enables us to translate questions about the general probability function  $w$  on the left hand side of (3) into questions about the very simple probability functions  $w_{\vec{x}}$  on the right hand side.

For example we immediately have that for real numbers  $b, c$ ,

$$b^2x_1^2 + c^2x_2^2 \geq 2bcx_1x_2$$

since  $(bx_1 + cx_2)^2 \geq 0$ . Hence

$$b^2w_{\vec{x}}(\alpha_1(a_1) \wedge \alpha_1(a_2)) + c^2w_{\vec{x}}(\alpha_2(a_1) \wedge \alpha_2(a_2)) \geq 2bcw_{\vec{x}}(\alpha_1(a_1) \wedge \alpha_2(a_2))$$

and from (3), for  $w$  satisfying Ex,

$$b^2w(\alpha_1(a_1) \wedge \alpha_1(a_2)) + c^2w(\alpha_2(a_1) \wedge \alpha_2(a_2)) \geq 2bcw(\alpha_1(a_1) \wedge \alpha_2(a_2)).$$

Thus at least one of

$$b^2w(\alpha_1(a_1) \wedge \alpha_1(a_2)) \geq bcw(\alpha_1(a_1) \wedge \alpha_2(a_2)),$$

$$c^2w(\alpha_2(a_1) \wedge \alpha_2(a_2)) \geq bcw(\alpha_1(a_1) \wedge \alpha_2(a_2)),$$

must hold. Putting  $b = c = 1$  and using Ex to permute constants this gives that at least one of

$$w(\alpha_1(a_2) \mid \alpha_1(a_1)) \geq w(\alpha_2(a_2) \mid \alpha_1(a_1)),$$

$$w(\alpha_2(a_2) \mid \alpha_2(a_1)) \geq w(\alpha_1(a_2) \mid \alpha_2(a_1)),$$

must hold. Similarly putting  $b = w(\alpha_2(a_1))$ ,  $c = w(\alpha_1(a_1))$  it gives that at least one of

$$w(\alpha_1(a_2) \mid \alpha_1(a_1)) \geq w(\alpha_1(a_2) \mid \alpha_2(a_1)),$$

$$w(\alpha_2(a_2) \mid \alpha_2(a_1)) \geq w(\alpha_2(a_2) \mid \alpha_1(a_1)),$$

must hold. In other words we cannot have  $\alpha_2(a_1)$  being stronger evidence for  $\alpha_1(a_2)$  than  $\alpha_1(a_1)$  is *and* at the same time have  $\alpha_1(a_1)$  being stronger evidence for  $\alpha_2(a_2)$  than  $\alpha_2(a_1)$  is (though it is easy to see that we can have both separately).

Indeed by a just such a simple proof Humburg [15] showed the following result, originally due to Gaifman [10]:<sup>7</sup>

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<sup>7</sup>See [26] for a proof in the current notation.



**Theorem 5** *Ex implies the:*

**Principle of Instantial Relevance, PIR**

For  $\theta(a_1, a_2, \dots, a_m) \in SL$ ,

$$w(\alpha_i(a_{m+2}) \mid \alpha_i(a_{m+1}) \wedge \theta(a_1, a_2, \dots, a_m)) \geq w(\alpha_i(a_{m+1}) \mid \theta(a_1, a_2, \dots, a_m)).$$

From a philosopher's point of view this *is* an interesting result (or at least it should be!) because it confirms one's intuition re induction that the more often you've seen something in the past the more probable you should expect it to be in the future. So this turns out to be simply a consequence of Ex. Whilst one might claim, in accord with Hume [16], that Ex is really the assumption of the uniformity of nature and in that sense can be equated with induction<sup>8</sup> what this argument does show is that *if you think Ex rational then you should think PIR rational*. Certainly this result can be seen as casting a favorable light on Carnap's Programme.

A further observation on de Finetti's Theorem concerns the functions  $w_{\vec{x}}$ : They can be characterized by a certain *Irrelevance Principle*:

**Proposition 6** *The probability function  $w$  satisfying Ex is of the form  $w_{\vec{x}}$  for some  $\vec{x}$  just if it satisfies the:*

**Constant Irrelevance Principle, IP**

For  $\theta(a_1, a_2, \dots, a_m), \phi(a_{m+1}, a_{m+2}, \dots, a_{m+n}) \in SL$ ,

$$\begin{aligned} w(\theta(a_1, a_2, \dots, a_m) \wedge \phi(a_{m+1}, a_{m+2}, \dots, a_{m+n})) \\ = w(\theta(a_1, a_2, \dots, a_m)) \cdot w(\phi(a_{m+1}, a_{m+2}, \dots, a_{m+n})). \end{aligned}$$

Again this is a principle with some rational content and although, perhaps, on further consideration, not particularly to be recommended. It is interesting in that it says that IP characterizes the extreme probability functions satisfying Ex – i.e. the only probability functions satisfying IP are the  $w_{\vec{x}}$  and these are 'extreme' solutions to Ex in the sense that the only mixture of functions satisfying Ex which gives you  $w_{\vec{x}}$  is the trivial mixture containing that same  $w_{\vec{x}}$  alone.

What we have seen then here is that the *symmetry* principle Ex has led us to the *relevance* principle PIR and the *irrelevance* principle IP. It turns out that this is far from an isolated phenomenon and suggests then that one might be able to explain the perceived rationality of relevance and irrelevance principles in terms of the rationality of (obeying) symmetry.<sup>9</sup>

As a final observation here concerning de Finetti's Representation Theorem we might try to argue that a case could be made for  $w$  as in (3) to be rational if the measure  $\mu$  was rational. In other words for  $w$  to acquire rationality through  $\mu$ . An immediate objection here could be that saying what it means for  $\mu$  to be rational is even more of a puzzle than it was for  $w$ . One suggestion however that one might come up with is that  $\mu$  should be as uniform, as unassuming, as possible, after all whatever reason could the agent have for thinking otherwise? (if indeed the agent ever thinks about de Finetti priors!) Such considerations suggest that we should we should take  $\mu$  to be simply the standard normalized Lebesgue measure.

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<sup>8</sup>The result does not go the other way however, see [31, footnote 6] for an example of a probability function satisfying PIR but not Ex.

<sup>9</sup>As we mentioned above IP is not perhaps a particularly attractive Irrelevance Principle. However we shall later see another Irrelevance Principle, WIP, which arises from a representation theorem in an analogous fashion and seems somewhat more acceptable.

If we do that then  $w$  comes out to be the probability function  $c_{2^q}^L$  from Carnap's Continuum of Inductive Methods which for this language  $L$  with  $q$  unary predicates is characterized by:

$$c_{2^q}^L(\alpha_j(a_{n+1}) \mid \bigwedge_{i=1}^n \alpha_{h_i}(a_i)) = \frac{m_j + 1}{n + 2^q}$$

where  $m_j$  is the number of times that the atom  $\alpha_j$  occurs amongst

$$\alpha_{h_1}, \alpha_{h_2}, \dots, \alpha_{h_n}.$$

We shall have more to say about  $c_{2^q}^L$  in the next tutorial.

## Tutorial II

### Polyadic Inductive Logic

As previously mentioned until the turn of the millennium ‘Inductive Logic’, with very few exceptions (for example [9], [11], [30], [21] and [14]) meant ‘Unary Inductive Logic’.

Of course there was an awareness of this further challenge, Carnap [1, pages 123-4] and Kemeny [18], [19] both made this point. There were at least three reasons why the move to the polyadic was so delayed. The first is that simple, everyday examples of induction with non-unary relations are rather scarce. However they do exist and we do seem to have some intuitions about them. For example suppose that you are planting an orchard and you read that apples of variety  $\mathcal{A}$  are good pollinators and apples of variety  $\mathcal{B}$  are readily pollinated. Then you might expect that if you plant an  $\mathcal{A}$  apple next to a  $\mathcal{B}$  apple you will be rewarded with an abundant harvest, at least from the latter tree. In this case one might conclude that you had applied some sort of polyadic induction to reach this conclusion, and that maybe it has a logical structure worthy of further investigation.

A second reason for a certain reluctance to venture into the polyadic is that the notation and mathematical complication increases significantly, at least when compared with the unary case. The key reason for this is that a state description, for  $a_1, a_2, \dots, a_n$  say, no longer tells us everything there is to know about these constants in the way it did in the unary situation. For example for a binary relation symbol  $R$  of  $L$  it gives no information about whether or not  $R(a_1, a_{n+1})$  holds. Thus we can never really ‘nail down’ the constants as we could before.

Finally, a third reason, we would suggest, is the relative lack of intuition when it comes to forming beliefs about polyadic relations. To take an example suppose you are told that

$$R(a_1, a_2) \wedge R(a_2, a_1) \wedge \neg R(a_1, a_3).$$

In this case which of  $R(a_3, a_1), \neg R(a_3, a_1)$  would you think the more likely? The first of these two options seems to be supported by the fact that we know of two positive occurrences of  $R$  (namely  $R(a_1, a_2), R(a_2, a_1)$ ) and only one occurrence of  $\neg R$  (namely  $\neg R(a_1, a_3)$ ). On the other hand the fact that  $R$  holds of both  $\langle a_1, a_2 \rangle$  and of  $\langle a_2, a_1 \rangle$  suggests maybe that  $R$  is somewhat symmetric in its two arguments and hence, by *analogy*, that the  $\neg R(a_1, a_3)$  should provide evidence for  $\neg R(a_3, a_1)$ .

This lack of intuition extends too to the problem of proposing rational principles in the polyadic case. We have Ex of course and also Predicate Exchangeability and Strong Negation but beyond that things look less clear. In fact one such principle can be obtained<sup>10</sup> by generalizing a principle which both Johnson and Carnap<sup>11</sup> espoused. To motivate this recall a state description  $\Theta(a_1, a_2, \dots, a_7)$  which we looked at in the unary case:

$$\begin{array}{ccccccc} R_1(a_1) & R_1(a_2) & \neg R_1(a_3) & R_1(a_4) & \neg R_1(a_5) & \neg R_1(a_6) & \neg R_1(a_7) \\ R_2(a_1) & R_2(a_2) & \neg R_2(a_3) & R_2(a_4) & \neg R_2(a_5) & \neg R_2(a_6) & R_2(a_7) \\ \neg R_3(a_1) & \neg R_3(a_2) & \neg R_3(a_3) & \neg R_3(a_4) & R_3(a_5) & R_3(a_6) & \neg R_3(a_7) \end{array}$$

<sup>10</sup>Though historically it arose from a different path.

<sup>11</sup>It amounts to Carnap’s *Attribute Symmetry* in the present context.

Say  $a_i, a_j$  are *indistinguishable w.r.t. this stated description* if it is consistent with  $\Theta$  that  $a_i, a_j$  are actually equal, i.e.

$$\Theta(a_1, a_2, \dots, a_7) \wedge a_i = a_j$$

is consistent in the Predicate Calculus with equality.

So here  $a_1, a_2, a_4$  are indistinguishable, as are  $a_5, a_6$ , but  $a_3, a_7$  are both distinguishable from all the other  $a_i$ . Note the thinking of these atoms as defining the colour of a constant,  $a_i, a_j$  being indistinguishable just says they have the same colour. Clearly indistinguishability with respect to a state description is an equivalence relation. In this specific case then the equivalence classes are

$$\{a_1, a_2, a_4\}, \{a_5, a_6\}, \{a_3\}, \{a_7\}$$

and according to the colour analogy the equivalence classes are just the sets of  $a_i$  with the same colour.

Now define the *Spectrum of a State Description* to be the multiset of sizes of these equivalence classes.

So for  $\Theta(a_1, a_2, \dots, a_7)$  as above its spectrum is  $\{3, 2, 1, 1\}$ .

## Spectrum Exchangeability

The principle alluded to above that both Johnson and Carnap adopted in their situation of unary languages we call:

### Atom Exchangeability, Ax

*For a state description  $\Theta$  the probability  $w(\Theta)$  should only depend on the spectrum of  $\Theta$ .*

Ax is really a symmetry principle. If we think of the atoms as corresponding to colours it says that the probability we assign to a state description  $\Theta$  should be invariant under a permutation, or renaming of the colours. In short that all that matters about a colour is that it is different from other colours, not whether it is called red, or blue, or green, etc..<sup>12</sup>

Atom Exchangeability, what Carnap called ‘Attribute Symmetry’, is now quite well studied and accepted in Unary Inductive Logic. Furthermore it extends smoothly to polyadic languages.

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<sup>12</sup>Ax is a consequence of a stronger principle that Johnson and Carnap both championed:

#### Johnson’s Sufficientness Principle

$$w(\alpha_j(a_{n+1}) \mid \bigwedge_{i=1}^n \alpha_{h_i}(a_i))$$

depends only on  $n$  and the number of times that the atom  $\alpha_j$  occurs amongst

$$\alpha_{h_1}, \alpha_{h_2}, \dots, \alpha_{h_n}.$$

Expressed in terms of colours then it says that having seen the colours of  $a_1, a_2, \dots, a_n$  the probability that  $a_{n+1}$  will be green depends only on the number,  $n$ , of previous  $a_i$  seen, and the number of those that were green – the distribution of colours amongst the non-green  $a_i$  is of no consequence. This seems reasonable if we think of these colours being determined by picking balls out of an urn (with replacement). However it is debateable whether our agent will feel it a reasonable assumption that his/her ambient structure was constructed in this fashion.

For taking  $L$  henceforth to be polyadic we define the indistinguishability of  $a_{i_k}, a_{i_r}$  w.r.t a state description  $\Theta(a_{i_1}, a_{i_2}, \dots, a_{i_m})$  as before, i.e. just if

$$\Theta(a_{i_1}, a_{i_2}, \dots, a_{i_m}) \wedge a_{i_k} = a_{i_r}$$

is consistent (in the Predicate Calculus with Equality). So, for example, taking the language  $L$  to have just a single binary relation symbol  $R$ , with respect to the state description (which is the conjunction of)

$$\begin{array}{cccc} R(a_1, a_1) & \neg R(a_1, a_2) & R(a_1, a_3) & R(a_1, a_4) \\ R(a_2, a_1) & \neg R(a_2, a_2) & R(a_2, a_3) & \neg R(a_2, a_4) \\ R(a_3, a_1) & \neg R(a_3, a_2) & R(a_3, a_3) & R(a_3, a_4) \\ R(a_4, a_1) & R(a_4, a_2) & R(a_4, a_3) & R(a_4, a_4) \end{array}$$

the ‘indistinguishability’ equivalence classes are

$$\{a_1, a_3\}, \{a_2\}, \{a_4\}$$

and the spectrum is  $\{2, 1, 1\}$

In the (possibly) polyadic setting we can now propose the following rational principle:

### **Spectrum Exchangeability, Sx**

*For a state description  $\Theta$  the probability  $w(\Theta)$  only depends on the spectrum of  $\Theta$ . If state descriptions  $\Theta, \Phi$  have the same spectrum then  $w(\Theta) = w(\Phi)$ .*

So for example, with  $L$  just having a single binary relation symbol the conjunctions of

$$\begin{array}{ccc} R(a_1, a_1) & \neg R(a_1, a_2) & R(a_1, a_3) \\ R(a_2, a_1) & \neg R(a_2, a_2) & R(a_2, a_3) \\ R(a_3, a_1) & \neg R(a_3, a_2) & R(a_3, a_3) \end{array}$$

and of

$$\begin{array}{ccc} \neg R(a_1, a_1) & \neg R(a_1, a_2) & R(a_1, a_3) \\ \neg R(a_2, a_1) & \neg R(a_2, a_2) & R(a_2, a_3) \\ R(a_3, a_1) & R(a_3, a_2) & R(a_3, a_3) \end{array}$$

get the same probability as both have spectrum  $\{2, 1\}$ .

Notice a difference here from the unary case. In the unary if  $a_1, a_2$  are indistinguishable with respect to the state description  $\Theta(a_1, a_2, a_3)$  they will still be indistinguishable with respect to  $\Phi(a_1, a_2, a_3, a_4)$  for any state description  $\Phi$  extending  $\Theta$ . In other words if  $a_1, a_2$  get the same colour then thereafter they have the same colour for all eternity. But for a not purely unary language that is not necessarily the case, indistinguishables can be distinguished by ‘later’ constants. For example the state description

$$\begin{array}{cccc} \neg R(a_1, a_1) & \neg R(a_1, a_2) & R(a_1, a_3) & R(a_1, a_4) \\ \neg R(a_2, a_1) & \neg R(a_2, a_2) & R(a_2, a_3) & \neg R(a_2, a_4) \\ R(a_3, a_1) & R(a_3, a_2) & R(a_3, a_3) & \neg R(a_3, a_4) \\ R(a_4, a_1) & R(a_4, a_2) & R(a_4, a_3) & R(a_4, a_4) \end{array}$$

extends the one immediately above it but has spectrum  $\{1, 1, 1, 1\}$ ,  $a_4$  has distinguished  $a_1$  and  $a_2$ .<sup>13</sup>

Sx has a number of pleasing consequences, for example Px, SN, simply because the permutations they specify when applied to a state description do not change its spectrum. However before going on to explain some of the deeper consequences of Sx we will consider an altogether different sort of principle, more a ‘meta-principle’, which appears to be both natural and desirable within Inductive Logic as a whole.

## Language Invariance

The enlargement of PIL to polyadic languages raises the issue of why we should fix, and work in, a particular language  $L$  in the first place. After all our agent could always *imagine* that the language contained relation symbols in addition to those in  $L$ . Moreover would not the agent then wish to adopt a probability function for that larger language which satisfied the same principle that s/he would have considered rational for  $L$  alone? Surely yes!

But then can this wish of the agent actually be realized? The problem is that the agent might follow his/her rational principles and pick the probability functions  $w$  on  $L$  and  $w^+$  on the (imagined) extended language  $L^+$  and find that the restriction of  $w^+$  to  $SL$ , denoted  $w^+ \upharpoonright SL$ , is not the same as  $w$ . In other words simply by imagining being rational in  $L^+$  the agent would have discredited  $w$ . Indeed looked at from this perspective  $w$  might seem a particularly bad choice if there was no extension at all of  $w$  to  $L^+$  which satisfied the agent’s favored rational principles.<sup>14</sup>

To make this more concrete suppose the agent felt Sx (+ Ex) was the (only) rational requirement that s/he was obliged to impose on his/her choice  $w$ . Then it might be that the agent made such a choice only to realize that there was no way to extend this probability function to a larger language and still maintain having Sx.

In fact this can happen for some bad choices of  $w$ , but fortunately it needn’t happen, there will be choices of probability function for which there are no such dead ends. These are the ones which satisfy:

### Language Invariance with Sx, Li+Sx

*A probability function  $w$  satisfies Language Invariance with Sx if there is a family of probability functions  $w^{\mathcal{L}}$ , one on each language  $\mathcal{L}$ , containing  $w$  (so  $w = w^L$ ) such that each member of this family satisfies Sx and whenever language  $\mathcal{L}_1, \mathcal{L}_2$  are such that  $\mathcal{L}_1 \subseteq \mathcal{L}_2$  then  $w^{\mathcal{L}_2} \upharpoonright S\mathcal{L}_1 = w^{\mathcal{L}_1}$ .*

It turns out that Li+Sx implies most (maybe even all) the of the desirable properties so far proposed for a rational polyadic probability function. Before mentioning some these however it would seem a good idea to give an example of a probability function satisfying Li+Sx, we’ve

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<sup>13</sup>This observation may raise the unpleasant worry in ones mind that maybe Sx is not even consistent, or is at best trivial! For maybe the two state descriptions for  $a_1, a_2, a_3$  above with the same spectra could have different multiplicities of spectra amongst their extensions to state descriptions for  $a_1, a_2, a_3, a_4$ . Fortunately this does not happen though the proof of that seems not entirely trivial, see [25], [22].

<sup>14</sup>There is just such a fault with the choice of the probability function  $c_{2^q}^L$  on a unary language  $L$  with  $q$  predicates. This was justified in terms of the measure  $\mu$  being standard Lebesgue measure. If we apply exactly the same reasoning to the language  $L^+$  formed by adding an extra unary predicate to  $L$  we obtain  $c_{2^{q+1}}^{L^+}$ . But if we restrict this probability function to  $SL$  we do not get back our ‘favored choice’  $c_{2^q}^L$  for that language! Seems to me this seriously dents the so called ‘rationality of Lebesgue measure’.

got this far without actually confirming that there's any probability function satisfying Li+Sx, equivalently confirming that Li+Sx is even consistent!

## The $u^{\bar{p},L}$

Let  $\bar{p}$  be the sequence

$$p_0, p_1, p_2, p_3, \dots$$

of real numbers such that

$$p_1 \geq p_2 \geq p_3 \geq \dots \geq 0 \quad \text{and} \quad \sum_{i=0}^{\infty} p_i = 1.$$

We think of the subscripts here  $0, 1, 2, 3, \dots$  as *colours*, with 0 being black, and  $p_i$  as the probability of picking colour  $i$  (with replacement).

Now consider a state description  $\Theta(a_{i_1}, a_{i_2}, \dots, a_{i_n})$  and a sequence of colours (not necessarily distinct)  $c_1, c_2, \dots, c_n$  (so these are really just natural numbers). We say that  $\Theta(a_{i_1}, a_{i_2}, \dots, a_{i_n})$  is *consistent* with this sequence if whenever  $c_j = c_k \neq 0$  then  $a_{i_j}, a_{i_k}$  are indistinguishable with respect to  $\Theta$ .

To take a specific example here suppose that the language has a single binary relation symbol  $R$ ,

$$p_0 = 1/4, p_1 = 2/3, p_2 = 1/12, p_3 = p_4 = \dots = 0,$$

and  $\Theta(a_1, a_2, a_3)$  is the conjunction of

$$\begin{array}{lll} \neg R(a_1, a_1) & \neg R(a_1, a_2) & R(a_1, a_3) \\ \neg R(a_2, a_1) & \neg R(a_2, a_2) & R(a_2, a_3) \\ R(a_3, a_1) & R(a_3, a_2) & R(a_3, a_3) \end{array}$$

so the indistinguishability equivalence classes are  $\{a_1, a_2\}$  and  $\{a_3\}$ . Then  $\Theta(a_1, a_2, a_3)$  is consistent with the sequence of colours 2, 2, 1, and with 0, 2, 1, and with 0, 0, 0, and with 1, 2, 0, but not with 1, 2, 1, nor with 0, 1, 1.

Now define  $u^{\bar{p},L}(\Theta(a_{i_1}, a_{i_2}, \dots, a_{i_n}))$  as follows:

- Pick a sequence of colours  $c_1, c_2, \dots, c_n$  according to the probabilities  $p_0, p_1, p_2, \dots$ , so the probability of picking  $c_1, c_2, \dots, c_n$  is

$$p_{c_1} \times p_{c_2} \times \dots \times p_{c_n}.$$

- Randomly (i.e. according to the uniform distribution) pick a state description  $\Phi(a_{i_1}, a_{i_2}, \dots, a_{i_n})$  consistent with  $c_1, c_2, \dots, c_n$ .
- $u^{\bar{p},L}(\Theta(a_{i_1}, a_{i_2}, \dots, a_{i_n}))$  is the probability that  $\Theta = \Phi$ .

Unfortunately even calculating  $u^{\bar{p},L}(\Theta)$  for the specific example above is horribly complicated, but here goes:

Initially there are 27 choices for  $c_1, c_2, c_3$ . Of these  $\Theta$  is not consistent with 1, 0, 1; 1, 1, 1; 1, 2, 1; 2, 0, 2; 2, 1, 2; 2, 2, 2; 0, 1, 1; 2, 1, 1 so none of these could possibly yield a  $\Phi$  which equalled  $\Theta$ .

In the case of the sequence of colours being one of

0, 0, 0; 1, 0, 0; 2, 0, 0; 0, 1, 0; 0, 2, 0; 0, 0, 1; 0, 0, 2;  
1, 0, 2; 2, 0, 1; 0, 1, 2; 0, 2, 1; 1, 2, 0; 2, 1, 0

any of the  $2^9$  possible state descriptions for  $a_1, a_2, a_3$  are consistent with the sequence of colours, so the probability in each case of picking  $\Theta$  is  $2^{-9}$ . The probability of picking one of these sequences of colours is

$$(1/4)^3 + 2 \times (2/3) \times (1/4)^2 + 2 \times (1/12) \times (1/4)^2 + 6 \times (1/4) \times (2/3) \times (1/12).$$

In the cases of 1, 1, 2; 2, 2, 1; 1, 1, 0; 2, 2, 0 (chosen with probabilities  $(2/3)^2 \times (1/12)$ ,  $(1/12)^2 \times (2/3)$ ,  $(2/3)^2 \times (1/4)$  and  $(1/12)^2 \times (1/4)$  respectively) there are  $2^4$  state descriptions consistent with them (including  $\Theta$ ) so the probability of picking  $\Theta$  is  $2^{-4}$  in each case.

So  $u^{\bar{p},L}(\Theta(a_1, a_2, a_3))$  is

$$2^{-9}[(1/4)^3 + 2 \times (2/3) \times (1/4)^2 + 2 \times (1/12) \times (1/4)^2 + 6 \times (1/4) \times (2/3) \times (1/12)] \\ + 2^{-4}[(2/3)^2 \times (1/12) + (1/12)^2 \times (2/3) + (2/3)^2 \times (1/4) + (1/12)^2 \times (1/4)]$$

which equals  $\frac{8707}{98304}$  (I think!).

Despite their painful birth one can, in time, make friends with the  $u^{\bar{p},L}$  and they turn out to be *the central building blocks* in the study of  $Sx$ , as the next section indicates.

## Paradise Gained

**Theorem 7** *The  $u^{\bar{p},L}$  satisfy Ex and Li+Sx.*

For Language Invariance notice that this definition of the  $u^{\bar{p},L}$  is completely uniform in the ‘parameter’  $L$ . A little work shows that if we fix the  $\bar{p}$  but vary the language then these  $u^{\bar{p},L}$ , as  $L$  ranges over languages, form a Language Invariance family (containing of course our  $u^{\bar{p},L}$ ). In other words, for  $\mathcal{L}_1 \subseteq \mathcal{L}_2$ ,  $u^{\bar{p},\mathcal{L}_2}$  restricted to  $S\mathcal{L}_1$  is just  $u^{\bar{p},\mathcal{L}_1}$ .

To see that  $u^{\bar{p},L}$  (and hence all members of the family of  $u^{\bar{p},L}$ ) satisfies  $Sx$  is quite easy: The definition of  $u^{\bar{p},L}(\Theta)$  actually only depends only on the spectrum of  $\Theta$ , so will be the same for any two state descriptions with the same spectrum. Similarly with Ex, the value of  $u^{\bar{p},L}(\Theta)$  does not depend on the actual constants involved (– except that we do need to check that this definition does not over specify  $u^{\bar{p},L}$ ).

Now for the pay-off, these  $u^{\bar{p},L}$  act a building blocks in the polyadic case for Li+Sx (+Ex) just as the  $w_{\bar{x}}$  did for Ex in the unary case:

**Theorem 8** *Any probability function  $w$  on  $L$  satisfying Li+Sx (+Ex) is a mixture of the  $u^{\bar{p},L}$ , i.e.*

$$w = \int u^{\bar{p},L} d\mu(\bar{p}).$$

Using this we can now show:

**Theorem 9** *The  $u^{\bar{p},L}$  are precisely the probability functions  $w$  on  $L$  satisfying Li+Sx which also satisfy the:*

### Weak Irrelevance Principle, WIP

*If  $\theta, \phi \in SL$  have no constant nor relation symbols in common then*

$$w(\theta \wedge \phi) = w(\theta) \cdot w(\phi).$$



Again then, as with the  $w_{\bar{x}}$  and Ex, these  $u^{\bar{p},L}$  are the extreme solutions to Li+Sx and they are characterized by an Irrelevance Principle, in this case WIP.

Similarly using Theorem 8 we can show the following *Relevance Principle*:

**Theorem 10** *Let  $w$  be a probability function on  $L$  satisfying Li+Sx, let  $\Theta(a_1, a_2, \dots, a_n)$  be a state description and suppose that amongst  $a_1, a_2, \dots, a_n$  there are at least as many  $a_i$  which are indistinguishable from  $a_1$  as there are  $a_i$  which are indistinguishable from  $a_2$ . Then given  $\Theta$ , the probability that  $a_{n+1}$  is indistinguishable from  $a_1$  is greater or equal to the probability that it is indistinguishable from  $a_2$ .*

As a particular application of this result recall the earlier question:

*Suppose you are told that*

$$R(a_1, a_2) \wedge R(a_2, a_1) \wedge \neg R(a_1, a_3).$$

*Which of  $R(a_3, a_1), \neg R(a_3, a_1)$  should you think the more probable?*

– it follows from Theorem 10 that if your  $w$  satisfies Li+Sx then  $\neg R(a_3, a_1)$  is at least as probable as  $R(a_3, a_1)$ . I.e. analogy wins out.

The proofs of these last three theorems may be found in [23], [28], [27] respectively.<sup>15</sup>

Finally we mention (informally) a consequence of Theorem 8 which seems far less expected.

**Corollary 11** *Let  $w$  satisfy Li+Sx.<sup>16</sup> Then the probability, according to  $w$ , that  $a_1, a_2$  are indistinguishable but distinguishable from all other constants  $a_i$  is zero.*

[This result parallels J.Kingman’s result on Genetic Diversity that one never meets exactly two examples of a species: One either meets exactly one or one meets infinitely many. See [20].]

Well, this has all become rather mathematical, let’s step back and see where we’re up to.

## Trying to Make Sense of it all

With apologies to many of you in the audience these tutorials have contained a fair bit of mathematics, and in fact the formal mathematics behind some of these last results is considerably worse. The payoff is that some, I hope most, of these results have some philosophical import. For example the result that if one accepts as rational Constant Exchangeability Ex then one is, by dint of mathematical proof, forced to accept the Principle of Instantial Relevance.

Nevertheless such results seem to me to raise a fundamental problem because it is not as if one can, once it is pointed out, simply *see* this implication *unless* one has the mathematical background to understand and be convinced by the proof. For example could our original, possibly innumerate, agent be blamed for accepting Ex but not PIR?

To counter this charge one might suggest here that there could be an alternate, much simpler, proof just waiting to be discovered. Well, maybe, but I very much doubt it, and furthermore this is but one of a number of similar results, surely they can’t all be trivialized.

<sup>15</sup>A broader coverage of recent results on Sx may be found in [24] or the (hopefully forthcoming) [29].

<sup>16</sup>Actually Li can be dropped in this case, just Sx suffices.

One might also suggest that whether or not one saw the connection between Ex and PIR both principles are rational in their own right, so both should be accepted even if in fact just accepting the former would actually suffice. Indeed Carnap already favored PIR before Gaifman revealed this link to Ex. That suggestion might hold weight here in this case but there are results where the hypothesis is arguably rational whilst the consequence, obtained via some complicated mathematics, is far less so, for example Corollary 11.

In conclusion then it seems that whilst the mathematics of PIL may uncover interesting new principles and connections between principles, it is questionable whether, in general, it currently provides any additional *philosophical* insight or understanding, even to those who fully comprehend the mathematics involved.

A second question that the work in PIL has raised concerns the relationship between the three main (to date) sources of rational principles, Symmetry, Relevance and Irrelevant. Over these two lectures I have discussed two, broadly speaking, symmetry principles, Ex and Li+Sx. In each case I have described a de Finetti style representation theorem and from that we obtained a relevance principle, PIR in the case of Ex and Theorem 10 in the case of Li+Sx. In addition the ‘building blocks’ of these representation theorems, the  $w_{\vec{x}}$  and the  $u^{\vec{p},L}$  respectively, were characterized by satisfying an irrelevance principle, IP and WIP respectively. Whether or not these irrelevance principles are at all desirable as expressions of rationality (I think this is debatable) the fact is that symmetry seems to be the basic principle which begets and relevance and irrelevance principles. This can in a way be explained for relevance, symmetry says that the future is like the past and relevance says that what happened in the past should be a guide to the future. But for irrelevance?

We finally mention an issue which was certainly close to Carnap’s heart – completeness. Even if our agent accepts that to be rational his/her chosen probability function  $w$  should satisfy Li+Sx (which is about as strong a principle as we have on the table right now) this still leaves him/her with a very wide range of probability functions to choose from. In a way this might be viewed as unfortunate because it suggests that two agents, both entirely rational according to this criterion, could still assign different beliefs/probabilities. Wouldn’t it be nice if we could find additional, acceptable, rational principles which cut out this choice, forced these two agents to agree. In other words that we had ‘completeness’.

Carnap spent some effort to achieve this in the case of Unary Inductive Logic (with his search for the ‘right’ value for the parameter  $\lambda$  in his *Continuum of Inductive Methods*). To date this search has proved fruitless and the situation in Polyadic Inductive Logic certainly looks no more hopeful. Having said that there are ostensibly rational principles which do ensure completeness. Namely we can argue that as far as our agent is concerned s/he has no reason to think that  $R(a_{i_1}, a_{i_2}, \dots, a_{i_n})$ , where  $R$  is an  $n$ -ary relation symbol of  $L$ , is any more probable than its negation, and furthermore has no reason for supposing that there is any dependence between different sentences of this simple form (i.e. that they are all stochastically independent). These assumptions alone cut down the choice of probability function to just one,  $u^{\vec{p},L}$  where  $\vec{p} = 1, 0, 0, 0, \dots$  in fact.

Unfortunately this probability function is completely bereft of any inductive inclinations For example for a unary  $R$  it gives

$$u^{\vec{p},L}(R(a_{n+1}) \mid R(a_1) \wedge R(a_2) \wedge \dots \wedge R(a_n)) = 1/2$$

no matter how large  $n$  is. In other words it takes no notice at all of the previous supportive evidence  $R(a_1) \wedge R(a_2) \wedge \dots \wedge R(a_n)$  in assigning a probability to  $R(a_{n+1})$ . This is no good, what we want here is a choice of probability function which gives greater than in place of

equality when  $n > 0$ . But how much greater than? The apparent arbitrariness of that answer suggests to me that acceptable completeness is an unattainable dream – but as ever I could be wrong!

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