A Guided Tour
of
Variation of Parameters Formulae
for
Continuous and Discretized DDEs.

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Abstract

In this report we discuss variation of parameters formulae (i) for systems of linear delay
differential equations (DDEs),
\[
\frac{dy(t)}{dt} - A(t)y(t) - B(t)y(t - \tau) = f(t), \quad \text{for } t \in [0, T],
\]
subject to an initial condition \( y(t) = \varphi(t) \) for \( t \in [-\tau, 0] \), and (ii) for the corresponding
discretized equations that result from the application of Euler methods.

The explicit Euler method for the above DDE, using a step \( h = \tau/N \) with \( T = Kh \) and
\( N, K \in \mathbb{N} \), yields the equations

\[
\bar{y}_{n+1} - \bar{y}_n - A_n \bar{y}_n - B_n \bar{y}_{n-N} = f_n, \quad \text{for } n \in \{0, 1, \cdots, K-1\},
\]

subject to the initial condition \( \bar{y}_n = \varphi_n \) for \( -N \leq n \leq 0 \), where \( B_n = B(nh) \), \( A_n = A(nh) \)
\( f_n = f(nh) \) and \( \varphi_n = \varphi(nh) \).

Of interest is the form of the solution to the DDE or its discretized form, in a representation
that depends upon the inhomogeneous term \( f \) or the initial function \( \varphi \). Various ways to derive
variation of parameters formulae will be pursued. In the continuous case, the fundamental
solution can be obtained as a solution of a DDE, a related integral equation, by seeking a
generalized Green’s function or by considering adjoint equations. (A number of results related
to the continuous case are presented in the literature and these are recalled in an appendix.)
We look for analogous results in the discretized case, drawing on parallels in linear algebra.
Keywords: Continuous and discretized delay differential equations, Continuous and discrete
Volterra equations, Resolvents, Generalized Green’s functions, Fundamental solutions.

1 Introduction

Suppose that

\[
f(t), \varphi(t) \in \mathbb{R}^{m \times 1}, \quad A(t), B(t) \in \mathbb{R}^{m \times m}
\]  

(1.1)

where we assume that \( A, B \in C[0, T] \), that \( \varphi \) is piecewise continuous on \([-\tau, 0]\), and that \( f \) is
piecewise continuous on \([0, T]\), and both are continuous from the right. We seek \( y(t) \in \mathbb{R}^{m \times 1} \) that

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satisfies the delay differential equation (DDE)
\[ \frac{dy(t)}{dt} - A(t)y(t) - B(t)y(t - \tau) = f(t), \quad \text{for } t \in [0, T], \] (1.2)
and also satisfies the initial condition \( y(t) = \varphi(t) \) for \( t \in [-\tau, 0] \). The derivative \( y'(t) \) in (1.2) is taken to be the right-hand derivative, where necessary.

We are also concerned with the solution of the discretized equations that result from the application of an Euler method. The explicit Euler equations for the DDE above, using a step \( h = \tau/N \) with \( T = Kh \) and \( N, K \in \mathbb{N} \), read
\[ \tilde{y}_{n+1} - \tilde{y}_n - A_n\tilde{y}_n - B_n\tilde{y}_{n-N} = f_n, \quad \text{for } n \in \{0, 1, \ldots, K-1\}, \] (1.3)
subject to the initial condition \( \tilde{y}_n = \varphi_n \) for \( -N \leq n \leq 0 \), where \( B_n = B(nh) \), \( A_n = A(nh) \)
\( f_n = f(nh) \) and \( \varphi_n = \varphi(nh) \).

We consider the DDEs in §2 and the discretized equations in §3. In either case, a variation of parameters formula is a formula that gives an expression for the solution in terms of \( \varphi(\cdot) \) and \( f(\cdot) \). Tools for investigating such formulae can be drawn from the theory of DDE, the theory of integral equations, by seeking a generalized Green’s function or by considering a rôle for adjoint equations, as well as from linear algebra.

Since we deal with systems of equations in \( m \)-dimensional space, we shall use some vector notation. We suppose that \( e_\lambda \in \mathbb{R}^{m \times 1} \) (for \( \lambda = 1, 2, \ldots, m \)) are the successive columns of the identity matrix \( I \) and that \( e^T_\rho \in \mathbb{R}^{1 \times m} \) (for \( \rho = 1, 2, \ldots, m \)) are the successive rows of \( I \). The superscript \( ^T \) denotes the transpose.

2 The continuous case

Consider the linear system
\[ \frac{dy(t)}{dt} - A(t)y(t) - B(t)y(t - \tau) = f(t), \quad \text{for } t \in [0, T], \] (2.1a)
subject to
\[ y(t) = \varphi(t) \quad \text{for } t \in [-\tau, 0]. \] (2.1b)

As a special case of (2.1) we have the homogeneous case in which \( f \) vanishes,
\[ \frac{du(t)}{dt} - A(t)u(t) - B(t)u(t - \tau) = 0, \quad \text{for } t \in [0, T], \] (2.2a)
subject to
\[ u(t) = \varphi(t) \quad \text{for } t \in [-\tau, 0]. \] (2.2b)
This equation will feature later. Observe that \( \varphi(t) \) is not required to vanish identically.

**Remark 2.1** We remark that (2.1) can be transformed into a problem in which \( \varphi(t) \) vanishes on \([-\tau, 0] \) (\( \varphi(0) \) need not vanish). When \( t \in [0, \tau] \), we can re-write (2.1), as
\[ \frac{dy(t)}{dt} - A(t)y(t) = f(t) + B(t)\varphi(t - \tau) \] (2.3)
subject to the condition \( y(0) = \varphi(0) \). Of course, (2.3) is a differential equation on \([0,T]\), not a delay differential equation. We leave (2.1) unchanged for \( \tau \leq t \leq T \). Thus, \( y(t) \) agrees, for \( t \in [0,T] \), with the solution \( w(t) \) of the problem

\[
\frac{dw(t)}{dt} - A(t)w(t) - B(t)w(t - \tau) = f_2(t), \quad t \in [0,T],
\]  
(2.4a)

where \( w(t) \) is subject to

\[
w(t) = 0 \quad \text{for} \quad t \in [-\tau,0), \quad w(0) = \varphi(0).
\]  
(2.4b)

Here, the derivative at \( t = 0 \) is taken as the right-hand derivative and

\[
f_2(t) = f(t) + B(t)\varphi(t - \tau), \quad \text{when} \quad 0 \leq t \leq \tau, \quad \text{and} \quad f_2(t) = f(t), \quad \text{when} \quad t > \tau.
\]  
(2.4c)

2.1 A related equation: the adjoint equation

Later in this report we shall need to consider another type of problem with deviating arguments, related to (2.1).

**Definition 2.1** Given functions \( p^T(t) \) and \( \psi^T(t) \in \mathbb{R}^{1 \times m} \) (for \( t \in [0,T] \) and \( t \in [T,T + \tau] \) respectively, the corresponding formal adjoint for (2.1) is

\[
\frac{dx^T(s)}{ds} + x^T(s)A(s) + x^T(s+\tau)B(s+\tau) = p^T(t), \quad s \in [0,T],
\]  
(2.5a)

subject to

\[
x^T(s) = \psi^T(s) \quad \text{for} \quad s \in [T,T + \tau],
\]  
(2.5b)

with a solution \( x^T(s) \in \mathbb{R}^{1 \times m} \). We shall refer to (2.5a) as a “formal adjoint equation” for (2.1a), and when \( p(t) \equiv 0 \) we refer to the homogeneous formal adjoint.

The derivative \( \frac{dx^T(s)}{ds} \) in (2.5a) is interpreted, if necessary, as the left-hand derivative at \( t = T \). If we replace \( s \) by \( T - t \) and take the transpose, then the adjoint equation is transformed into a DDE of the form (2.2).

2.2 A resolvent \( V \) for a Volterra integral equations

We remark on (and exploit) a connection with Volterra integral equations. The equation

\[
\frac{dw(t)}{dt} - A(t)w(t) - B(t)w(t - \tau) = f_2(t) \quad (t \in [0,T])
\]

encountered in (2.4) allows us, since \( w(t) = w(0) + \int_0^t w'(s)ds \), to write

\[
w(t) = \{w(0) + \int_0^t f_2(s)ds\} + \int_0^t A(s)w(s)ds + \int_0^{t-\tau} B(s+\tau)w(s)ds.
\]

This is a Volterra integral equation of the form

\[
w(t) = g(t) + \int_0^t K(t,s)w(s)ds \quad (t \in [0,T])
\]  
(2.6)
where $K(t, s)$ is expressible in terms of $A(\cdot)$ and $B(\cdot)$:

$$K(t, s) = \begin{cases} A(s) + B(s + \tau) & (0 \leq s \leq t - \tau), \\ A(s) & (t - \tau < s \leq t), \end{cases} \quad (2.7a)$$

and, since $w(0) = \varphi(0)$,

$$g(t) = \varphi(0) + \int_0^t f_2(s)ds. \quad (2.7b)$$

The solution of (2.6) can be expressed in terms of an appropriate resolvent kernel:

$$w(t) = g(t) + \int_0^t R(t, s)g(s)ds \quad (2.8a)$$

where

$$R(t, s) = \int_0^t K(t, \sigma)R(\sigma, s)d\sigma = K(t, s). \quad (2.8b)$$

Alternatively$^1$ (evaluating $\int_0^t R(t, s)g(s)ds$ using integration by parts)

$$w(t) = V(t, 0)g(0) + \int_0^t V(t, s)g'(s)ds, \quad (2.9a)$$

since $g'(s)$ exist, where, for $t \in [0, T]$,

$$V(t, s) - \int_s^t K(t, \sigma)V(\sigma, s)d\sigma = I \quad \text{for } 0 \leq s \leq t, \text{ and } V(t, s) = 0 \quad \text{for } s > t. \quad (2.9b)$$

**Remark 2.2** It is convenient to regard $K(t, s)$, $R(t, s)$ and $V(t, s)$ as all vanishing if $s > t$ (though not all writers do so).

We note that $\frac{\partial}{\partial s}V(t, s) = -R(t, s)$, $V(t, t) = I$. Further, $R(t, s)$ (like $K(t, s)$) has a possible jump at $s = t - \tau$ and $V(t, s) = I + \int_{t-\tau}^t R(t, \sigma)d\sigma + \int_t^s R(t, \sigma)d\sigma$ for $s < t - \tau$ while $V(t, s) = I + \int_s^t R(t, \sigma)d\sigma$ for $s \in [t - \tau, t]$. Thus, $V(t, s)$ is continuous for $0 \leq s \leq t$, $t \in [0, T]$.

With $g(t)$ given as above, we have

$$g(0) = \varphi(0), \quad g'(t) = f_2(t), \quad (2.10a)$$

$$f_2(t) = f(t) + B(t)\varphi(t - \tau), \quad \text{when } 0 \leq t \leq \tau, \text{ and } f_2(t) = f(t), \quad \text{when } t \in [\tau, T], \quad (2.10b)$$

and we therefore find (via the theory of integral equations) that, for $t \in [0, T]$,

$$w(t) = V(t, 0)\varphi(0) + \int_0^T V(t, s)B(s)\varphi(s - \tau)ds + \int_0^t V(t, s)f(s)ds. \quad (2.11)$$

In view of Remark 2.1, we have the following result.

**Theorem 2.1** There exists a function $V : [0, T] \times [0, T] \rightarrow \mathbb{R}^{m \times m}$, such that the solution $y(t)$ of (2.1) satisfies

$$y(t) = V(t, 0)\varphi(0) + \int_{-\tau}^0 V(t, s + \tau)B(s + \tau)\varphi(s)ds + \int_0^t V(t, s)f(s)ds \text{ for } t \in [0, T]. \quad (2.11)$$

We shall see later that $V(t, s)$ satisfies a delay differential equation.

---

$^1$Observe that $R(\cdot, \cdot)$ is normally termed the resolvent kernel for $K(\cdot, \cdot)$, but $V(\cdot, \cdot)$ (which is often called a fundamental solution) also has a claim to be termed a “resolvent” or a “solvent” kernel.
2.3 The fundamental solution \( X \) for the DDE

The preceding theorem is a variation of parameters result. We state a result that is expressed in terms of the usual definition of the fundamental solution.

**Definition 2.2** The function \( X : [0,T] \times [0,T] \to \mathbb{R}^{m \times m} \), defined for arbitrary \( T > 0 \) by the equations

\[
\frac{\partial}{\partial s} X(s,t) + X(s,t)A(s) + X(s + \tau, t)B(s + \tau) = 0 \quad (s < t, \ t \in [0,T]) \tag{2.12a}
\]

where, for \( 0 \leq t \leq T \),

\[
X(t,t) = I \tag{2.12b}
\]

and

\[
X(s,t) = 0 \quad (t < s \leq t + \tau) \tag{2.12c}
\]

is the fundamental solution for equation (2.1).

**Remark 2.3** We may extend the definition of \( X \) so that \( X(t,s) \) vanishes for \( 0 \leq t < s \leq T \).

In the above equations, \( \frac{\partial}{\partial s} X(s,t) \) is the left-hand derivative at \( s = t \). Clearly, the fundamental solution \( X(s,t) \) is continuous for \( 0 \leq s \leq t \leq T \) (see, for example, [2, p.363]).

**Remark 2.4** With the vector notation introduced in §1, the successive rows \( x_p^T(s,\sigma) \) of \( X(s,\sigma) \) can be denoted \( e^T_p X(s,\sigma) \). It is clear, from (2.12), that if we fix \( \sigma \) by assigning it a value \( \hat{\sigma} \) then the function \( s \mapsto x_p^T(s,\hat{\sigma}) \) satisfies the version of (2.5) that is obtained by replacing \( T \) by \( \sigma \), if we also set \( \psi^T(s) = 0 \) for \( s \in (\sigma, \hat{\sigma} + \tau) \) and \( \psi^T(\hat{\sigma}) = e^T_p \) (for every \( \hat{\sigma} \in [0,T] \)).

It is known from the literature on DDEs that that the solution \( y \) of (2.1) can be expressed in terms of the fundamental solution \( X \) and that the following result is true.

**Theorem 2.2** If \( X \) is the fundamental solution in Definition 2.2, the solution \( y \) of (2.1) is given for \( t \in [0,T] \) by

\[
y(t) = X(0,t) \varphi(0) + \int_0^t X(s + \tau, t)B(s + \tau) \varphi(s)ds + \int_0^t X(s,t)f(s)ds. \tag{2.13}
\]

For a proof, we may refer to Bellman & Cooke [1, pp.306-308], Halanay [2, p.361], Hale [3, p.150], and Lakshmikantham & Deo, [4, p.123]), but for completeness we shall provide a proof in §2.3.1 below.

It is clear on comparing the previous two theorems that (2.11) and (2.13) are effectively the same, the second being obtained on identifying \( V(t,s) \) in the first equation with \( X(s,t) \). Since \( \varphi(t) \) and \( f(t) \) are arbitrary we deduce

**Theorem 2.3** The function \( V \) in Theorem 2.1 is obtained from the fundamental solution \( X \) on setting \( V(t,s) = X(s,t) \) for \( 0 \leq s \leq t \leq T \).
Proof: Consider the cases where \( \phi(t) \equiv 0 \) for \( t \in [-\tau,0] \), and, for arbitrary \( \sigma \in [0,T] \), \( f(t) = 1 \) for \( t \in [0,\sigma] \), \( f(t) = 0 \) for \( t \in (\sigma,T] \). Then we find from the two representations of the solution, that for arbitrary \( \sigma \in [0,T] \)

\[
\int_0^\sigma \{ V(t,s) - X(s,t) \} ds = 0.
\]

From our earlier remarks, we know the integrand to be continuous in the region of integration. Differentiating with respect to \( \sigma \) gives \( V(t,\sigma) = X(\sigma,t) \) for arbitrary \( \sigma \in [0,T] \).

Remark 2.5 If we apply Theorem 2.2 to (2.4) we obtain from the expression for \( w(t) \) the result

\[
y(t) = X(0,t)\phi(0) + \int_0^t X(s,t)f(s)ds \quad (t \in [0,T])
\]

so that \( y(t) = X(0,t)\phi(0) + \int_0^\tau X(s,t)B(s)\phi(s-\tau)ds + \int_0^t X(s,t)f(s)ds \quad (t \in [0,T]) \). Hence (as expected), \( y(t) = X(0,t)\phi(0) + \int_{-\tau}^t X(s+\tau,t)B(s+\tau)\phi(s)ds + \int_0^t X(s,t)f(s)ds \quad (t \geq 0) \).

2.3.1 A more general result, that establishes Theorem 2.2

We can establish Theorem 2.2 as a consequence of a more general version (which is also known) that follows.

Theorem 2.4 Suppose that \( s \in [0,T] \) and

\[
\frac{dy(t)}{dt} - A(t)y(t) - B(t)y(t-\tau) = f(t), \quad \text{for } t \in [s,T],
\]

where \( y(t) \) is known for \( t \in [s-\tau,s] \). Suppose, further, that \( X \) satisfies equations (2.12). Then \( y(\cdot) \) is given by

\[
y(t) = X(s,t)y(s) + \int_{s-\tau}^s X(\mu+\tau,t)B(\mu+\tau)y(\mu)d\mu + \int_s^t X(\mu,t)f(\mu)d\mu.
\]

Proof: To prove this result we may introduce, with \( \sigma \in [0,T] \), the function

\[
u(t,\sigma) := X(\sigma,t)y(\sigma) + \int_{\sigma-\tau}^\sigma X(\mu+\tau,t)B(\mu+\tau)y(\mu)d\mu + \int_\sigma^t X(\mu,t)f(\mu)d\mu \quad (t \in [0,T])
\]

and establish that \( u(t,\sigma) \equiv y(t) \) for all \( \sigma \in [0,t] \). Clearly \( u(t,t) = y(t) \), and we can show, from equation (2.15), that the derivative \( \frac{\partial}{\partial \sigma} u(t,\sigma) \) with respect to \( \sigma \) vanishes; thus \( u(t,\sigma) \equiv y(t) \).

To provide the details needed in this proof, we observe that

\[
\frac{\partial}{\partial \sigma} u(t,\sigma) = \frac{\partial X(\sigma,t)}{\partial \sigma} y(\sigma) + X(\sigma,t) \frac{dy(\sigma)}{d\sigma} + X(\sigma+\tau,t)B(\sigma+\tau)y(\sigma) - X(\sigma,t)B(\sigma)y(\sigma-\tau) - X(\sigma,t)f(\sigma).
\]

However, \( y'(\sigma) - A(\sigma)y(\sigma) - B(\sigma)y(\sigma-\tau) = f(\sigma) \). We can therefore write (2.17) as

\[
\frac{\partial}{\partial \sigma} u(t,\sigma) = \frac{\partial X(\sigma,t)}{\partial \sigma} y(\sigma) + X(\sigma,t) \left( A(\sigma)y(\sigma) + B(\sigma)y(\sigma-\tau) + f(\sigma) \right) + X(\sigma+\tau,t)B(\sigma+\tau)y(\sigma) +
\]
\[-X(\sigma, t)B(\sigma)g(\sigma - \tau) - X(\sigma, t)f(\sigma) = \frac{\partial X(\sigma, t)}{\partial \sigma} y(\sigma) + X(\sigma, t)A(\sigma)g(\sigma) + X(\sigma + \tau, t)B(\sigma + \tau)g(\sigma)\]

or

\[\frac{\partial}{\partial \sigma} u(t, \sigma) = \left( \frac{\partial X(\sigma, t)}{\partial \sigma} + X(\sigma, t)A(\sigma) + X(\sigma + \tau, t)B(\sigma + \tau) \right) y(\sigma).\]

Since \(X(s, t)\) satisfies (2.12), we have

\[\frac{\partial}{\partial \sigma} u(t, \sigma) = 0\]

and the result follows.

If we set \(\sigma = 0\), and assign \(y(s) = \varphi(s)\) for \(s \in [-\tau, 0]\), then Theorem 2.2 follows immediately, from Theorem 2.4.

### 2.4 A generalized Green’s function \(Y\)

We shall explore yet another formulation, which has echoes of the Green’s function approach to ordinary differential equations\(^2\).

**Definition 2.3** Define the function \(Y(t, s)\) such that, for \(s, t \in [0, T]\),

\[
\frac{\partial Y(t, s)}{\partial t} - A(t)Y(t, s) - B(t)Y(t - \tau, s) = 0, \text{ for } t > s\]  \hspace{1cm} (2.18a)

\[Y(t, t) = I, \text{ and } Y(t, s) = 0 \text{ when } t < s.\]  \hspace{1cm} (2.18b)

We shall call \(Y(t, s)\) the generalized Green’s function for (2.1).

**Remark 2.6** Using the vector notation introduced in §1, the successive columns \(y_\lambda(t, \sigma)\) of \(Y(t, \sigma)\) can be denoted \(Y(t, \sigma)e_\lambda\), and it is clear from (2.18) that if we fix \(\sigma\) by assigning it a value \(\hat{\sigma}\) then the function \(t \mapsto y_\lambda(t, \hat{\sigma})\) satisfies the version of (2.2) obtained on setting \(\varphi(t) = 0\) for \(t \in [\hat{\sigma} - \tau, \hat{\sigma}]\) and \(\varphi(\hat{\sigma}) = e_\lambda\).

**Theorem 2.5** Suppose \(Y(t, s)\) to be defined by (2.18). Then the solution of the system (2.1) is given by

\[y(t) = Y(t, 0)\varphi(0) + \int_{-\tau}^{0} Y(t, s + \tau)B(s + \tau)\varphi(s)ds + \int_{0}^{t} Y(t, s)f(s)ds \quad (0 \leq t \leq T).\]  \hspace{1cm} (2.19)

Equation (2.19) is of the same form as (2.11).

### 2.4.1 A proof of Theorem 2.5

We shall prove Theorem 2.5 above, and to do so it is convenient to note that (2.18) implies, by elementary manipulation, that the following results hold for arbitrary \(f\) and \(\varphi\):

\[
\left[ \frac{\partial Y(t, 0)}{\partial t} - A(t)Y(t, 0) - B(t)Y(t - \tau, 0) \right] \varphi(0) = 0; \]  \hspace{1cm} (2.20a)

\[
\int_{-\tau}^{0} \left[ \frac{\partial Y(t, s + \tau)}{\partial t} - A(t)Y(t, s + \tau) - B(t)Y(t - \tau, s + \tau) \right] B(s + \tau)\varphi(s)ds = 0; \]  \hspace{1cm} (2.20b)

---

\(^2\)Green’s functions arise most commonly, but not exclusively, in the context of self-adjoint boundary value problems.
\[
\int_0^t \left[ \frac{\partial Y(t, s)}{\partial t} - A(t)Y(t, s) - B(t)Y(t - \tau, s) \right] f(s) ds = 0; \quad (2.20c)
\]

\[
\int_{t-\tau}^t B(t)Y(t - \tau, s) f(s) ds = 0. \quad (2.20d)
\]

Now, to establish Theorem 2.5, introduce the function

\[
u(t) := Y(t, 0)\varphi(0) + \int_0^{t-\tau} Y(t, s + \tau)B(s + \tau)\varphi(s) ds + \int_0^t Y(t, s) f(s) ds \quad (0 \leq t \leq T). \quad (2.21)
\]

We observe that \( \nu(0) = \phi(0) \) and we shall show that for \( t \in [0, \tau] \) we have \( \nu(t) = A(t)u(t) + B(t)\varphi(t - \tau) + f(t) \) while for \( t \in [\tau, T] \) we have \( \nu(t) = A(t)u(t) + B(t)u(t - \tau) + f(t) \). Then, by the uniqueness of the solution \( y(t) \) it follows that \( u(t) = y(t) \).

To provide the details of this argument, we first find the derivative of the function \( u(t) \) in (2.21). We consider separately \( t \in [0, \tau] \) and \( t \in [\tau, T] \). For \( t \in [0, \tau] \), we obtain, from (2.21) and taking into account (2.18b),

\[
u(t) = Y(t, 0)\varphi(0) + \int_{t-\tau}^0 Y(t, s + \tau)B(s + \tau)\varphi(s) ds + \int_0^t Y(t, s) f(s) ds \quad (0 \leq t \leq T).
\]

On differentiating, it follows that for \( t \in [0, \tau] \),

\[
\frac{du(t)}{dt} = \frac{\partial Y(t, 0)}{\partial t} \varphi(0) + \int_0^t \frac{\partial Y(t, s)}{\partial t}B(s + \tau)\varphi(s) ds + \left[ f(t) + \int_0^t \frac{\partial Y(t, s)}{\partial t} f(s) ds \right].
\]

(We have replaced \( Y(t, t) \) by the identity and replace \( s + \tau \) by \( s \) in the first integral.) For \( t \in [\tau, T] \) we have

\[
\frac{du(t)}{dt} = \frac{\partial Y(t, 0)}{\partial t} \varphi(0) + \int_{t-\tau}^0 \frac{\partial Y(t, s + \tau)}{\partial t}B(s + \tau)\varphi(s) ds + Y(t, t) f(t) + \int_0^t \frac{\partial Y(t, s)}{\partial t} f(s) ds.
\]

Further, \( A(t)u(t) \) and \( B(t)u(t - \tau) \) have the form

\[
A(t)u(t) = A(t)Y(t, 0)\varphi(0) + \int_{t-\tau}^0 A(t)Y(t, s + \tau)B(s + \tau)\varphi(s) ds + \int_0^t A(t)Y(t, s) f(s) ds \quad \text{for } t \geq 0; \quad (2.24)
\]

\[
B(t)u(t - \tau) = \begin{cases} B(t)\phi(t - \tau) \quad \text{for } t \in [0, \tau]; \\ B(t)Y(t - \tau, 0)\varphi(0) + \int_{t-\tau}^0 B(t)Y(t - \tau, s + \tau)B(s + \tau)\varphi(s) ds + \int_0^t B(t)Y(t - \tau, s) f(s) ds \quad \text{for } t \geq \tau.
\end{cases} \quad (2.25)
\]

Substitute (2.23), (2.24) and (2.25) into (a) the expression \( \nu'(t) = A(t)u(t) + B(t)\varphi(t - \tau) + f(t) \) for \( t \in [0, \tau] \) and into (b) the expression \( \nu'(t) = A(t)u(t) + B(t)u(t - \tau) + f(t) \) for \( t \in [\tau, T] \). On exploiting (2.20), we establish that \( u \) satisfies (2.1) and hence is the solution \( y \) as required.
2.5 The relationship between $V$, $X$ and $Y$

In this section we shall provide our own (direct) proof of the following result.

**Theorem 2.6** The functions $Y$ and $X$ defined respectively by (2.12) and (2.18) satisfy $X(t, s) = Y(s, t)$.

In order to establish this theorem, we consider a preliminary lemma.

Let us define for each $t$ the inner product (see Halanay, [2, p. 369] or Hale [3, p. 151])

$$ (u, v, t) = u^T(t)v(t) + \int_t^{t+\tau} u^T(\mu)B(\mu)v(\mu - \tau) d\mu. \tag{2.26} $$

We are concerned with this inner product when $v(t) = y(t)$, where $y(t)$ satisfies equation (2.1) for any $\varphi(t)$ and $u(t) = x(t)$, where $x^T(t)$ satisfies an adjoint equation associated with (2.1), namely,

$$ -\frac{dx^T(t)}{dt} - x^T(t)A(t) - x^T(t + \tau)B(t + \tau) = 0, \text{ for } t \in [0, T], \tag{2.27a} $$

subject to

$$ x^T(t) = \psi^T(t) \text{ for } t \in [T, T + \tau], \tag{2.27b} $$

and where $\psi^T(t)$ is an arbitrary initial function.

**Lemma 2.1** Suppose that $y$ is a solution of (2.2) and that $x^T$ is a solution of the adjoint problem (2.27). Then using the notation (2.26) $(x, y, t)$ is constant for $0 \leq t \leq T$.

The value of the constant in the above result depends upon $\varphi(t)$ and $\psi(t)$.

**Proof:** From equation (2.26) we have

$$ \frac{d(x, y, t)}{dt} = \frac{dx^T(t)}{dt}y(t) + x^T(t)\frac{dy(t)}{dt} + x^T(t + \tau)B(t + \tau)y(t) - x^T(t)B(t)y(t - \tau). $$

Therefore, we have

$$ \frac{d(x, y, t)}{dt} = \left( \frac{dx^T(t)}{dt} + x^T(t + \tau)B(t + \tau) \right) y(t) + x^T(t) \left( \frac{dy(t)}{dt} - B(t)y(t - \tau) \right). $$

Now if we add and subtract the term $x^T(t)A(t)y(t)$, we obtain

$$ \frac{d(x, y, t)}{dt} = \left( \frac{dx^T(t)}{dt} + x^T(t + \tau)B(t + \tau) \right) y(t) + x^T(t) \left( \frac{dy(t)}{dt} - B(t)y(t - \tau) \right) + x^T(t)A(t)y(t) - x^T(t)A(t)y(t) = $$

$$ = \left( \frac{dx^T(t)}{dt} + x^T(t + \tau)B(t + \tau) + x^T(t)A(t) \right) y(t) + x^T(t) \left( \frac{dy(t)}{dt} - B(t)y(t - \tau) - A(t)y(t) \right). $$

Since $y(t)$ and $x^T(t)$ are solutions of the homogeneous problems (2.2) and (2.27), we have $\frac{d(x, y, t)}{dt} = 0$ and $(x, y, t)$ is therefore independent of $t$. The lemma is therefore established.

Let us now consider the fundamental solution $Y(t, s)$ such that $Y(t, s) = 0$, when $s - \tau \leq t < s$, $Y(s, s) = I$. We write $y_\lambda(t, s) = Y(t, s)e_\lambda$ and take the value of $s$ to be $\hat{s}$. Then, for each $\hat{s} \in [0, T]$, $y_\lambda(t, \hat{s})$ satisfies the equation (2.2) as indicated in Remark 2.6. In a like manner, consider $X(s, t)$.

such that $X(s, t) = 0$, when $t < s \leq t + \tau$, $X(t, t) = I$. For fixed $\hat{t} \in [0, T]$, the successive rows $x^T_{\mu}(s, \hat{t}) = e^T_{\mu}X(s, \hat{t})$ of $X(s, \hat{t})$ satisfy an equation (2.5) as indicated in Remark 2.4. We now apply Lemma 2.1 to

$$(x_{\mu}(t, \hat{t}), y_{\lambda}(t, \hat{s}), t) = x^T_{\mu}(t, \hat{t})y_{\lambda}(t, \hat{s}) + \int_{t}^{t+\tau} x^T_{\mu}(\mu, \hat{t})B(\mu)y_{\lambda}(\mu - \tau, \hat{s})d\mu$$

and see that

$$X(t, \hat{t})Y(t, \hat{s}) + \int_{t}^{t+\tau} X(\mu, \hat{t})B(\mu)Y(\mu - \tau, \hat{s})d\mu$$

is independent of $t$,

so that we obtain the same value if we set $t = \hat{s}$ or $t = \hat{t}$. Hence

$$X(\hat{t}, \hat{t})Y(\hat{t}, \hat{s}) + \int_{t}^{t+\tau} X(\mu, \hat{t})B(\mu)Y(\mu - \tau, \hat{s})d\mu = X(\hat{s}, \hat{t})Y(\hat{s}, \hat{s}) + \int_{\hat{s}}^{\hat{s}+\tau} X(\mu, \hat{t})B(\mu)Y(\mu - \tau, \hat{s})d\mu$$

Taking into account $X(\hat{t}, \hat{t}) = I$ and $Y(\hat{s}, \hat{s}) = I$, we obtain $Y(\hat{t}, \hat{s}) = X(\hat{s}, \hat{t})$, when $\hat{t}, \hat{s} \in [0, T]$. This establishes Theorem 2.6.

The same type of argument (exploiting (2.11) in place of (2.19)) shows that $V(t, s) = X(s, t)$.

### 3 The discrete case

Let us now consider a discrete version of the equation (2.1). We have

$$\frac{\tilde{y}_{n+1} - \tilde{y}_n}{h} - A_n\tilde{y}_n - B_n\tilde{y}_{n-N} = f_n$$

for $n = 0, 1, \ldots, K - 1$, and the initial condition

$$\tilde{y}_n = \varphi_n \quad \text{(for } n = -N, \ldots, 0),$$

where

$$B_n = B(nh), \ A_n = A(nh), \ f_n = f(nh) \quad \text{and} \quad \varphi_n = \varphi(nh).$$

#### 3.1 Formulation as a problem in linear algebra

We may reformulate (3.1) in matrix-vector notation as

$$L\tilde{y} = g \quad \text{where} \quad y = [y_1^T, y_2^T, \ldots, y_K^T]^T,$$
in which the form of the augmented matrix \([L|g]\) expressed in terms of sub-matrices reads

\[
\begin{bmatrix}
I, & 0, & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & g_1 \\
I - hA_1, & I, & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & g_2 \\
0, & I - hA_2, & I, & \cdots & \cdots & \cdots & \cdots & 0 & g_3 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0, & 0, & \cdots & \cdots & I - hA_N, & I, & \cdots & \cdots & 0 & g_{N+1} \\
-hB_{N+1}, & 0, & \cdots & \cdots & I - hA_{N+1}, & I, & \cdots & 0 & h f_{N+1} \\
0, & -hB_{N+2}, & 0, & \cdots & I - hA_{N+2}, & I, & \cdots & 0 & h f_{N+1} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0, & \cdots & 0, & \cdots & 0, & -hB_{K-1}, & 0, \cdots, 0, & I - hA_{K-1}, & I & h f_{K-1} \\
\end{bmatrix}
\]

with

\[
\begin{align*}
g_1 &= h f_0 + (I + hA_0)\varphi(0) + h B_0 \varphi_{-N}, \\
g_2 &= h f_1 + h B_1 \varphi_{-N}, \\
g_3 &= h f_2 + h B_2 \varphi_{-N}, \\
& \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \vdots \\
g_{N+1} &= h f_N + h B_N \varphi_0.
\end{align*}
\]

We therefore have

\[
\begin{align*}
g_1 &= h f_2(0) + (I + hA_0)\varphi(0), \\
g_\ell &= h f_2((\ell - 1)h), \quad \text{for } \ell = 2, \ldots, K,
\end{align*}
\]

where the function \(f_2\) is defined in (2.4) and it follows that \(\tilde{y} = L^{-1} g\). Here, \(L\) being unit lower triangular (lower triangular with ones on the diagonal), \(M = L^{-1}\) is also unit lower triangular. We represent the \((\mu, \nu)\)-th submatrix of \(L\) by \(L_{\mu \nu}\), where \(L_{\mu \mu} = I\), \(L_{\mu \mu - 1} = I - h A_{\mu - 1}\), and where \(L_{\mu \mu - (N+1)} = -hB_{\mu - 1}\) when \(\mu \in \{N + 2, \ldots, K\}\) while \(L_{\mu \nu} = 0\) otherwise. We also represent the \((\mu, \nu)\)-th submatrix of \(M\) by \(M_{\mu \nu}\); the submatrices \(M_{\mu \nu}\) satisfy the equations \(M_{\mu \nu} = 0\) if \(\nu > \mu\) and, for \(\mu = 1, 2, \ldots, K\), \(\sum_{\nu=1}^{K} L_{\mu \nu} M_{\nu \nu} = \delta_{\mu I}\). Thus \(\tilde{y}_n = \sum_{\ell=1}^{n} M_{n\ell} g_\ell \) (\(n = 1, 2, \ldots, K\)) or

\[
\tilde{y}_n = M_{n1} \left\{ h f_0 + (I + hA_0)\varphi_0 + h B_0 \varphi_{-N} \right\} + \sum_{k=2}^{N+1} M_{nk} \left\{ h f_{k-1} + h B_{k-1} \varphi_{k-N-1} \right\} + h \sum_{k=N+2}^{n} M_{nk} f_{k-1}.
\]

For \(n = 1, 2, \ldots, N + 1\) some of the terms above vanish.

Rearranging (3.3) we can write (for \(n = 1, 2, \ldots, K\))

\[
\tilde{y}_n = \left[ M_{n1} (I + hA_0) \right] \varphi_0 + h \sum_{k=1}^{N+1} M_{nk} B_{k-1} \varphi_{k-N-1} + h \sum_{k=1}^{n} M_{nk} f_{k-1}
\]

or

\[
\tilde{y}_n = \left[ M_{n1} (I + hA_0) \right] \varphi_0 + h \sum_{k=1}^{0} M_{n,k+N+1} B_{k+N} \varphi_k + h \sum_{k=0}^{n-1} M_{nk+1} f_{k}
\]
where $M_{nn} = I$ and $M_{nk} = 0$ if $k > n$. This result will guide us.

### 3.2 Further linear algebra

Associated with the equations $Ly = g$ considered above are the algebraic equations

$$
\tilde{x}^T L = q^T
$$

where $L$ is the matrix defined in equation (3.2b) in §3.1. We shall define the right-hand side $q^T = [q_1^T, q_2^T, \ldots, q_K^T]$ by setting:

$$
\begin{align*}
q_1^T &= h\psi_1^T, \\
& \quad \vdots \\
q_{K-(N+1)}^T &= h\psi_{K-(N+1)}^T, \\
q_{K-N}^T &= h\psi_K^T B_K + h\psi_{K-N}^T, \\
& \quad \vdots \\
q_{K-1}^T &= h\psi_{K+N-1}^T B_{K+N-1} + h\psi_K^T, \\
q_K^T &= \psi_K^T (I + hA_K) + h\psi_{K+N}^T B_{K+N} + h\psi_K^T.
\end{align*}
$$

(3.6a)

We have $\tilde{x}^T = q^T L^{-1} = q^T M$. Since the submatrices $M_{\mu\nu}$ satisfy the equations $M_{\mu\nu} = 0$ if $\nu > \mu$ and $\sum_{\ell=1}^K L_{\mu\ell} M_{\ell\nu} = \delta_{\mu\nu} I$ we can write the solution of (3.5) in the form $\tilde{x}^T_n = \sum_{\ell=1}^K q^T \ell M_n$ or

$$
\begin{align*}
\tilde{x}^T_n &= \{\psi_K^T (I + hA_K) + h\psi_{K+N}^T B_{K+N} + h\psi_K^T\} M_{Kn} + \\
& \quad + h \sum_{j=K-N}^{K-1} \{\psi_{j+N}^T B_{j+N} + h\psi_j^T\} M_{jn} + h \sum_{j=n}^{K-N-1} p_j^T M_{jn}.
\end{align*}
$$

(3.7)

Rearranging (3.7) we can write (for $n = 1, 2, \ldots, K$)

$$
\tilde{x}^T_n = [\psi_K^T (I + hA_K)] M_{Kn} + h \sum_{j=K-N}^{K} \psi_{j+N}^T B_{j+N} M_{jn} + h \sum_{j=n}^{K} p_j^T M_{jn},
$$

or

$$
\tilde{x}^T_n = \psi_K^T [(I + hA_K)] M_{Kn} + h \sum_{j=K}^{K+N} \psi_j^T B_j M_{j-n,n} + h \sum_{j=n}^{K} p_j^T M_{jn},
$$

(3.8)

where $M_{nn} = I$ and $M_{jn} = 0$ if $n > j$ (so that some terms in (3.8) may vanish). This is an analogue of (3.4).

### 3.3 The discrete adjoint equation by linear algebra

If we discretize the nonhomogeneous adjoint equation (2.5) we can obtain an equation of the form

$$
\frac{\tilde{x}_{n}^T - \tilde{x}_{n-1}^T}{h} - \tilde{x}_{n}^T A_n - \tilde{x}_{n+N}^T B_{n+N} = p_n^T
$$

(3.9)

for $n = K, \ldots, 1$, where we assume the initial condition

$$
\tilde{x}_{n}^T = \psi_n^T \quad (\text{for } n = K, \ldots, K+N).
$$
If we define \( \tilde{z}_i = \tilde{z}_{i-1} \), \( i = 1, \ldots, K \), then we can write equation (3.9) in the form (3.6). From our preceding results, the solution of the equation (3.9) has the form

$$\tilde{z}_n^T = \psi_K^T [ (I + h A_K) ] M_{K,n+1} + h \sum_{j=K}^{K+N} \psi_j^T B_j M_{j-N,n+1} + h \sum_{j=n+1}^{K} p_j^T M_{j,n+1}, \quad n = 0, 1, \ldots, K - 1$$

(3.10)

Thus, we can formulate the following result.

**Lemma 3.1** There exists a function \( \tilde{W}(j,n) \), such that the solution of the system (3.9) is given by

$$\tilde{z}_n^T = \psi_K^T [ (I + h A_K) ] \tilde{W}(K,n) + h \sum_{j=K}^{K+N} \psi_j^T B_j \tilde{W}(j - N,n) + h \sum_{j=n+1}^{K} p_j^T \tilde{W}(j,n).$$

(3.11)

**Proof:** Equation (3.10) shows (3.11) to be true if we set

$$\tilde{W}(j,n) = M_{j,n+1}.$$

**Remark 3.1** Let us consider the sub-matrices \( M_{\mu\nu} \) of \( M = L^{-1} \) appearing in (3.3). From §3.1 we have \( M_{\mu\nu} = \hat{Y}(\mu,\nu - 1) \) and from Lemma 3.1 we have \( \tilde{W}(\mu,\nu - 1) = M_{\mu\nu} \). We therefore have \( \tilde{W}(i,j) = \hat{Y}(i,j) \) for \( i,j = 0, \ldots, K \).

With \( n = 0, 1, \ldots, K - 1 \), we can write equation (3.10) in the form

$$\tilde{z}_n = [M_{K,n+1}]^T [(I + h A_K)] \psi_K + h \sum_{j=K}^{K+N} [M_{j-N,n+1}]^T B_j \psi_j + h \sum_{j=n+1}^{K} [M_{j,n+1}]^T p_j.$$

Let us define \( \tilde{X}(n,j) = [M_{j,n+1}]^T \). Then

$$\tilde{z}_n = \tilde{X}(n,K) [(I + h A_K)] \psi_K + h \sum_{j=K}^{K+N} \tilde{X}(n,j - N) B_j \psi_j + h \sum_{j=n+1}^{K} \tilde{X}(n,j) p_j.$$

(3.12)

### 4 Parallels with the continuous case

Let us return to Theorem 2.5, which applies to the continuous case. If we consider its discrete analogues, we might be led to speculate the following: **There exists a function \( \tilde{Y}(n,j) \) such that the solution of the system (3.1) is given by \( \tilde{y}_n = \tilde{Y}(n,0) \varphi_0 + h \sum_{j=-N}^{0} \tilde{Y}(n,j + N) B_{j+N} \varphi_j + h \sum_{j=0}^{n-1} \tilde{Y}(n,j) f_j. \)** However, if we refer back to (3.4) we are led instead to the following modified result:

**Theorem 4.1** There exists a function \( \tilde{Y}(n,j), Y : \mathbb{Z}_+ \times \mathbb{Z}_+ \rightarrow \mathbb{R}^{m \times m} \) such that the solution of the system (3.1) is given by

$$\tilde{y}_n = \tilde{Y}(n,0) \{1 + h A_0 \} \varphi_0 + h \sum_{j=-N}^{0} \tilde{Y}(n,j + N) B_{j+N} \varphi_j + h \sum_{j=0}^{n-1} \tilde{Y}(n,j) f_j.$$

(4.1)
Equation (3.4) shows (4.1) to be true if we set
\[ \tilde{Y}(n,j) := M_{n,j+1} \text{ for } j = 0,1,\ldots,n-1. \]  
(4.2)

We can proceed in one of two ways to investigate (4.1) further; we can compute the submatrices \( \{M_{ij}\} \) from linear algebra, or we can obtain recurrence relations for \( \tilde{Y}(n,j) \) that parallel the differential equations for \( Y(t,s) \). We shall now pursue the second path.

From (4.1), we have the result
\[ \tilde{y}_{n+1} = \tilde{Y}(n+1,0) \{ I + hA_0 \} \varphi_0 + h \sum_{j=-N}^{0} \tilde{Y}(n+1,j+N)B_{j+N} \varphi_j + h \sum_{j=0}^{n} \tilde{Y}(n+1,j)f_j \]  
(4.3)

The terms \( A_n \tilde{y}_n \) and \( B_n \tilde{y}_{n-N} \) have the form
\[ A_n \tilde{y}_n = A_n \tilde{Y}(n,0) \{ I + hA_0 \} \varphi_0 + h \sum_{j=-N}^{0} A_n \tilde{Y}(n,j+N)B_{j+N} \varphi_j + h \sum_{j=0}^{n-1} A_n \tilde{Y}(n,j)f_j \]  
(4.4)
\[ B_n \tilde{y}_{n-N} = B_n \tilde{Y}(n-N,0) \{ I + hA_0 \} \varphi_0 + h \sum_{j=-N}^{0} B_n \tilde{Y}(n-N,j+N)B_{j+N} \varphi_j + h \sum_{j=0}^{n-N-1} B_n \tilde{Y}(n-N,j)f_j \]  
(4.5)

Thus, if we substitute (4.1), (4.3) and (4.4), (4.5) to (3.1) we obtain
\[
\begin{align*}
&h \sum_{j=-N}^{0} \left[ \frac{\tilde{Y}(n+1,j+N)}{h} - A_n \tilde{Y}(n,j) - B_n \tilde{Y}(n-N,j+N) \right] \cdot \varphi_j + \\
&h \sum_{j=0}^{n-1} \left[ \frac{\tilde{Y}(n+1,j)}{h} - A_n \tilde{Y}(n,j) - B_n \tilde{Y}(n-N,j) \right] \cdot f_j + \\
&\tilde{Y}(n+1,n)f_n + \sum_{j=n-N}^{n} B_n \tilde{Y}(n-N,j)f_j = f_n
\end{align*}
\]

This formula must be true for any \( \{\varphi_n\} \) and \( \{f_n\} \). Therefore we require
\[ \frac{\tilde{Y}(n+1,j+N)}{h} - A_n \tilde{Y}(n,j) - B_n \tilde{Y}(n-N,j+N) = 0, \text{ when } n > j, \]  
(4.6a)
and
\[ \tilde{Y}(n+1,n) = I, \]  
(4.6b)
\[ \tilde{Y}(n,j) = 0, \text{ when } n \leq j. \]  
(4.6c)

All the preceding steps are reversible; that is to say, if \( \tilde{Y}(\cdot,\cdot) \) satisfies (4.6) then the solution \( \{y_n\} \) can be expressed as in (4.1).

**Theorem 4.2** Suppose that \( \tilde{Y}(\cdot,\cdot) \) satisfies (4.6). Then the sequence \( \{\tilde{y}_n\}^K \) satisfying (3.1) is given by (4.1). If, for arbitrary \( \{f_n\} \) and \( \{\varphi_n\} \) the sequence \( \{\tilde{y}_n\}^K \) satisfying (3.1) is given by (4.1) then \( \tilde{Y}(\cdot,\cdot) \) satisfies (4.6).
Let us now obtain recurrence relations for $\tilde{X}(j, n)$. Using formula (3.12) for $\tilde{z}_n$ we repeat the method that we applied to $\tilde{y}_n$. Thus we obtain the following result

**Lemma 4.1** Suppose that $\tilde{X}(\cdot, \cdot)$ satisfies the equation

$$
\frac{\tilde{X}(n, j) - \tilde{X}(n-1, j)}{h} - \tilde{X}(n, j)A_n - \tilde{X}(n + N, j)B_{n+N} = 0, \quad \text{when } n < j,
$$

and

$$
\tilde{X}(n-1, n) = I, \quad \text{and } \tilde{X}(n, j) = 0, \quad \text{when } n \geq j.
$$

Then the sequence $\{\tilde{x}_n\}_1^K$ satisfying (3.9) is given by (3.12). If, for arbitrary $\{p_n\}$ and $\{\psi_n\}$ the sequence $\{\tilde{x}_n\}_1^K$ satisfying (3.9) is given by (3.12) then $\tilde{X}(\cdot, \cdot)$ satisfies (4.7).

### 4.1 The relationship between $\tilde{Y}$ and $\tilde{X}$

Let us consider a homogeneous version of the equation (3.1), where $f_n \equiv 0$ for $n = 0, K - 1$. Here $hN = \tau$ and $hK = T$.

The discrete analog of a formal adjoint equation (2.27) has the form

$$
\frac{\tilde{z}_n^T - \tilde{z}_{n-1}^T}{h} - \tilde{z}_n^T A_n - \tilde{z}_{n+1}^T B_{n+1} = 0,
$$

for $n = K, \ldots, 1$, and the initial condition

$$
\tilde{z}_n^T(n) = \psi^T(n) \quad \text{for } n = K, \ldots, K + N.
$$

Let us now consider a discrete analog of the inner product (2.26). We can write

$$
F_n \equiv F(\tilde{z}, \tilde{y}, n) = \tilde{z}_{n-1}^T \tilde{y}_n + h \sum_{j=n}^{n+N-1} \tilde{z}_j^T B_j \tilde{y}_{(j-N)},
$$

**Lemma 4.2** $F_{n+1} = F_n$ for $N \leq n \leq K$.

**Proof.** Let us consider $F_{n+1} - F_n$ we have

$$
F_{n+1} - F_n = \tilde{z}_{n-1}^T \tilde{y}_n - \tilde{z}_{n}^T \tilde{y}_{n+1} + h \tilde{z}_{n+1}^T B_{n+1} \tilde{y}_n - h \tilde{z}_n^T B_n \tilde{y}_{n-N}.
$$

Let us add and subtract following term $\tilde{z}_{n}^T \tilde{y}_{n+1} + h \tilde{z}_n^T A_n \tilde{y}_n$. Then we have

$$
F_{n+1} - F_n = \tilde{z}_{n}^T [\tilde{y}_{n+1} - \tilde{y}_n] + [\tilde{z}_{n}^T - \tilde{z}_{n-1}^T] \tilde{y}_n - \tilde{z}_n^T [h B_n \tilde{y}_{n-N} + h A_n \tilde{y}_n] + [h \tilde{z}_{n+1}^T B_{n+1} + h \tilde{z}_n^T A_n] \tilde{y}_n =
$$

$$
\tilde{z}_{n}^T [\tilde{y}_{n+1} - \tilde{y}_n - h B_n \tilde{y}_{n-N} - h A_n \tilde{y}_n] + [\tilde{z}_{n}^T - \tilde{z}_{n-1}^T + h \tilde{z}_{n+1}^T B_{n+1} + h \tilde{z}_n^T A_n] \tilde{y}_n.
$$

Thus, using equations (3.1) and (4.8) we obtain

$$
F_{n+1} - F_n = 0.
$$

Let us consider the matrix solution $\tilde{X}(s, n)$ of the equation (4.8) defined by condition

$$
\tilde{X}(s, n) = 0, \quad \text{when } s \geq n,
\quad \tilde{X}(n-1, n) = I.
$$

(4.10)
Since $F(\tilde{X}(s,n), \tilde{y}, n)$ is independent of $n$ we can write $F(\tilde{X}(n,n), \tilde{y}, n) = F(\tilde{X}(s,n), \tilde{y}, s)$. Thus we obtain

\[
\tilde{X}(n-1,n)\tilde{y}_n + h \sum_{j=n}^{n+N-1} \tilde{X}(j,n)B_j \tilde{y}_{j-N} = \tilde{X}(s-1,n)\tilde{y}_s + h \sum_{j=s}^{s+N-1} \tilde{X}(j,n)B_j \tilde{y}_{j-N}.
\]

Taking into account (4.10) we can write

\[
\tilde{y}_n = \tilde{X}(s-1,n)\tilde{y}_s + h \sum_{j=s}^{s+N-1} \tilde{X}(j,n)B_j \tilde{y}_{j-N}.
\]

Now, if $\tilde{Y}(n,s)$ is a fundamental solution for equation (3.1), then $\tilde{Y}(n,s)$ satisfies (4.6). Using (4.11), we obtain

\[
\tilde{Y}(n,s-1) = \tilde{X}(s-1,n)\tilde{Y}(s,s-1) + h \sum_{j=s}^{s+N-1} \tilde{X}(j,n)B_j \tilde{Y}(j-N,s-1)
\]

It follows that

\[
\tilde{Y}(n,s-1) = \tilde{X}(s-1,n), \text{ for } n \in \mathbb{N}, s \in \mathbb{N}.
\]

### 4.2 Linear algebraic equations revisited

Consider (with $\tilde{y}_{-N} = \tilde{\phi}(-Nh)$, $\tilde{y}_{-N} = \tilde{\phi}(h-Nh)$, ..., $\tilde{y}_0 = \tilde{\phi}(0)$) the relation

\[
\tilde{y}_{(r+1)} - \tilde{y}_r = hA_r \tilde{y}_n + hB_r \tilde{y}_{r-N} + hf_r \quad (r = 0,1,\ldots,K-1)
\]

deduced from that found in (3.1). Summing over $r$, we find the result

\[
\tilde{y}_{(n+1)} = \varphi(0) + h \sum_{r=0}^{n} f_r + h \sum_{r=0}^{n} A_r \tilde{y}_r + h \sum_{r=0}^{n} B_r \tilde{y}_{r-N}.
\]

The contribution $h \sum_{r=0}^{min(n,N)} B_r \tilde{y}_{r-N}$ to the last term $h \sum_{r=0}^{n} B_r \tilde{y}_{r-N}$ can be written $h \sum_{r=0}^{n} B_r \tilde{\phi}_{r-N}$, and, when $n > N$, we have

\[
h \sum_{r=0}^{n} B_r \tilde{y}_{r-N} = h \sum_{r=0}^{N} B_r \tilde{\phi}_{r-N} + h \sum_{r=N+1}^{n} B_r \tilde{y}_{r-N}.
\]

The equation (4.12) may be written

\[
\tilde{y}_{n+1} = \tilde{g}_{n+1} + h \sum_{r=1}^{n} K((n+1)h, r h) \tilde{y}_r,
\]

where

\[
\tilde{g}_{n+1} = (I + hA_0)\tilde{\phi}(0) + h \sum_{r=0}^{n} f_r(h).
\]

and

\[
K((n+1)h, rh) = \begin{cases} A_r + B_{r+N}, & r = 1,2,\ldots,n-N \\ A_r, & r = n-N+1,\ldots,n \end{cases}
\]

as in (2.7).

The equations (4.12) are discrete Volterra ‘summation’ equations. A discrete resolvent theory can be constructed for such equations (see for example [5, 6, 7], for example) to parallel the continuous version found in § 2.2.
References


A Appendix

Here we present some formulae as they appear in the literature, using the original notation but with a change of font.

A.1 The presentation of Bellman & Cooke

We consider the “variation of constants formula” of Bellman and Cooke [11], pp. 306-308. Let us consider a system of “differential-difference equations” having the form

\[ \sum_{n=0}^{m} \mathcal{A}_n(t) \varphi_n(t+h_n) + \sum_{n=0}^{m} \mathcal{B}_n(t) \psi_n(t+h_n) = \mathbf{w}(t), \]

Suppose that

\[ 0 = h_0 < h_1 < \ldots < h_m; \]

If \( \mathcal{A}_m(t) \) is nonsingular for \( t > t_0 \), whereas \( \mathcal{A}_0(t), \ldots, \mathcal{A}_{m-1}(t) \) are identically zero, then the system takes the form

\[ \varphi_n(t+h_n) + \sum_{n=0}^{m} \mathcal{B}_n(t) \psi_n(t+h_n) = \mathbf{w}(t), \quad t > t_0 \quad (A.1) \]

The adjoint equation and kernel \( \mathcal{X}(s, t) \) are in this case defined as follows by Bellman & Cooke:

**Definition A.1** Let \( \mathcal{X}(s, t) \) denote the unique matrix function, defined for \( t > t_0, t_0 \leq s \leq t+h_m \), which is continuous for \( t_0 \leq s \leq t \) and satisfies the initial condition \( \mathcal{X}(s, t) = 0, \quad t < s \leq t+h_m \) and \( \mathcal{X}(s, t) = I, \quad s = t \) and the adjoint equation

\[ -\frac{\partial \mathcal{X}(s, t)}{\partial s} + \sum_{n=0}^{m} \mathcal{X}(s+h_n-h_n, t) \mathcal{B}_n(s+h_m-h_n) = 0, \quad t > t_1, \quad t_0 < s < t. \]
Then, the basic theorem on the representation of solutions of (A.1) can be formulated as follows.

**Theorem A.1** Suppose that $\mathbf{w}(t)$ is a continuous vector function and $\mathbf{B}_n(t)$ a continuous matrix function $(n = 0, 1, \ldots, m)$ for $t > t_0$. Let $\mathcal{X}(s, t)$ denote the kernel matrix defined above. Then the unique continuous solution of equation (A.1) for $t > t_0$ which satisfies the initial condition

$$z(t) = 0, \quad t_0 \leq t \leq t_0 + h_m$$

is given by the formula

$$z(t + h_m) = \int_{t_0}^{t} \mathcal{X}(s, t)\mathbf{w}(s)ds, \quad t > t_0.$$

### A.2 The presentation of Halanay

Following Halanay [2, pp.359-362], let us consider the linear delay differential system

$$\mathbf{z}'(t) = \mathbf{A}(t)\mathbf{z}(t) + \mathbf{B}(t)\mathbf{z}(t - \tau) + \mathbf{f}(t) \quad (\tau > 0) \quad (A.2)$$

defined for $t \geq \sigma$ by an initial condition given in $[\sigma - \tau, \sigma]$ and let $\mathcal{X}(\alpha, t)$ be a matrix which satisfies system for $s < t$

$$\mathcal{Y}_s^\prime(s, t) = -\mathcal{Y}(s, t)\mathbf{A}(s) - \mathcal{Y}(s + \tau, t)\mathbf{B}(s + \tau),$$

(where $\mathcal{Y}_s^\prime(s, t)$ is the partial derivative with respect to $s$ of $\mathcal{Y}(s, t)$) and $\mathcal{Y}(t, t) = I$, $\mathcal{Y}(s, t) \equiv 0$ for $s > t$. Then, the solution of equation (A.2) can be expressed in the form

$$\mathbf{z}(t) = \mathcal{Y}(\sigma, t)\mathbf{z}(\sigma) + \int_{\sigma - \tau}^{\sigma} \mathcal{Y}(\alpha + \tau, t)\mathbf{B}(\alpha + \tau)\mathbf{z}(\alpha)\,d\alpha + \int_{\sigma}^{t} \mathcal{Y}(\alpha, t)\mathbf{f}(\alpha)\,d\alpha.$$

From this formula it is obvious that if $\mathcal{X}(t, \sigma)$ is the solution of system (A.2) with $\mathbf{f}(t) \equiv 0$ which verifies the condition $\mathcal{X}(\sigma, \sigma) = I$, $\mathcal{X}(t, \sigma) \equiv 0$ for $t < \sigma$, then $\mathcal{X}(t, \sigma) \equiv \mathcal{Y}(\sigma, t)$.

If $\mathbf{y}(t)$ is a solution of the system

$$\mathbf{y}'(t) = -\mathbf{y}(t)\mathbf{A}(t) - \mathbf{y}(t + \tau)\mathbf{B}(t + \tau) \quad (A.3)$$

defined for $t \leq \sigma$ by the initial conditions given in $[\sigma, \sigma + \tau]$. Then if $\mathcal{X}(\alpha, t)$ is a matrix of solutions of system (A.2) with $\mathbf{f}(t) \equiv 0$ with initial condition $\mathcal{X}(t, t) = I$, $\mathcal{X}(\alpha, t) \equiv 0$ for $\alpha < t$ the solution of the equation of the equation (A.3) has the form

$$\mathbf{y}(t) = \mathbf{y}(\sigma)\mathcal{X}(\sigma, t) + \int_{\sigma}^{\sigma + \tau} \mathbf{y}(s)\mathbf{B}(s)\mathcal{X}(s, t)\,ds.$$

### A.3 The presentation of Hale

We now refer to Hale [3, pp.150-151], who considers delay differential equations in the form

$$\mathbf{z}'(t) \equiv \mathcal{L}(t, x(t)) + h(t) = \sum_{k=1}^{N} \mathbf{A}_k(t)\mathbf{z}(t - \omega_k) + \int_{-r}^{0} \mathbf{A}(t, \theta)\mathbf{z}(t + \theta)\,d\theta + h(t), \quad 0 \leq \omega_k \leq r, \quad t \geq \sigma \quad (A.4)$$
with initial function \( \varphi \). Here each \( \mathfrak{A}_k(t), \mathfrak{A}(t, \theta) \) is continuous in \( t, \theta \).

The formal adjoint in this case is

\[
\frac{d\eta(s)}{ds} = - \sum_{k=1}^{N} \eta(s + \omega_k)\mathfrak{A}_k(s + \omega_k) - \int_{-\tau}^{0} \eta(s - \xi)\mathfrak{A}(s - \xi, \xi)d\xi \tag{A.5}
\]

and the representation formula is given as

\[
\eta(t) = \mathfrak{U}(t, \sigma)\varphi(0) + \sum_{k=1}^{N} \int_{\sigma - \omega_k}^{\omega} \mathfrak{U}(t, \alpha + \omega_k)\mathfrak{A}_k(\alpha + \omega_k)\varphi(\alpha - \sigma)d\alpha + \int_{\sigma}^{\sigma + \tau} \left[ \int_{\sigma - r}^{\sigma} \mathfrak{U}(t, s)\mathfrak{A}(s, \alpha - s)ds \right] \varphi(\alpha - \sigma)d\alpha.
\]

[see Hale, p.150: it appears that \( \varphi(0) \) should be replaced by \( \varphi(\sigma) \)] where \( \mathfrak{U}(t, s) \) satisfies the system

\[
\frac{\partial \mathfrak{U}(t, s)}{\partial t} = \mathfrak{L}(t, \mathfrak{U}_0(t, s)), \quad t \geq s
\]

with initial condition \( \mathfrak{U}(t, s) = 0 \), when \( s - r \leq t < s \), \( \mathfrak{U}(s, s) = I \).

### A.4 The presentation of Lakshmikantham and Deo

Lakshmikantham and Deo [4, p.110-111], consider the linear delay differential equation in the form

\[
\eta'(t) = \mathfrak{A}(t)\eta(t) + \mathfrak{B}(t)\eta(t - \tau). \tag{A.6}
\]

The related perturbed system is of the form

\[
\eta'(t) = \mathfrak{A}(t)\eta(t) + \mathfrak{B}(t)\eta(t - \tau) + \mathfrak{H}(t). \tag{A.7}
\]

The formal adjoint system for (A.6) is given by

\[
\eta'(s) = -\mathfrak{A}^T(s)\eta^T(s) - \mathfrak{B}^T(s + \tau)\eta^T(s + \tau). \tag{A.8}
\]

Let \( \mathfrak{Z}(s, t) \) an \( m \times m \) continuous matrix function, be such that

\[
\mathfrak{Z}(s, t) = 0, \quad t < s \leq t + \tau
\]

\[
\mathfrak{Z}(t, t) = I. \tag{A.9}
\]

Then the authors formulate following result.

**Theorem A.2** Consider the difference-differential equation (A.7) where \( \mathfrak{A}(t) \) and \( \mathfrak{B}(t) \) are \( m \times m \) continuous matrices on \( \mathbb{R}_+ \) and \( \mathfrak{H} \in C[\mathbb{R}_+, \mathbb{R}^n] \). Let \( \mathfrak{Z}(s, t) \) be an \( m \times m \) continuous matrix function satisfying (A.9) which is the fundamental solution of (A.8). Then the solution of (A.7) is given by formula

\[
\eta(t) = \mathfrak{Z}(t_0, t)\varphi_0(t_0) + \int_{t_0 - \tau}^{t_0} \mathfrak{Z}(s + \tau, t)\mathfrak{B}(s + \tau)\varphi_0(s)ds + \int_{t_0}^{t} \mathfrak{Z}(s, t)\mathfrak{H}(s)ds.
\]