Notes on XXX

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1 Introduction

It can be shown that

$$\widehat{C} = AB + \Delta C, \qquad |C| \le nu|A||B| + O(u^2). \tag{1}$$

Proving equation (1) is left as an exercise for the reader. For details on flaoting point arithmetic see [1].

If $D = \mathrm{diag}(d_i)$ then $D^{-1} = \mathrm{diag}(d_i^{-1})$.

In general, we have

$$\min_{E\in\mathcal{E}} \|E\|_F = \min\left\{ \frac{\|r\|_2}{\|x\|_2}, \sigma_{\min}([A,\ R]) \right\}.$$

This modified formula requires the SVD of an $m \times 2m$ matrix instead of an eigendecomposition of a symmetric $m \times m$ matrix. We note that if an SVD of A is available, it can be exploited to reduce the cost of evaluating or estimating $\sigma_{\min}([A,\ R])$, but we will not go into the details.

Lemma 1.1 *Let $A \in \mathbb{C}^{m\times n}$ $(m \ge n)$ have the polar decomposition $A = UH$. Then*

$$\frac{\|A^*A - I\|}{\|A\|_2 + 1} \le \|A - U\| \le \|A^*A - I\|,$$

for any unitarily invariant norm $\|\cdot\|$.

Proof. Exercise. □

Lemma 1.1 relates $A - U$ to the departure of the columns of A from orthogonality.

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1.1 Basic Linear Algebra Definitions

$\mathbb{R}^{m\times n}$ denotes the vector space of all real $m \times n$ matrices.

$$A \in \mathbb{R}^{m\times n} \qquad \Longleftrightarrow \qquad A = (a_{ij}) = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}.$$

$\mathbb{R}^n$ denotes the vector space of real n-vectors. Generally, we use

capital letters	A, B, C, Δ, Λ	for matrices,
lower case letters	$a_{ij}, b_{ij}, c_{ij}, \delta_{ij}, \lambda_{ij}$	for matrix elements,
lower case letters	x, y, z, c, g, h	for vectors,
lower case Greek letters	$\alpha, \beta, \gamma, \theta, \pi$	for scalars.

Basic matrix operations include:

$$\begin{aligned} C = A + B &\quad\Longleftrightarrow\quad c_{ij} = a_{ij} + b_{ij} \\ C = \alpha A &\quad\Longleftrightarrow\quad c_{ij} = \alpha a_{ij} \\ \underbrace{C}_{m\times p} = \underbrace{A}_{m\times n}\underbrace{B}_{n\times p} &\quad\Longleftrightarrow\quad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \\ C = A^T &\quad\Longleftrightarrow\quad c_{ij} = a_{ji} \quad (A \in \mathbb{R}^{n\times n}) \\ C = A^* = A^H &\quad\Longleftrightarrow\quad c_{ij} = \overline{a_{ji}} \quad (A \in \mathbb{C}^{n\times n}) \end{aligned}$$

The matrix

$$A = \begin{bmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{bmatrix} \qquad \lambda \neq (2k+1)\pi,$$

has the principal logarithm

$$\log(A) = \begin{bmatrix} 0 & -\lambda \\ \lambda & 0 \end{bmatrix}.$$

If $A, B \in \mathbb{R}^{n\times n}$ satisfy $AB = I$ then B is the *inverse* of A, written $B = A^{-1}$. If A^{-1} exists A is *nonsingular*; otherwise A is *singular*.

2 Algorithm

We now state our algorithm.

Algorithm 1 ($F = \mathbf{F}(t, A, B, \mathbf{balance})$) *Given $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times n_0}$, $t \in \mathbb{C}$, and a tolerance* tol, *this algorithm produces an approximation $F \approx e^{tA}B$. The logical variable* balance *indicates whether or not to apply balancing.*

if balance
  $\widetilde{A} = D^{-1}AD$
  if $\|\widetilde{A}\|_1 < \|A\|_1$, $A = \widetilde{A}$, $B = D^{-1}B$, else balance = false, end
end
$\mu = \text{trace}(A)/n$

6 A very long line, a very long line,
a very long line, a very long line. Continuation is not numbered.
7 $A = A - \mu I$
8 if $t\|A\|_1 = 0$
9 $\quad m_* = 0,\ s = 1$ % The case $tA = 0$.
10 else
11 $\quad [m_*, s] = \mathrm{parameters}(tA, \mathrm{tol})$
12 end
13 $F = B,\ \eta = e^{t\mu/s}$
14 for $i = 1\colon s$
15 $\qquad c_1 = \|B\|_\infty$
16 $\qquad$ for $j = 1\colon m_*$
17 $\qquad\qquad B = tAB/(sj),\ c_2 = \|B\|_\infty$
18 $\qquad\qquad F = F + B$
19 $\qquad\qquad$ if $c_1 + c_2 \le \mathrm{tol}\|F\|_\infty$, break, end
20 $\qquad\qquad c_1 = c_2$
21 $\qquad$ end
22 $\qquad F = \eta F,\ B = F$
23 end
24 if balance, $F = DF$, end

Line 7 shifts by the mean of the eigenvalues. Line 19 tests for convergence.

References

[1] Nicholas J. Higham. *Accuracy and Stability of Numerical Algorithms*. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, second edition, 2002.