# Weyl modules and the mod 2 Steenrod algebra

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**Abstract**

This paper continues our study of the action of the mod 2 Steenrod algebra $\mathcal{A}_2$ on the polynomial algebra $P(n) = \mathbb{F}_2[t_1, \ldots, t_n]$. We obtain further partial results on the 'hit problem' of F. P. Peterson, which asks for a minimal generating set for $P(n)$ as an $\mathcal{A}_2$-module. We also study the structure of the quotient by the 'hit elements' as a graded representation of the finite general linear group $\mathbf{G}(n) = GL(n, \mathbb{F}_2)$, i.e. as a module over the finite group algebra $\mathbb{F}_2\mathbf{G}(n)$. These results were obtained in [29] for the special case of the Steinberg module for $\mathbf{G}(n)$.

By extending the scalars to $\overline{\mathbb{F}}_2$, the algebraic closure of $\mathbb{F}_2$, we obtain commuting actions of $\mathcal{A}_2$ and $\mathbf{G}(n)$ on $\mathbf{P}(n) = \overline{\mathbb{F}}_2[t_1, \ldots, t_n]$. While this makes no essential difference to the representation theory of $\mathbf{G}(n)$ or to the hit problem, it allows us to treat the action of $\mathbf{G}(n)$ on $\mathbf{P}(n)$ as the restriction of that of the algebraic group $\overline{\mathbf{G}}(n) = GL(n, \overline{\mathbb{F}}_2)$. In particular, we make use of tilting modules for $\overline{\mathbf{G}}(n)$ to show that for every irreducible representation $\mathbf{L}(\lambda)$ of $\mathbf{G}(n)$, a minimal set of $\mathcal{A}_2$-generators of $\mathbf{P}(n)$ must contain a copy of the corresponding dual Weyl module $\nabla(\lambda)$.

## 1 Introduction

Our main objective is to study the action of the mod 2 Steenrod algebra $\mathcal{A}_2$ on the polynomial algebra $P(n) = \mathbb{F}_2[t_1, \ldots, t_n]$. The 'hit problem' of F. P. Peterson asks for a minimal generating set for $P(n)$ as an $\mathcal{A}_2$-module, i.e. a basis for the quotient space $Q(n) = P(n)/\mathcal{A}_2^+ P(n)$, where $\mathcal{A}_2^+$ denotes the positively graded part of $\mathcal{A}_2$. We study the structure of $Q(n)$ as a graded representation of the finite general linear group $\mathbf{G}(n) = GL(n, \mathbb{F}_2)$, i.e. as a module over the finite group algebra $\mathbb{F}_2\mathbf{G}(n)$.

The topological applications of this problem are based on the isomorphism $P(n) \cong H^*(BV)$, the mod 2 cohomology of the classifying space of a vector space $V$ of dimension $n$ over $\mathbb{F}_2$ with basis $u_1, \ldots, u_n$. There is a dual homology version:

$H_*(BV) \cong D(n)$, a divided power algebra over $\mathbb{F}_2$ in $n$ variables. By identifying $H_1(BV)$ with $V$ and $H^1(BV)$ with $V^* = \mathrm{Hom}(V, \mathbb{F}_2)$, we regard the generators $t_1, \ldots, t_n$ of $P(n)$ as the basis of $V^*$ dual to $u_1, \ldots, u_n$. The vector space $P^d(n)$ of homogeneous polynomials of degree $d$ in $P(n)$ is thus identified with $H^d(BV)$ and its dual $D^d(n)$ with $H_d(BV)$: for each monomial $m$ in $P(n)$ we thus have a dual divided monomial $m^*$ in $D(n)$, defined with respect to the basis of monomials in $P(n)$. As an algebra over $\mathbb{F}_2$, $D(n)$ is the exterior algebra with $n$ generators $u_1^{(2^s)}, \ldots, u_n^{(2^s)}$ for $s \geq 0$, where $m^* = u_i^{(2^s)}$ is dual to $m = t_i^{2^s}$. The dual action of $Sq^k \in \mathcal{A}_2$ on $D(n)$ lowers degree by $k$, and the dual of the hit problem is to determine the kernel $K(n)$ of this dual action, i.e. the elements of $D(n)$ which are annihilated by all elements of $\mathcal{A}_2^+$. Because of the Cartan formula, $K(n)$ is a subalgebra of $D(n)$. This extra structure has been exploited in [2, 7, 24], but it does not appear to make the dual problem easier than the original one. Guided by representation theory, we take the point of view that the two versions should be studied side by side. Both versions of the problem have a substantial history, and some progress has been made on their generalization to odd primes [6, 20].

Much of the difficulty in applying representation theory to the hit problem is that the coefficient field $\mathbb{F}_2$ is too small to allow the application of general results from the modular theory for finite groups. However, as a finite group of Lie type, success in the study of modular representations of $\mathbf{G}(n)$ in the defining characteristic 2 has been achieved mainly by restriction of representations of the corresponding algebraic group $\overline{\mathbf{G}}(n) = GL(n, \overline{\mathbb{F}}_2)$, where $\overline{\mathbb{F}}_2$ is the algebraic closure of $\mathbb{F}_2$ [15]. Extension of the coefficients to $\overline{\mathbb{F}}_2$ makes no essential difference to the representation theory of $\mathbf{G}(n)$, since $\mathbb{F}_2$ is a splitting field for $\mathbf{G}(n)$ [15, Section 5.2]: this means that for every simple $\mathbb{F}_2\mathbf{G}(n)$-module $M$ its extension $\overline{\mathbb{F}}_2 \otimes_{\mathbb{F}_2} M$ remains simple. We thus consider the right action of $\overline{\mathbf{G}}(n)$ on $\mathbf{P}(n) = \overline{\mathbb{F}}_2[t_1, \ldots, t_n]$ by linear substitutions of the variables, and regard the restriction of this action to the finite group $\mathbf{G}(n)$ as the extension to the coefficient field $\overline{\mathbb{F}}_2$ of the action we wish to study over $\mathbb{F}_2$.

It is straightforward to extend the action of the Steenrod algebra $\mathcal{A}_2$ to $\mathbf{P}(n)$, so that $Sq^k(\alpha f) = \alpha Sq^k(f)$ for $\alpha \in \overline{\mathbb{F}}_2$ and $f \in P(n)$. Thus we obtain commuting actions of $\mathcal{A}_2$ and $\mathbf{G}(n)$ on $\mathbf{P}(n)$, and it is in this context that we work throughout this article. It is important to note that this action of $\mathcal{A}_2$ does *not* commute with the action of $\overline{\mathbf{G}}(n)$ on $\mathbf{P}(n)$. Since the number of variables $n$ will be fixed throughout, we write $\mathbf{G}$, $\mathbf{P}$, $\mathbf{D}$ etc. for $\mathbf{G}(n)$, $\mathbf{P}(n)$, $\mathbf{D}(n)$ etc. from now on.

There are $2^{n-1}$ inequivalent simple $\overline{\mathbb{F}}_2\mathbf{G}$-modules $\mathbf{L}(\lambda)$, for column 2-regular partitions $\lambda$ of length $\leq n-1$ [15, Section 2.11], and each $\mathbf{L}(\lambda)$ occurs as a composition factor in $\mathbf{P}^d$ for infinitely many degrees $d$. Here a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a sequence of integers with $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$. The **modulus** of $\lambda$ is $|\lambda| = \sum_{i=1}^n \lambda_i$ and its **length** is the number of $i$ such that $\lambda_i \neq 0$. The **diagram** of $\lambda$ is the set $\{(i,j) \mid 1 \leq j \leq \lambda_i\}$, using matrix conventions, so that $(i,j)$ refers to row $i$ and column $j$. As usual, we think of the $(i,j)$'s as 'boxes' which may be filled by integers to give Young tableaux [12]. If no two columns have equal length, then $\lambda$ is **column 2-regular**: equivalently, the conjugate partition $\omega = \lambda'$ is strictly decreasing.

The first occurrence of $\mathbf{L}(\lambda)$ is in degree $d(\lambda) = \sum_{i=1}^{n}(2^{\lambda_i} - 1)$ [4]. Since $\mathcal{A}_2$ acts on $\mathbf{P}$ by maps of $\overline{\mathbb{F}}_2\mathbf{G}$-modules, this composition factor in degree $d(\lambda)$ cannot be hit. The representation $\mathbf{L}(\lambda)$ occurs with multiplicity 1 in degree $d(\lambda)$, and the monomial $s(\lambda) = t_1^{2^{\lambda_1}-1} \cdots t_n^{2^{\lambda_n}-1}$ generates this composition factor. Monomials such as $s(\lambda)$, with all exponents of the form $2^k - 1$, are called **spikes** following Singer [28]: a monomial $m$ can not appear in any polynomial in $\mathcal{A}_2^+\mathbf{P}$ if and only if $m$ is a spike. Thus a monomial basis for $\mathbf{Q} = \mathbf{P}/\mathcal{A}_2^+\mathbf{P}$ must include all the spikes. For the dual action of $\mathcal{A}_2$ on $\mathbf{D}$, the corresponding divided monomials $s^*(\lambda)$ are linearly independent elements of the kernel $\mathbf{K} = \{h \in \mathbf{D} \mid \mathcal{A}_2^+(h) = 0\}$.

In the case of one variable $t$, it is easy to prove that the spikes $t^{2^k-1}$ for $k \geq 0$ are a basis for $\mathbf{Q}$. For $n = 2$, Peterson showed [23] that $\mathbf{Q}^d \neq 0$ if and only if $\mathbf{P}^d$ contains a spike, and conjectured that this is true for all $n$. Peterson's conjecture was proved in [32] by use of the **$\chi$-trick**, which uses the Cartan formula and the properties of the canonical anti-automorphism $\chi$ of $\mathcal{A}_2$ to show that for $u, v \in \mathbf{P}$, $u \cdot Sq^k(v) \equiv \chi(Sq^k)u \cdot v \mod \mathcal{A}_2^+\mathbf{P}$. This elementary lemma has proved to be central to nearly all significant progress on the hit problem.

In his 1990 Ph.D. dissertation, Masaki Kameko solved the hit problem for $n = 3$ [17], and showed in particular that $\mathbf{Q}(3)$ has a basis of monomials which are related to spikes by having the same **$\omega$-vector** (see Definition 2.1). We shall see in Section 2 that this result has a natural interpretation in terms of the structure of $\mathbf{P}$ as a $\overline{\mathbf{G}}$-module. For arbitrary $n$, we define a filtration on $\mathbf{Q}^d$ with quotients $\mathbf{Q}^\omega$ by ordering the $\omega$-vectors $\omega = (\omega_1, \ldots, \omega_r)$ of degree $d$.

A weakly decreasing $\omega$ can be regarded as a partition, but it is necessary to include cases where $\omega$ is non-decreasing. For a partition $\lambda$, the $\omega$-vector of the spike $s(\lambda)$ is the conjugate partition $\omega = \lambda'$. Thus for $n \leq 3$, Kameko's work shows that $\mathbf{Q}^\omega \neq 0$ if and only if $\omega$ is weakly decreasing. However, this is known to be false for $n \geq 4$: in particular, a preprint of Kameko [18] asserts that for $n = 4$, with the left lexicographic order on $\omega$-vectors, there are exactly three non-decreasing $\omega$ with $\mathbf{Q}^\omega \neq 0$. These are $(1,3)$, $(2,3)$ and $(2,3,2)$, with degrees 7, 8 and 16 and with $\mathbf{Q}^\omega$ having dimension $1, 4, 4$ respectively. What we have to say about $\mathbf{Q}^\omega$ is almost entirely confined to the case where $\omega$ is weakly decreasing.

Thus we approach the hit problem one $\omega$-vector at a time. More precisely, we have a vector space isomorphism $\mathbf{Q}^d \cong \sum_\omega \mathbf{Q}^\omega$, where the sum is taken over all $\omega$-vectors of degree $d$. Moreover, $\mathbf{Q}^d$ and $\sum_\omega \mathbf{Q}^\omega$ have the same $\overline{\mathbb{F}}_2\mathbf{G}$-composition factors. We emphasise that this decomposition of $\mathbf{Q}^d$ depends on the choice of ordering on $\omega$-vectors.

Our results are of two kinds, 'lower bounds' and 'upper bounds' for $\mathbf{Q}^\omega$. It is important to note that the definition of $\mathbf{Q}^\omega$ depends on the ordering chosen for the whole set of $\omega$-vectors in the same degree. The 'lower bound' result (Theorem

4.3) which follows is valid for any such ordering which refines the partial ordering by dominance (see Definition 2.2).

**Theorem 1.1** *Let $\lambda$ be a column 2-regular partition of length $\leq n-1$, with associated Weyl module $\Delta(\lambda)$ and dual Weyl module $\nabla(\lambda)$ for $\mathbf{G} = GL(n, \mathbb{F}_2)$, and let $\omega = \lambda'$. Then $\mathbf{Q}^\omega$ has a quotient module isomorphic to $\nabla(\lambda)$, and $\mathbf{K}^\omega$ has a submodule isomorphic to $\Delta(\lambda)$.*

In contrast, the following 'upper bound' result (Theorem 5.2) requires the use of an ordering which gives preference to the chosen $\omega$.

**Theorem 1.2** *If $\omega$ is weakly decreasing, then the $\omega$-vectors with the same degree can be ordered so that $\mathbf{Q}^\omega$ is spanned by monomials which correspond to semi-standard Young tableaux with diagram $\lambda = \omega'$ and content $\{1, 2, \ldots, n\}$.*

The correspondence between monomials and Young tableaux is explained in Section 5, and the conditions on the ordering are made precise in Theorem 5.2. The modules $\Delta(\lambda)$ and $\nabla(\lambda)$ have dimension $\delta(\lambda)$, the number of semi-standard Young tableaux with diagram $\lambda$: this is given by the 'hook formula' (Section 5). We may combine these results as follows (Theorem 5.7).

**Theorem 1.3** *If $\omega$ is strictly decreasing, with conjugate $\lambda$, then the $\omega$-vectors with the same degree can be ordered so that $\mathbf{Q}^\omega \cong \nabla(\lambda)$, with a basis given by monomials which correspond to semi-standard Young tableaux with diagram $\lambda = \omega'$ and content $\{1, 2, \ldots, n\}$, and $\mathbf{K}^\omega \cong \Delta(\lambda)$, with generator the dual spike $s^*(\lambda)$. In particular, $\dim \mathbf{Q}^\omega = \dim \mathbf{K}^\omega = \delta(\lambda)$, the hook number.*

Theorem 1.3 generalizes the main result of [29], which applies when there is only one **spike-type** or weakly decreasing $\omega$-vector $\omega$ in degree $d$. In this case, $\mathbf{Q}^\omega = \mathbf{Q}^d$.

In the remainder of Section 1, we seek to put these results in context. Singer [28] began the study of the spike types arising in a given degree, or equivalently the possible decompositions of an integer $d$ as a sum

$$d = (2^{\lambda_1} - 1) + (2^{\lambda_2} - 1) + \ldots + (2^{\lambda_l} - 1) \qquad (1)$$

where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l > 0$ and $l \leq n$. For $d \geq 0$ we write $\mu(d)$ for the minimum length $l$ of $\lambda$, so that in the $n$-variable case $\mathbf{P}^d$ contains a spike if and only if $\mu(d) \leq n$. Thus $\mu(0) = 0$ and $\mu(2^k - 1) = 1$ for $k \geq 1$.

The 'greedy algorithm', which chooses successively for $i = 1, 2, 3, \ldots$ the largest possible exponent $\lambda_i$, yields a particular $\lambda = \lambda^{\min}$ of minimum length $\mu(d)$. A spike with exponents $2^{\lambda_i} - 1$ for $1 \leq i \leq \mu(d)$ is called a **minimal spike**. For example when $d = 27$, $\lambda^{\min} = (4, 3, 2, 1, 1)$ and $t_1^{15} t_2^7 t_3^3 t_4 t_5$ is a minimal spike: the spike $t_1^7 t_2^7 t_3^7 t_4^3 t_5^3$ with $\lambda = (3, 3, 3, 2, 2)$ also has length $\mu(27) = 5$, but is not minimal. The greedy algorithm gives a decomposition of $d$ as a sum (1) where only the last two may be equal, as in the case $27 = 15+7+3+1+1$. This property characterises the minimal spike type uniquely [28].

Let $\omega^{\min}$ be the partition conjugate to $\lambda^{\min}$, i.e. the $\omega$-vector of a minimal spike. Then $\omega_i^{\min} - \omega_{i+1}^{\min} \leq 2$ for $i \geq 1$, and if $\omega_k^{\min} - \omega_{k+1}^{\min} = 2$ then $\omega_i^{\min} = \omega_{i+1}^{\min}$ for all $i < k$. In other words, a difference of 2 between successive parts of $\omega^{\min}$ can occur only once, and if it does occur, there can be no previous difference of 1. For example when $d = 27$, $\omega^{\min} = (5,3,2,1)$. This property characterises $\omega^{\min}$ uniquely. In particular, $\omega_1^{\min} = \mu(d)$. Writing $d' = (d-\mu(d))/2$, the properties $\mu(d) \equiv d \bmod 2$ and $\mu(d') \leq \mu(d)$ of $\mu$ give a 'greedy algorithm' to construct $\omega^{\min}$, i.e. $\omega^{\min}(d) = (\mu(d), \omega^{\min}(d'))$.

The $\omega$-vector $\omega^{\min}$ is minimal in the dominance partial order (see Definition 2.2) in $\mathbf{P}^d$ among all weakly decreasing $\omega$-vectors (cf. [25, p.96]). To see this, note that by the remarks above any weakly decreasing $\omega$-vector other than $\omega^{\min}$ has either a difference $\geq 3$ between successive parts, or a difference $\geq 1$ with a subsequent difference of 2. In the first case, $\omega$ immediately dominates the $\omega$-vector of another spike-type, e.g. $(4,1,1) \succeq (2,2,1)$. The second case can be reduced to the first by a sequence of dominance steps (see Lemma 2.3) which move the difference $\geq 1$ to the right, e.g. $(4,3,3,1) \succeq (2,4,3,1) \succeq (2,2,4,1)$.

The following result was conjectured by Silverman and Singer in [25, Conjecture 1.2] and was proved in [26, Theorem 1.2].

**Theorem** (D. M. Meyer, J .H. Silverman) *Let $P$ be a polynomial of the form $E \cdot F^{2^k}$ for some polynomials $E$ and $F$ of degrees $e$ and $f$ respectively, and suppose that $e < (2^k - 1)\mu(f)$. Then $P$ is hit.*

In fact the conjecture of Silverman and Singer was stated in two forms, which they proved to be equivalent. Thus [25, Conjecture 1.1] can be stated as the theorem which follows.

**Theorem** (D. M. Meyer, J .H. Silverman, W. M. Singer) *Let $m$ be a monomial in $\mathbf{P}^d$ such that $\omega(m)$ does not dominate $\omega^{\min}$. Then $m$ is hit.*

This result strengthens the earlier result of Singer [28, Theorem 1.2], which states that the monomial $m$ is hit when $\omega(m) <_l \omega^{\min}$.

Given $d \geq 0$ and $n \geq \mu(d)$, the 'greedy algorithm', which chooses successively for $i = 1,2,3\ldots$ the *largest* possible $\omega_i$ such that $\omega_i \leq \omega_{i-1}$ for $i > 1$ yields a particular weakly decreasing $\omega$-vector $\omega^{\max}$. If $\lambda$ is the conjugate partition to $\omega^{\max}$, a spike with exponents $2^{\lambda_i} - 1$ is called a **maximal spike**. For example when $d = 27$ and $n = 5$, $t_1^7 t_2^7 t_3^7 t_4^3 t_5^3$ is maximal. The maximal spike type $\omega^{\max} = (\omega_1^{\max}, \ldots \omega_r^{\max})$ has the property that $\omega_i^{\max} - \omega_{i+1}^{\max} \leq 1$ for $i < r-1$. This property characterises the maximal spike type uniquely: it depends on $n$ as well as $d$, and $\omega^{\max} = (d)$ for $n \geq d$. In particular, $\omega_1^{\max} = \nu(d,n)$ where

$$\nu(d,n) = \begin{cases} \min(d,n), & \text{if } d \equiv n \bmod 2 \\ \min(d, n-1), & \text{otherwise.} \end{cases}$$

The 'greedy algorithm' for $\omega^{\max}$ can be given in terms of the function $\nu$ by $\omega^{\max}(d,n) = (\nu(d,n), \omega^{\max}(d', \nu(d,n)))$, where as above $d' = (d-\mu(d))/2$. By a similar argument to that for $\omega^{\min}$, it is easily shown that if $\omega$ has degree $d$ and is the $\omega$-vector of a spike in $n$ variables, then $\omega \preceq \omega^{\max}$. An analogous result to that of Singer is known for $\omega^{\max}$: a generating set for $\mathbf{P}^d$ can be chosen from the set of monomials $m$ for which $\omega(m) \leq_l \omega^{\max}$ [5, 21]. We give a proof below (Proposition 5.9). When exactly one spike-type $\omega$ appears in $\mathbf{P}^d$, so that $\omega^{\min} = \omega^{\max}$, the results quoted above imply that $\mathbf{Q}^d = \mathbf{Q}^\omega$. However, the problem of determining the hit polynomials is difficult even in this case: for example, it can be shown that for $n=3$ the monomial $m = t_1^{11}t_2^{21}t_3^{38}$ is hit. The test $e < (2^k-1)\mu(f)$ fails in this case, since $\omega(m) = (2,2,2,1,1,1)$ is a spike-type.

# 2 The action of $GL(n, \overline{\mathbb{F}}_2)$ on polynomials

Let $\overline{\mathbb{F}}_2$ be the algebraic closure of $\mathbb{F}_2$, $V$ the vector space over $\overline{\mathbb{F}}_2$ with basis $\{u_1, \ldots, u_n\}$, and $\{t_1, \ldots, t_n\}$ the dual basis for $V^*$. Let $\overline{\mathbf{G}}$ denote the algebraic group $GL(n, \overline{\mathbb{F}}_2)$ of all non-singular linear transformations of $V$, acting on the left. We identify the polynomial algebra $\overline{\mathbb{F}}_2[t_1, \ldots, t_n]$ with the symmetric algebra $S(V^*)$, with $\overline{\mathbf{G}}$ acting on the right by linear substitutions. Thus for each $d \geq 0$ we have a finite dimensional polynomial $\overline{\mathbf{G}}$-module $\overline{\mathbf{P}}^d$ with basis the set of all monomials $m = t_1^{a_1} \cdots t_n^{a_n}$ of degree $d = \sum_{i=1}^n a_i$. We write $\overline{\mathbf{P}}$ for the graded module $\sum_{d \geq 0} \overline{\mathbf{P}}^d$.

Recall [13, 15] that simple polynomial $\overline{\mathbf{G}}$-modules are parametrised (up to equivalence) by partitions $\gamma = (\gamma_1, \ldots, \gamma_n)$, where $\gamma_1 \geq \ldots \geq \gamma_n \geq 0$. We identify these partitions with dominant weights in the usual way. The simple module $\overline{\mathbf{L}}(\gamma)$ is a highest weight module of highest weight $\gamma$. The structure of $\overline{\mathbf{P}}$ as a polynomial $\overline{\mathbf{G}}$-module was determined by Doty [10]. Only certain special simple modules $\overline{\mathbf{L}}(\gamma)$ occur in $\overline{\mathbf{P}}$, and those which do occur do so only once, this single occurrence being in degree $|\gamma| = \sum_{i=1}^n \gamma_i$. Every submodule of $\overline{\mathbf{P}}^d$ has a $\overline{\mathbb{F}}_2$-basis consisting of monomials. A combinatorial datum can be assigned to each monomial in such a way that the simple composition factors of $\overline{\mathbf{P}}^d$ correspond to the values of the datum on monomials $m$ of degree $d$. In Doty's formulation, this datum is defined as the 'carry pattern' of $m$, i.e. the vector $c = (c_1, \ldots, c_r)$ where $c_k \geq 0$ is the integer carried to the $2^k$ column when $a_1, \ldots, a_n$ are added in base 2 arithmetic.

We shall write base 2 expansions from the left, so that the non-negative integer $a = \sum_{j \geq 0} a(j)2^j$ is denoted by $a(0)a(1)\cdots a(r)$, trailing 0's being allowed. Thus 10011 and 100110 both denote 25, while 11001 denotes 19 and 011001 denotes 38. The number of $j$ such that $a(j) = 1$ is usually denoted by $\alpha(a)$, and we adapt this notation to the monomial $m = t_1^{a_1} \cdots t_n^{a_n}$ by writing $\alpha(m) = \sum_{i,j} a_i(j)$.

For convenience in working with the Steenrod algebra action, we replace the carry pattern by an equivalent datum defined as follows (cf. [30, Section 4]).

**Definition 2.1** The **$\omega$-vector** of a monomial $m = t_1^{a_1} \cdots t_n^{a_n}$ is the finite sequence $\omega(m) = (\omega_1, \ldots, \omega_r)$, where $\omega_j = \sum_{i=1}^n a_i(j-1)$. Its **degree** $\deg \omega = \sum_{i=1}^n a_i = \sum_{j=1}^r 2^{j-1}\omega_j$, and its **modulus** $|\omega| = \sum_{j=1}^r \omega_j$, so that $\alpha(m) = |\omega(m)|$.

Thus the monomial $m$ corresponds to a **binary block** $B$ of 0's and 1's with $n$ rows given by the binary expansions of the exponents, and $\omega(m)$ is the vector obtained by summing the columns as integers, with no 'carrying'. We call this the $\omega$-vector of $B$, and we also refer to $\alpha(m)$ as the **$\alpha$-count** of $B$: this is simply the number of 1's in the block $B$. Example 5.1 shows the binary blocks for the monomials $m$ with $\omega(m) = (2, 1)$ in the case $n = 3$. A block $B$ such that no 0 is followed by a 1 in the same row corresponds to a spike. Example 2.4 shows $\omega$-vectors of degree 7 (for $n \geq 7$), with corresponding monomials and simple $\overline{\mathbf{G}}$-modules.

Following [10], the submodule structure of $\overline{\mathbf{P}}^d$ can be described as follows. The set of $\omega$-vectors which occur in degree $d$ are the solutions of $\sum_{j\geq 1} 2^{j-1}\omega_j = d$. There is a natural partial order on this set, as follows.

**Definition 2.2** Let $\rho$ and $\sigma$ be $\omega$-vectors of degree $d$. Then $\rho$ **dominates** $\sigma$ (written $\rho \succeq \sigma$ or $\sigma \preceq \rho$), if and only if

$$\sum_{j=1}^{k} 2^{j-1}\rho_j \geq \sum_{j=1}^{k} 2^{j-1}\sigma_j, \text{ for all } k \geq 1. \tag{2}$$

This corresponds to the partial order defined by Doty on the set of carry patterns, i.e. $c \geq c'$ if and only if $c_k \geq c'_k$ for all $k$. Although we regard a weakly decreasing $\omega$-vector as a partition, we emphasise that this notion of dominance is *not* the usual one for partitions, because the $j$th component of the $\omega$-vector is weighted by the factor $2^{j-1}$. If $\deg \omega = d$, then $|\omega|$ takes all values between its minimum and maximum: the minimum is $\alpha(d)$ for all $n$, and the maximum is $d$ for $n \geq d$.

**Lemma 2.3** *Let $\rho$ and $\sigma$ be $\omega$-vectors with $\deg \rho = \deg \sigma$. Then $\rho \succeq \sigma$ if and only if $\rho$ can be transformed into $\sigma$ by a finite sequence of 'binary addition' steps, in which 2 is subtracted from some component and 1 is added to the next.*

**Proof** By adding trailing zeros, we may assume that $\rho$ and $\sigma$ have equal length $r$. If $\rho \succ \sigma$ then by adding the $r$ inequalities (2) with weightings $2^{r-2}, 2^{r-3}, \ldots, 2, 1, 1$ we obtain $|\rho| > |\sigma|$. If $\rho \succeq \sigma$ and $\rho_1 = \sigma_1$, then using (2) again we have $(\rho_2, \ldots, \rho_r) \succeq (\sigma_2, \ldots, \sigma_r)$. Hence we may assume by induction on $r$ that $\rho_1 > \sigma_1$. Since $\deg \rho = \deg \sigma$, $\rho_1 \equiv \sigma_1 \bmod 2$, and so $\rho_1 \geq \sigma_1 + 2$. Thus $\rho \succeq \overline{\rho} \succeq \sigma$ where $\overline{\rho} = (\rho_1 - 2, \rho_2 + 1, \rho_3, \ldots, \rho_r)$. The result follows by induction on $|\rho| - |\sigma|$. $\square$

Doty's results show that the monomials with a given $\omega$ of degree $d$ form a basis for a $\overline{\mathbf{G}}$-composition factor $\overline{\mathbf{P}}^\omega$ in $\overline{\mathbf{P}}^d$. More precisely, if $\deg \omega = d$ let $\overline{\mathbf{P}}^{\preceq\omega}$, $\overline{\mathbf{P}}^{\prec\omega}$

be the $\overline{\mathbf{G}}$-submodules of $\overline{\mathbf{P}}^d$ with $\overline{\mathbb{F}}_2$-bases given by the monomials with $\omega$-vectors $\preceq \omega$, $\prec \omega$ respectively. Then $\overline{\mathbf{P}}^\omega = \overline{\mathbf{P}}^{\preceq\omega}/\overline{\mathbf{P}}^{\prec\omega}$. The module $\overline{\mathbf{P}}^\omega$ is isomorphic to the simple polynomial $\overline{\mathbf{G}}$-module $\Lambda^{\omega_1} \otimes (\Lambda^{\omega_2})^{(1)} \otimes \cdots \otimes (\Lambda^{\omega_r})^{(r-1)}$, where $\Lambda^k$ is the $k$th exterior power of $V$, and for a $\overline{\mathbf{G}}$-module $M$ we denote its $i$th Frobenius twist by $M^{(i)}$. In particular, we identify $\overline{\mathbf{L}}(1^k)$ with $\Lambda^k$ for $1 \leq k \leq n$, where $(1^k)$ denotes the partition $(1, \ldots, 1)$ of length $k$. In general, if $t^\gamma = t_1^{\gamma_1} \cdots t_n^{\gamma_n}$ is the highest monomial in lexicographic order (with $t_1 > t_2 > \ldots > t_n$) such that $\omega(m) = \omega$, then $\overline{\mathbf{P}}^\omega \cong \overline{\mathbf{L}}(\gamma)$. It follows that no two of the $\overline{\mathbf{P}}^\omega$ are isomorphic: in particular, each $\overline{\mathbf{P}}^d$ is a multiplicity-free $\overline{\mathbf{G}}$-module.

**Example 2.4** For $\overline{\mathbf{P}}^7$, the diagrams below show the dominance order on $\omega$-vectors, with corresponding monomials $t^\gamma$ and simple $\overline{\mathbf{G}}$-modules $\overline{\mathbf{L}}(\gamma)$.

$$
\begin{array}{ccc}
 & (7) & \\
 & \downarrow & \\
 & (5,1) & \\
 & \downarrow & \\
 & (3,2) & \\
\swarrow & & \searrow \\
(1,3) & & (3,0,1) \\
\searrow & & \swarrow \\
 & (1,1,1) &
\end{array}
\qquad
\begin{array}{ccc}
 & t_1t_2t_3t_4t_5t_6t_7 & \\
 & \downarrow & \\
 & t_1^3t_2t_3t_4t_5 & \\
 & \downarrow & \\
 & t_1^3t_2^3t_3 & \\
\swarrow & & \searrow \\
t_1^3t_2^2t_3^2 & & t_1^5t_2t_3 \\
\searrow & & \swarrow \\
 & t_1^7 &
\end{array}
\qquad
\begin{array}{ccc}
 & \overline{\mathbf{L}}(1^7) & \\
 & \downarrow & \\
 & \overline{\mathbf{L}}(3,1^4) & \\
 & \downarrow & \\
 & \overline{\mathbf{L}}(3,3,1) & \\
\swarrow & & \searrow \\
\overline{\mathbf{L}}(3,2,2) & & \overline{\mathbf{L}}(5,1,1) \\
\searrow & & \swarrow \\
 & \overline{\mathbf{L}}(7) &
\end{array}
$$

The $\overline{\mathbf{G}}$-submodules of $\overline{\mathbf{P}}^d$ correspond to downward-closed subsets $S$ of $\omega$-vectors in the dominance order, the submodule corresponding to $S$ being given by the set of all polynomials in which every monomial that appears has $\omega$-vector belonging to $S$ [10]. Thus the lattice of $\omega$-vectors describes the join-irreducible submodules. For example, for $n \geq 7$ there are exactly 8 submodules of $\overline{\mathbf{P}}^7$, the two not visible in the lattice being the zero submodule and the submodule spanned by all monomials $m$ with $\omega(m) = (1,3), (3,0,1)$ or $(1,1,1)$.

For $\alpha \geq 0$ the monomials $m$ of degree $d$ with $\alpha(m) \leq \alpha$ span a $\overline{\mathbf{G}}$-submodule of $\overline{\mathbf{P}}^d$. Hence $\overline{\mathbf{P}}^d$ has an increasing filtration by $\alpha(m) = |\omega(m)|$, which we call the **$\alpha$-filtration**. Since the simple $\overline{\mathbf{G}}$-modules which correspond to the $\omega$-vectors which appear in $\overline{\mathbf{P}}$ are all distinct, the successive quotients in the $\alpha$-filtration are semisimple. Thus the non-trivial stages in the filtration give a Loewy series for $\overline{\mathbf{P}}^d$. The results of [9] imply that this is the unique Loewy series, i.e. it coincides with the socle and the radical series, and $\overline{\mathbf{P}}^d$ is rigid as a $\overline{\mathbf{G}}$-module. (We refer to [15, Chapter 13] for background on Loewy series.)

We shall consider various total orderings on the set of $\omega$-vectors in each degree $d$ which refine the partial ordering by dominance. We are particularly concerned with the **left** order $>_l$ (the left lexicographic order) and the **right** order $>_r$, (the *reverse* right lexicographic order). Both orderings totally order the set of all $\omega$-vectors, and in particular they totally order the $\omega$-vectors which occur in a given degree $d$. In Example 2.4 above, $(3,0,1) >_l (1,3)$ but $(3,0,1) <_r (1,3)$.

We define $\mathbf{G}$-submodules $\overline{\mathbf{P}}^{\leq_l \omega}$, $\overline{\mathbf{P}}^{<_l \omega}$, and $\overline{\mathbf{P}}^{\leq_r \omega}$, $\overline{\mathbf{P}}^{<_r \omega}$ using the left and right orderings. It is clear that $\overline{\mathbf{P}}^{\leq_l \omega}/\overline{\mathbf{P}}^{<_l \omega}$ and $\overline{\mathbf{P}}^{\leq_r \omega}/\overline{\mathbf{P}}^{<_r \omega}$ are isomorphic to $\overline{\mathbf{P}}^{\omega}$, and that the same is true for any total ordering $\leq$ on the set of $\omega$ vectors in each degree $d$ such that $\omega_1 \geq \omega_2$ when $\omega_1 \succeq \omega_2$. In fact, $\overline{\mathbf{P}}^{\omega}$ can be constructed directly by defining an action of $\overline{\mathbf{G}}$ on the vector space with $\overline{\mathbb{F}}_2$-basis given by the monomials $m$ with $\omega(m) = \omega$.

The left and right orderings do not in general refine the $\alpha$-filtration on $\overline{\mathbf{P}}^{d}$: already in degree 10, we have $(2,0,2) \geq_l (0,5)$ and $(0,5) \geq_r (4,1,1)$. However, as $\omega_1 \succ \omega_2$ implies that $|\omega_1| > |\omega_2|$ (see Lemma 2.3), any total ordering which refines the $\alpha$-filtration also refines dominance, and so gives rise to the same filtration quotients $\overline{\mathbf{P}}^{\omega}$. We shall require such orderings in Section 5.

# 3 Restriction to $GL(n, \mathbb{F}_2)$

Let $\mathbf{G}$ denote the finite general linear group $GL(n, \mathbb{F}_2)$, which we identify as usual with the subgroup of $\overline{\mathbf{G}}$ consisting of the fixed points under the Frobenius map. We denote by $\mathbf{P}$ the right $\overline{\mathbb{F}}_2\mathbf{G}$-module obtained by restriction of the $\overline{\mathbf{G}}$-module $\overline{\mathbf{P}}$, so that $\mathbf{P}$ and $\overline{\mathbf{P}}$ have the same underlying vector space $\overline{\mathbb{F}}_2[t_1, \ldots, t_n]$. For each $\omega$, we similarly denote by $\mathbf{P}^{\omega}$, $\mathbf{P}^{\preceq\omega}$ etc. the restrictions of $\overline{\mathbf{P}}^{\omega}$, $\overline{\mathbf{P}}^{\preceq\omega}$ etc. Since $\overline{\mathbf{P}}^{\omega} \cong \Lambda^{\omega_1} \otimes (\Lambda^{\omega_2})^{(1)} \otimes \cdots \otimes (\Lambda^{\omega_r})^{(r-1)}$ as a $\overline{\mathbf{G}}$-module, and the Frobenius twists disappear on restriction to $\mathbf{G}$, it follows that $\mathbf{P}^{\omega} \cong \Lambda^{\omega_1} \otimes \Lambda^{\omega_2} \otimes \cdots \otimes \Lambda^{\omega_r}$ as a $\overline{\mathbb{F}}_2\mathbf{G}$-module.

For any partition $\lambda$, we denote by $\mathbf{L}(\lambda)$ the restriction to $\mathbf{G}$ of the simple $\overline{\mathbf{G}}$-module $\overline{\mathbf{L}}(\lambda)$. The $\overline{\mathbb{F}}_2\mathbf{G}$-module $\mathbf{L}(\lambda)$ is simple if and only if $\lambda$ is column 2-regular. We can use Steinberg's tensor product theorem [15, Section 2.7] to write an arbitrary $\mathbf{L}(\lambda)$ as the tensor product of simple $\overline{\mathbb{F}}_2\mathbf{G}$-modules, as follows. Recall that partitions $\lambda$ and $\mu$ are added componentwise and are multiplied by concatenation and sorting, so that $(\lambda + \mu)' = \lambda' \cdot \mu'$. By considering the binary expansions of the exponents when the conjugate partition $\omega = \lambda'$ is written in multiset notation, we observe that $\lambda$ has a unique expansion $\lambda = \lambda(0) + 2\lambda(1) + \ldots + 2^{r-1}\lambda(r-1)$, where $\lambda(i)$ is column 2-regular for all $i$. The tensor product theorem gives $\overline{\mathbf{L}}(\lambda) \cong \overline{\mathbf{L}}(\lambda(0)) \otimes \overline{\mathbf{L}}(\lambda(1))^{(1)} \otimes \cdots \otimes \overline{\mathbf{L}}(\lambda(r-1))^{(r-1)}$, and hence $\mathbf{L}(\lambda) \cong \mathbf{L}(\lambda(0)) \otimes \mathbf{L}(\lambda(1)) \otimes \cdots \otimes \mathbf{L}(\lambda(r-1))$. For example, $\lambda = (11,5,2)$ gives $\omega = (3^2 2^3 1^6) = (2)^1 \cdot (3,2,1)^2 \cdot (1)^4$, giving $(11,5,2) = (1,1) + 2(3,2,1) + 4(1)$ and so $\overline{\mathbf{L}}(11,5,2) \cong \overline{\mathbf{L}}(1,1) \otimes \overline{\mathbf{L}}(3,2,1)^{(1)} \otimes \overline{\mathbf{L}}(1)^{(2)}$.

We recall [14, Section 6] that $\mathbb{F}_2$ is a splitting field for $\mathbf{G}$. Thus the finite group algebra $\mathbb{F}_2\mathbf{G}$ already has $2^{n-1}$ inequivalent simple modules, which correspond to our modules $\mathbf{L}(\lambda)$ by extending the field of coefficients to $\overline{\mathbb{F}}_2$, and we lose no information by studying the action of $\mathbf{G}$ on $\mathbf{P}$ rather than on the polynomial algebra $P = \mathbb{F}_2[t_1, \ldots, t_n]$. Thus for each $\omega$ of degree $d$ we have a corresponding subquotient $\overline{\mathbb{F}}_2\mathbf{G}$-module $\mathbf{P}^\omega$ of $\mathbf{P}$, and the lattice of $\omega$-vectors of degree $d$ is isomorphic to a lattice of $\overline{\mathbb{F}}_2\mathbf{G}$-submodules of $\mathbf{P}$. We regard this as a first approximation to its full submodule structure. In particular, we may obtain composition series for $\mathbf{P}$ which refine any total ordering on the set of $\omega$-vectors of degree $d$ which refines dominance, such as the left order $\leq_l$ or the right order $\leq_r$.

For all partitions $\lambda$, we denote by $\overline{\Delta}(\lambda)$ the Weyl module of highest weight $\lambda$ for $\overline{\mathbf{G}}$ (cf. [8]), and by $\overline{\nabla}(\lambda)$ the corresponding dual Weyl module, and we write $\Delta(\lambda)$ and $\nabla(\lambda)$ for their respective restrictions to $\mathbf{G}$. We refer to [15] for information about the general theory of Weyl modules. In particular, we require the following result [31], [15, Section 5.9]:

**Theorem** (W. J. Wong) *For each column 2-regular partition $\lambda$, the Weyl module $\Delta(\lambda)$ is an indecomposable $\overline{\mathbb{F}}_2\mathbf{G}$-module with unique simple quotient $\mathbf{L}(\lambda)$.*

Since $\nabla(\lambda)$ is the transpose dual of $\Delta(\lambda)$, we have the equivalent statement that when $\lambda$ is column 2-regular $\nabla(\lambda)$ is an indecomposable $\overline{\mathbb{F}}_2\mathbf{G}$-module with unique simple submodule $\mathbf{L}(\lambda)$.

Our argument makes use of some facts about tilting modules [8] for $\overline{\mathbf{G}}$, which we collect here. For each partition $\lambda$ of length $\leq n$, there is a corresponding $\overline{\mathbf{G}}$-module $\overline{\mathbf{T}}(\lambda)$, called the **(partial) tilting module** of highest weight $\lambda$, with the following defining properties.

- $\overline{\mathbf{T}}(\lambda)$ is indecomposable.
- $\overline{\mathbf{T}}(\lambda)$ has a $\lambda$-weight space of dimension 1, and all other weights $\mu$ of $\overline{\mathbf{T}}(\lambda)$ are lower than $\lambda$ in left lexicographic order $\leq_l$.
- $\overline{\mathbf{T}}(\lambda)$ has a 'good' filtration, i.e. an increasing filtration with each quotient isomorphic to $\nabla(\mu)$ for some $\mu \leq_l \lambda$.
- $\overline{\mathbf{T}}(\lambda)$ is self-dual under transpose duality, so that it also has an increasing filtration with each quotient isomorphic to $\Delta(\mu)$ for some $\mu \leq_l \lambda$.

By a **tilting module** we mean any direct sum of partial tilting modules. Since each exterior power $\Lambda^k$ of $V$ is a tilting module, and since the tensor product of two tilting modules is tilting, the $\overline{\mathbf{G}}$-module $\Lambda^\omega \cong \Lambda^{\omega_1} \otimes \ldots \otimes \Lambda^{\omega_r}$ is a tilting module. In addition, $\Lambda^\omega$ is a highest weight module. If $\omega$ is weakly decreasing, the highest weight of $\Lambda^\omega$ is the conjugate partition $\lambda = \omega'$, and in general it is the conjugate $\lambda$ of the partition obtained by sorting the components of $\omega$ in decreasing order. It follows that $\Lambda^\omega$ contains $\overline{\mathbf{T}}(\lambda)$ as a direct summand.

Thus the $\overline{\mathbf{G}}$-modules $\overline{\mathbf{P}}^\omega$ and $\Lambda^\omega$ restrict to the *same* $\overline{\mathbb{F}}_2\mathbf{G}$-module $\mathbf{P}^\omega$, and we can study $\mathbf{P}^\omega$ by starting with the tilting module $\Lambda^\omega$, rather than with the simple $\overline{\mathbf{G}}$-module $\overline{\mathbf{P}}^\omega$.

**Example 3.1** The diagram below shows the tilting modules $\Lambda^\omega$ and the simple composition factors $\overline{\mathbf{P}}^\omega$ for $\overline{\mathbf{P}}^7$, expressed as tensor products of Frobenius twists of simple modules for column 2-regular partitions. Note that the diagram of tilting modules does *not* represent the $\overline{\mathbf{G}}$-module $\overline{\mathbf{P}}^7$. We regard the restriction of either diagram as a 'first approximation' to the submodule lattice for $\mathbf{P}^7$.

$$\begin{array}{ccc} & \Lambda^7 & \\ & \downarrow & \\ & \Lambda^5 \otimes \Lambda^1 & \\ & \downarrow & \\ & \Lambda^3 \otimes \Lambda^2 & \\ \swarrow & & \searrow \\ \Lambda^1 \otimes \Lambda^3 & & \Lambda^3 \otimes \Lambda^1 \\ \searrow & & \swarrow \\ & \Lambda^1 \otimes \Lambda^1 \otimes \Lambda^1 & \end{array} \qquad \begin{array}{ccc} & \overline{\mathbf{L}}(1^7) & \\ & \downarrow & \\ & \overline{\mathbf{L}}(1^5) \otimes \overline{\mathbf{L}}(1)^{(1)} & \\ & \downarrow & \\ & \overline{\mathbf{L}}(1^3) \otimes \overline{\mathbf{L}}(1^2)^{(1)} & \\ \swarrow & & \searrow \\ \overline{\mathbf{L}}(1^3)^{(1)} \otimes \overline{\mathbf{L}}(1) & & \overline{\mathbf{L}}(1^3) \otimes \mathbf{L}(1)^{(2)} \\ \searrow & & \swarrow \\ & \overline{\mathbf{L}}(1) \otimes \overline{\mathbf{L}}(1)^{(1)} \otimes \overline{\mathbf{L}}(1)^{(2)} & \end{array}$$

# 4 The hit problem and its dual

The 'hit problem' of F. P. Peterson [23] asks for a minimal generating set for $\mathbb{F}_2[t_1, \ldots, t_n]$ as an module over the Steenrod algebra $\mathcal{A}_2$. There is an extensive literature on this problem and on its natural extension to the problem of describing the structure of such a minimal generating set in terms of the modular representation theory of $\mathbf{G}$. We extend the action of $\mathcal{A}_2$ to $\overline{\mathbb{F}}_2[t_1, \ldots, t_n]$ by making it act trivially on the coefficients $\overline{\mathbb{F}}_2$. Thus the action of $\mathcal{A}_2$ on the left of $\mathbf{P}$ commutes with the action of $\mathbf{G}$ on the right, so that we obtain $\overline{\mathbb{F}}_2\mathbf{G}$-maps $Sq^k : \mathbf{P}^d \to \mathbf{P}^{d+k}$ for all $k, d \geq 0$. We emphasise that the action of $\mathcal{A}_2$ on $\overline{\mathbb{F}}_2[t_1, \ldots, t_n]$ does *not* commute with the action of the infinite group $\overline{\mathbf{G}}$.

Following Singer [28] and others, we say that a homogeneous polynomial $f$ of degree $d$ is **hit** if it can be written in the form $f = \sum_{k>0} Sq^k h_k$, where $h_k$ is homogeneous of degree $d - k$. For each degree $d$, the hit polynomials form a vector subspace $\mathbf{H}^d = \mathbf{P}^d \cap \mathcal{A}_2^+\mathbf{P} = \sum_{k>0} \mathcal{A}_2^k \mathbf{P}^{d-k}$. This is in fact a $\overline{\mathbb{F}}_2\mathbf{G}$-submodule of $\mathbf{P}^d$, since the action of the Steenrod algebra on $\mathbf{P}$ commutes with linear substitutions of the variables. The quotient space $\mathbf{Q}^d = \mathbf{P}^d/\mathbf{H}^d$ is also a $\overline{\mathbb{F}}_2\mathbf{G}$-module, which we call the **cohit module** in degree $d$. Thus Peterson's problem asks for a description of the cohit modules $\mathbf{Q}^d$ for all $n$ and $d$.

Let $\leq$ denote any total ordering on the set of $\omega$-vectors in degree $d$ which refines the partial order by dominance. This gives an increasing filtration of $\mathbf{P}^d$ by $\overline{\mathbb{F}}_2\mathbf{G}$-submodules $\mathbf{P}^{\leq\omega}$, which depend on the choice of $\leq$, with quotients $\mathbf{P}^\omega = \mathbf{P}^{\leq\omega}/\mathbf{P}^{<\omega}$, which are independent of this choice. By intersection with $\mathcal{A}_2^+\mathbf{P}$ we obtain a corresponding filtration on $\mathbf{H}^d$ with $\mathbf{H}^{\leq\omega} = \mathbf{H}^d \cap \mathbf{P}^{\leq\omega}$, $\mathbf{H}^{<\omega} = \mathbf{H}^d \cap \mathbf{P}^{<\omega}$, $\mathbf{H}^\omega = \mathbf{H}^{\leq\omega}/\mathbf{H}^{<\omega}$. and by taking quotients we have a filtration on $\mathbf{Q}^d$ with $\mathbf{Q}^{\leq\omega} = \mathbf{P}^{\leq\omega}/\mathbf{H}^{\leq\omega}$, $\mathbf{Q}^{<\omega} = \mathbf{P}^{<\omega}/\mathbf{H}^{<\omega}$, $\mathbf{Q}^\omega = \mathbf{Q}^{\leq\omega}/\mathbf{Q}^{<\omega}$. We emphasise that the quotients in these filtrations depend, in general, on the ordering $\leq$. We call the quotient $\mathbf{Q}^\omega$ the $\omega$-**cohit module** for the ordering $\leq$, and use $\mathbf{Q}^\omega_l$, $\mathbf{Q}^\omega_r$, respectively to indicate the left and right orders. Using standard isomorphism theorems, $\mathbf{Q}^{\leq\omega} = \mathbf{P}^{\leq\omega}/(\mathbf{H}^d \cap \mathbf{P}^{\leq\omega}) \cong (\mathbf{P}^{\leq\omega} + \mathbf{H}^d)/\mathbf{H}^d$, and so the filtration quotient $\mathbf{Q}^\omega \cong (\mathbf{P}^{\leq\omega} + \mathbf{H}^d)/(\mathbf{P}^{<\omega} + \mathbf{H}^d)$. By noting that $\mathbf{P}^{<\omega}/\mathbf{H}^{<\omega} = (\mathbf{P}^{<\omega} + \mathbf{H}^{\leq\omega})/\mathbf{H}^{\leq\omega}$, we see that $\mathbf{Q}^\omega \cong \mathbf{P}^{\leq\omega}/(\mathbf{P}^{<\omega} + \mathbf{H}^{\leq\omega}) \cong (\mathbf{P}^{\leq\omega}/\mathbf{P}^{<\omega}) \,/\, ((\mathbf{P}^{<\omega} + \mathbf{H}^{\leq\omega})/\mathbf{P}^{<\omega})$. Now $\mathbf{P}^{\leq\omega}/\mathbf{P}^{<\omega} = \mathbf{P}^\omega$ and $(\mathbf{P}^{<\omega} + \mathbf{H}^{\leq\omega})/\mathbf{P}^{<\omega} \cong \mathbf{H}^{\leq\omega}/(\mathbf{P}^{<\omega} \cap \mathbf{H}^{\leq\omega}) \cong \mathbf{H}^{\leq\omega}/\mathbf{H}^{<\omega} = \mathbf{H}^\omega$. Thus we may regard $\mathbf{Q}^\omega$ as the quotient $\overline{\mathbb{F}}_2\mathbf{G}$-module $\mathbf{P}^\omega/\mathbf{H}^\omega$. In this way, the ordered list of all $\mathbf{Q}^\omega$ with $\deg \omega = d$ provides a filtration of $\mathbf{Q}^d$ as a $\overline{\mathbb{F}}_2\mathbf{G}$-module, with the nonzero $\mathbf{Q}^\omega$ as quotients.

We now turn to the dual problem. The graded Hopf algebra dual of $P = \mathbb{F}_2[t_1, \ldots, t_n]$ is the divided polynomial algebra $D$, with generators which we identify with the basis elements $u_1, \ldots, u_n$ of $V$. Thus $D^d = \mathrm{Hom}(P^d, \mathbb{F}_2)$ for $d \geq 0$, and we write $\langle\ ,\ \rangle : D^d \otimes P^d \to \mathbb{F}_2$ for the evaluation map. The (divided) monomials $m^* = u_1^{(a_1)} \cdots u_n^{(a_n)}$ form the $\mathbb{F}_2$-basis dual to the monomial basis of $P$, and we use the same notation of binary blocks of 0's and 1's to encode corresponding monomials $m$ and $m^*$ (see Example 5.1). Thus $\omega$-vectors are defined in the same way for $D$ and $P$. The binary block notation affords a convenient definition of the product in $D$, since the product of monomials $m_1$ and $m_2$ is defined by 'superposition of blocks', i.e. $m_1^* m_2^* = 0$ if the blocks representing $m_1^*$ and $m_2^*$ both have a 1 in the same position, and otherwise $m_1^* m_2^*$ is represented by the block with 1's in all positions where either block has a 1. In particular, let $u_i^{(2^j)} \in D$ be the element which evaluates to 1 on $t_i^{2^j}$ and evaluates to 0 on all other monomials in $t_1, \ldots, t_n$ of degree $2^j$. Then $D$ is the exterior algebra over $\mathbb{F}_2$ generated by the elements $u_i^{(2^j)}$ for $1 \leq i \leq n$ and $j \geq 0$. The 'divided binomial theorem' $(u+v)^{(k)} = \sum_{i+j=k} u^{(i)} v^{(j)}$ is useful in calculations in $D$.

As for $P$, we extend the scalars and consider the corresponding divided power algebra $\overline{\mathbb{F}}_2 \otimes_{\mathbb{F}_2} D$ over $\overline{\mathbb{F}}_2$. For $1 \leq k \leq n$ we identify $\Lambda^k(V)$ with $\overline{\mathbb{F}}_2 \otimes_{\mathbb{F}_2} D^k$, so that the products $u_{i_1} \cdots u_{i_k}$ with $1 \leq i_1 \leq \ldots \leq i_k \leq n$ form a $\overline{\mathbb{F}}_2$-basis for $\Lambda^k(V)$. Thus we regard the exterior algebra on $V$ as the subalgebra of $D$ which is represented by binary blocks with one column.

The algebra $\overline{\mathbb{F}}_2 \otimes_{\mathbb{F}_2} D$ has the structure of a graded left $\overline{\mathbf{G}}$-module by dualising the graded right $\overline{\mathbf{G}}$-module $\overline{\mathbf{P}}$. In order to work consistently with right $\overline{\mathbf{G}}$-modules, we define a right action of $\overline{\mathbf{G}}$ on divided polynomials by $\langle f \cdot \gamma, h \rangle = \langle f, h \cdot \gamma^t \rangle$, where $f$ is a divided polynomial, $h$ is an ordinary polynomial, and $\gamma \mapsto \gamma^t$ denotes matrix transposition in $\overline{\mathbf{G}}$. We denote the resulting graded right $\overline{\mathbf{G}}$-module by $\overline{\mathbf{D}} = \sum_{d \geq 0} \overline{\mathbf{D}}^d$. Thus $\overline{\mathbf{D}}^d$ is the transpose dual of $\overline{\mathbf{P}}^d$. This duality preserves simple $\overline{\mathbf{G}}$-modules but reverses composition series, so that submodules of $\overline{\mathbf{P}}^d$ correspond to isomorphic quotient modules of $\overline{\mathbf{D}}^d$, and vice versa. As for $\overline{\mathbf{P}}$, the action of $\overline{\mathbf{G}}$ on $\overline{\mathbf{D}}$ is given by linear substitutions of the basis $\{u_1, \ldots, u_n\}$ of $V = \overline{\mathbf{D}}^1$. We regard $u_1, \ldots, u_n$ as variables when considering the elements of $\overline{\mathbf{D}}$ as divided polynomials.

The action of an element of $\overline{\mathbf{G}}$ on a divided monomial $m^*$ is to replace it by a sum of divided monomials $m_i^*$ with $\omega(m_i^*) \succeq \omega(m^*)$ for all $i$. Thus for each $\omega$ of degree $d$, we define $\overline{\mathbf{G}}$-submodules $\overline{\mathbf{D}}^{\succeq\omega}$, $\overline{\mathbf{D}}^{\succ\omega}$ of $\overline{\mathbf{D}}^d$ as the set of divided polynomials all of whose terms have $\omega$-vectors $\succeq \omega$, $\succ \omega$ respectively. We define $\overline{\mathbf{D}}^{\omega} = \overline{\mathbf{D}}^{\succeq\omega}/\overline{\mathbf{D}}^{\succ\omega}$. We similarly define $\overline{\mathbf{G}}$-submodules $\overline{\mathbf{D}}^{\geq\omega}$, $\overline{\mathbf{D}}^{>\omega}$ for any total ordering $\geq$ which refines dominance, and observe that $\overline{\mathbf{D}}^{\geq\omega}/\overline{\mathbf{D}}^{>\omega} \cong \overline{\mathbf{D}}^{\omega}$.

The simple $\overline{\mathbf{G}}$-modules $\overline{\mathbf{P}}^{\omega}$ and $\overline{\mathbf{D}}^{\omega}$ correspond under transpose duality of $\overline{\mathbf{P}}$ and $\overline{\mathbf{D}}$, and so they are isomorphic. In Proposition 4.2 below we show that an isomorphism is given by the 'natural' correspondence of monomials $m$ and $m^*$ represented by the same binary block. On restriction to $\mathbf{G}$, we obtain graded right $\overline{\mathbb{F}}_2\mathbf{G}$-modules $\mathbf{D} = \sum_{d\geq 0}\mathbf{D}^d$, and similarly for $\mathbf{D}^{\succeq\omega}$ etc. Thus $\mathbf{P}^{\omega}$ and $\mathbf{D}^{\omega}$ are isomorphic, self-dual $\overline{\mathbb{F}}_2\mathbf{G}$-modules for each $\omega$. The self-duality of $\mathbf{P}^{\omega}$ also follows from the fact that it is the restriction to $\mathbf{G}$ of the tilting module $\Lambda^{\omega}$.

The left action of $\mathcal{A}_2$ on $P$ gives a right action of $\mathcal{A}_2$ on $D$. Since we prefer to work with left actions, we shall write the element of the opposite algebra $\mathcal{A}_2^{\mathrm{op}}$ corresponding to $Sq^k$ as $Sq_k$. The (down) action of $\mathcal{A}_2^{\mathrm{op}}$ on $D$ is then determined by

$$Sq_k(u_i^{(j)}) = \binom{j-k}{k} u_i^{(j-k)} \tag{3}$$

together with the Cartan formula $Sq_k(uv) = \sum_{i+j=k} Sq_i(u)Sq_j(v)$, for $u, v \in D$. In this way we obtain commuting actions of $\mathcal{A}_2^{\mathrm{op}}$ on the left and of $\mathbf{G}$ on the right of $D$. As for $\mathbf{P}$, we extend the action of $\mathcal{A}_2^{\mathrm{op}}$ to $\mathbf{D}$, so that the action of every Steenrod operation fixes the scalars $\overline{\mathbb{F}}_2$, and thus $Sq_k : \mathbf{D}^d \to \mathbf{D}^{d-k}$ is a map of right $\overline{\mathbb{F}}_2\mathbf{G}$-modules for $0 \leq k \leq d$. (It is *not* a map of $\overline{\mathbf{G}}$-modules if $k > 0$.) As the canonical anti-automorphism $\chi$ of $\mathcal{A}_2$ gives an isomorphism of algebras $\mathcal{A}_2 \cong \mathcal{A}_2^{\mathrm{op}}$, we follow the practice in algebraic topology [2, 7] of referring to this action of $\mathcal{A}_2^{\mathrm{op}}$ on $\mathbf{D}$ as a left action of $\mathcal{A}_2$.

We now consider the dual of the hit problem, namely to determine the set of all divided polynomials $f$ which are annihilated by $\mathcal{A}_2^+$, or equivalently $Sq_k(f) = 0$ for all $k \geq 1$. By the Cartan formula, this set is closed under the product in $\mathbf{D}$, and so it is a subalgebra of $\mathbf{D}$, which we call the **Steenrod kernel K**. By duality of the actions of $\mathcal{A}_2$ on $\mathbf{P}$ and $\mathbf{D}$, the orthogonal complement of the kernel $\mathbf{K}^d$ is isomorphic to the subspace of hit polynomials $\mathbf{H}^d$, and so $\mathbf{K}^d$ has the same dimension as $\mathbf{Q}^d$ as a vector space over $\overline{\mathbb{F}}_2$. By duality of the group actions, $\mathbf{K}^d$ and $\mathbf{Q}^d$ are transpose duals as $\overline{\mathbb{F}}_2\mathbf{G}$-modules.

Let $\geq$ be a total ordering on the set of $\omega$-vectors in degree $d$ which refines dominance. Then $\geq$ gives an increasing filtration of $\mathbf{D}^d$ by $\overline{\mathbb{F}}_2\mathbf{G}$-submodules $\mathbf{D}^{\geq\omega}$ which depend on the choice of $\geq$, with quotients $\mathbf{D}^{\omega} = \mathbf{D}^{\geq\omega}/\mathbf{D}^{>\omega}$ which are independent of this choice. By intersection with the Steenrod kernel $\mathbf{K}$, we obtain a corresponding filtration $\mathbf{K}^{\geq\omega} = \mathbf{K}^d \cap \mathbf{D}^{\geq\omega}$ on $\mathbf{K}^d$. We call the quotient module $\mathbf{K}^{\omega} = \mathbf{K}^{\geq\omega}/\mathbf{K}^{>\omega}$ the **$\omega$-kernel**. Again we emphasise that $\mathbf{K}^{\omega}$ depends on the choice of ordering, which we indicate in the notation, as for $\mathbf{Q}^{\omega}$, only when necessary.

Since $\mathbf{K}^\omega = (\mathbf{D}^{\geq\omega} \cap \mathbf{K}^d)/((\mathbf{D}^{\geq\omega} \cap \mathbf{K}^d) \cap \mathbf{D}^{>\omega}) \cong ((\mathbf{D}^{\geq\omega} \cap \mathbf{K}^d) + \mathbf{D}^{>\omega})/\mathbf{D}^{>\omega}$, we may regard $\mathbf{K}^\omega$ as a $\overline{\mathbb{F}}_2\mathbf{G}$-submodule of $\mathbf{D}^\omega = \mathbf{D}^{\geq\omega}/\mathbf{D}^{>\omega}$. In this way the ordered list of all $\mathbf{K}^\omega$ with $\omega$ of degree $d$ provides a filtration of $\mathbf{K}^d$ as a $\overline{\mathbb{F}}_2\mathbf{G}$-module, with the nonzero $\mathbf{K}^\omega$ as quotients. The following result follows by straightforward arguments from the definitions.

**Proposition 4.1** *The $\overline{\mathbb{F}}_2\mathbf{G}$-modules $\mathbf{K}^\omega$ and $\mathbf{Q}^\omega$, defined with respect to the same total order on $\omega$-vectors, are transpose duals. In particular, $\mathbf{K}^\omega$ and $\mathbf{Q}^\omega$ have the same dimension as vector spaces over $\overline{\mathbb{F}}_2$.* □

We observed above that $\mathbf{P}^\omega \cong \mathbf{D}^\omega$ is a self-dual $\overline{\mathbb{F}}_2\mathbf{G}$-module for all $\omega$. We use the following result to identify the submodule $\mathbf{K}^\omega$ of $\mathbf{D}^\omega$ with a submodule of $\mathbf{P}^\omega$, and the quotient $\mathbf{Q}^\omega$ of $\mathbf{P}^\omega$ with a quotient of $\mathbf{D}^\omega$.

**Proposition 4.2** *For all $\omega$, the $\overline{\mathbf{G}}$-modules $\overline{\mathbf{P}}^\omega$ and $\overline{\mathbf{D}}^\omega$ are canonically isomorphic via the 'identity' map which sends each monomial $m$ to the corresponding divided monomial $m^*$.*

**Proof** The (right) actions of $\overline{\mathbf{G}}$ on both $\overline{\mathbf{P}}$ and $\overline{\mathbf{D}}$ are defined by linear substitutions of $t_1, \ldots, t_n$ and $u_1, \ldots, u_n$ respectively. Since permutations of the variables correspond in the obvious way, we need only verify that the action of the transvection $g$ defined on $\overline{\mathbf{P}}$ by $g(t_1) = t_1 + \alpha t_2$ and $g(t_i) = t_i$ for $i > 1$, and on $\overline{\mathbf{D}}$ by $g(u_1) = u_1 + \alpha u_2$ and $g(u_i) = u_i$ for $i > 1$ commutes with the 'identity' map, where $\alpha \in \overline{\mathbb{F}}_2$. For this purpose, there is no loss of generality in assuming that $n = 2$.

Thus let $m = t_1^{a_1} t_2^{a_2}$ be a monomial in $\overline{\mathbf{P}}^\omega$, so that $mg = (t_1 + \alpha t_2)^{a_1} t_2^{a_2}$. By the mod 2 binomial theorem, this can be written $\sum t_1^{a_1'} (\alpha t_2)^{a_1''} \cdot t_2^{a_2}$, where the sum is over all decompositions $a_1 = a_1' + a_1''$ of $a_1$ where $a_1'$ and $a_1''$ have disjoint binary expansions. If the binary expansions of $a_1''$ and $a_2$ are not disjoint, then the corresponding term in the sum has $\omega$-vector $\prec \omega$. Thus in $\overline{\mathbf{P}}^\omega$ we have $mg = \sum t_1^{a_1'} (\alpha t_2)^{a_1'' + a_2}$, where the sum is over all decompositions $a_1 = a_1' + a_1''$ such the binary expansion of $a_1''$ is disjoint from that of both $a_1'$ and $a_2$.

This calculation is more transparent in block notation: each column $\begin{smallmatrix}1\\0\end{smallmatrix}$ in the block representing $m$ gives rise to two blocks with columns $\begin{smallmatrix}1\\0\end{smallmatrix}$ and $\begin{smallmatrix}0\\1\end{smallmatrix}$ in the sum, for example

$$\begin{matrix}1&0&1&0&1\\1&0&0&1&0\end{matrix} \mapsto \begin{matrix}1&0&1&0&1\\1&0&0&1&0\end{matrix} + \begin{matrix}1&0&0&0&1\\1&0&1&1&0\end{matrix} + \begin{matrix}1&0&1&0&0\\1&0&0&1&1\end{matrix} + \begin{matrix}1&0&0&0&0\\1&0&1&1&1\end{matrix}$$

Let $m^* = u_1^{(a_1)} u_2^{(a_2)}$ be the monomial dual to $m$. Then $m^* g = (u_1 + \alpha u_2)^{(a_1)} u_2^{(a_2)}$. By the divided binomial theorem, this can be written $\sum u_1^{(a_1')} (\alpha u_2)^{(a_1'')} \cdot u_2^{(a_2)}$, where the sum is over all decompositions $a_1 = a_1' + a_1''$ of $a_1$. Now $u_2^{(a_1'')} \cdot u_2^{(a_2)} = 0$ if the binary expansions of $a_1''$ and $a_2$ are not disjoint. If the binary expansions of

$a_1''$ and $a_1'$ are not disjoint, then those of $a_1''$ and $a_1' + a_2$ are also not disjoint, and so the corresponding term in the sum has $\omega$-vector $\succ \omega$. Thus in $\overline{\mathbf{D}}^\omega$ we have $m^* g = \sum u_1^{(a_1')} (\alpha u_2)^{(a_1''+a_2)}$, where the sum is over all decompositions $a_1 = a_1' + a_1''$ such the binary expansion of $a_1''$ is disjoint from that of both $a_1'$ and $a_2$. □

With these preliminaries, we can now prove our 'lower bound' result Theorem 1.1, using the representation theory outlined in Sections 2 and 3.

**Theorem 4.3** *Let $\omega$ be strictly decreasing with conjugate $\lambda$. Let $\mathbf{Q}^\omega$ and $\mathbf{K}^\omega$ be defined using an ordering on $\omega$-vectors in degree $\deg \omega$ which refines dominance. Then the $\omega$-cohit module $\mathbf{Q}^\omega$ has a $\overline{\mathbb{F}}_2\mathbf{G}$-quotient $\mathbf{Q}(\lambda)$ isomorphic to the dual Weyl module $\nabla(\lambda)$, where $\lambda = \omega'$. The simple submodule of $\mathbf{Q}(\lambda)$ isomorphic to $\mathbf{L}(\lambda)$ is generated by the spike monomial $s(\lambda) = t_1^{a_1} \cdots t_n^{a_n}$, where $a_i = 2^{\lambda_i} - 1$ for $1 \leq i \leq n$.*

*Dually, the $\omega$-kernel $\mathbf{K}^\omega$ has a $\overline{\mathbb{F}}_2\mathbf{G}$-submodule $\mathbf{K}(\lambda)$ isomorphic to the Weyl module $\Delta(\lambda)$, and $\mathbf{K}(\lambda)$ has simple quotient isomorphic to $\mathbf{L}(\lambda)$, generated by $s^*(\lambda) = u_1^{(a_1)} \cdots u_n^{(a_n)}$.*

*In particular,* $\dim \mathbf{Q}^\omega = \dim \mathbf{K}^\omega \geq \dim \Delta^(\lambda) = \delta(\lambda)$.

**Proof** Since $\omega$ is weakly decreasing, it is a partition with conjugate partition $\omega' = \lambda$. By [8, Lemma 3.4(ii)] the indecomposable summands of the tilting module $\Lambda^\omega$ have the form $\Lambda^\omega \cong \overline{\mathbf{T}}^\lambda \oplus \sum_{\mu \prec \lambda} m_{\lambda,\mu} \overline{\mathbf{T}}^\mu$ for some non-negative integers $m_{\lambda,\mu}$, where $\mu \prec \lambda$ means that $\mu$ is lower than $\lambda$ in the usual dominance order for partitions. Writing $\omega = (\omega_1, \ldots, \omega_r)$, the $\overline{\mathbf{G}}$-module $\Lambda^\omega$ has a 1-dimensional highest weight space generated by $v_1 \otimes \cdots \otimes v_r$, where $v_j = u_1 \wedge \cdots \wedge u_{\omega_j}$ for $1 \leq j \leq r$. This vector corresponds to the highest weight $\lambda$, and by our identifications of $\Lambda^\omega$ with $\overline{\mathbf{P}}^\omega$ and with $\overline{\mathbf{D}}^\omega$, to the monomial $s(\lambda)$ and the divided monomial $s^*(\lambda)$ respectively. Hence the top good filtration quotient of $\overline{\mathbf{P}}^\omega$ is isomorphic to $\overline{\nabla}(\lambda)$, and its socle $\overline{\mathbf{L}}(\lambda)$ is generated by $s(\lambda)$; dually, the top Weyl filtration quotient of $\overline{\mathbf{D}}^\omega$ is isomorphic to $\overline{\Delta}(\lambda)$, and its head $\overline{\mathbf{L}}(\lambda)$ is generated by $s^*(\lambda)$.

We now restrict to $\mathbf{G}$ and apply the hypothesis that $\omega$ is strictly decreasing. Then $\lambda$ is column 2-regular and it follows from Wong's theorem (Section 3) that $s(\lambda)$ generates the socle of the indecomposable $\overline{\mathbb{F}}_2\mathbf{G}$-module $\nabla(\lambda)$. Now $s(\lambda)$ does not appear as a term in any polynomial which is the image of a Steenrod operation of positive degree. Thus it can not be hit mod $\mathbf{P}^{<\omega}$ for any choice of ordering $\leq$ on $\omega$-vectors. It follows that $\nabla(\lambda)$ is mapped injectively under the projection of $\mathbf{P}^\omega$ to a quotient module of $\mathbf{Q}^\omega$.

Dually, $s^*(\lambda)$ is in the Steenrod kernel $\mathbf{K}^d$ and, since it is a monomial, it gives a generator of $\mathbf{K}^\omega$ for any choice of ordering $\leq$. Thus $s^*(\lambda)$ generates the indecomposable $\overline{\mathbb{F}}_2\mathbf{G}$-module given by the restriction $\Delta(\lambda)$ of $\overline{\Delta}(\lambda)$. It follows that $\Delta(\lambda)$ is a submodule of $\mathbf{K}^\omega$. □

# 5 Monomials and Young tableaux

This section extends [29]. For a monomial $m$ in $\mathbf{P}$ or $m^*$ in $\mathbf{D}$, we have a corresponding block $B$ of 0's and 1's. If $\omega(m)$ is weakly decreasing, then we can encode $B$ as a Young tableau $T$ with diagram $\lambda = \omega'$ and content $\{1, \ldots, n\}$. Thus $T$ assigns to each $(i, j)$ with $1 \leq j \leq \lambda_i$ an integer $T_{i,j} \in \{1, \ldots, n\}$ . The $j$th column of $T$ lists the rows of $B$ such that $B(i, j) = 1$ in increasing order. We call this bijection between blocks and tableaux which are increasing on columns the **column-position correspondence**.

The tableau $T$ is **semi-standard** (abbreviated to SSYT) if the entries not only increase strictly on columns but also increase weakly on rows. We also call a block or a monomial **semi-standard** if it corresponds in the way to a SSYT. By [29, Lemma 1.3], a block is semi-standard if and only if for $1 \leq k \leq n$ the $\omega$-vector of the subblock given by its first $k$ rows is weakly decreasing.

**Example 5.1** Let $n = 3$ and let $\omega = (2, 1) = \lambda$. There are nine Young tableaux with increasing columns

$$\begin{array}{ll} 1 & 1 \\ 2 & \end{array} \quad \begin{array}{ll} 1 & 1 \\ 3 & \end{array} \quad \begin{array}{ll} 1 & 2 \\ 2 & \end{array} \quad \begin{array}{ll} 1 & 3 \\ 3 & \end{array} \quad \begin{array}{ll} 2 & 2 \\ 3 & \end{array} \quad \begin{array}{ll} 2 & 3 \\ 3 & \end{array} \quad \begin{array}{ll} 1 & 2 \\ 3 & \end{array} \quad \begin{array}{ll} 1 & 3 \\ 2 & \end{array} \quad \begin{array}{ll} 2 & 1 \\ 3 & \end{array}$$

and these correspond to the binary blocks

$$\begin{array}{ll} 1 & 1 \\ 1 & \\ 0 & \end{array} \quad \begin{array}{ll} 1 & 1 \\ 0 & \\ 1 & \end{array} \quad \begin{array}{ll} 1 & \\ 1 & 1 \\ 0 & \end{array} \quad \begin{array}{ll} 1 & \\ 0 & \\ 1 & 1 \end{array} \quad \begin{array}{ll} 0 & \\ 1 & 1 \\ 1 & \end{array} \quad \begin{array}{ll} 0 & \\ 1 & \\ 1 & 1 \end{array} \quad \begin{array}{ll} 1 & \\ 0 & 1 \\ 1 & \end{array} \quad \begin{array}{ll} 1 & \\ 1 & \\ 0 & 1 \end{array} \quad \begin{array}{ll} 0 & 1 \\ 1 & \\ 1 & \end{array}$$

and to the monomials

$$t_1^3t_2, \quad t_1^3t_3, \quad t_1t_2^3, \quad t_1t_3^3, \quad t_2^3t_3, \quad t_2t_3^3, \quad t_1t_2^2t_3, \quad t_1t_2t_3^2, \quad t_1^2t_2t_3.$$

The first eight tableaux are semi-standard, and the corresponding eight monomials represent a basis for $\mathbf{Q}^{(2,1)}$. The equation $Sq^1(t_1t_2t_3) = t_1^2t_2t_3 + t_1t_2^2t_3 + t_1t_2t_3^2$ shows that the last monomial is not required.

The Weyl module $\overline{\Delta}(\lambda)$ and its dual $\overline{\nabla}(\lambda)$ have $\overline{\mathbb{F}}_2$-bases indexed by the set of SSYT's with diagram $\lambda$ and content $\{1, \ldots, n\}$ [12, 13]. The dimension $\delta(\lambda)$ of these modules (the number of SSYT's) is given by the **hook formula**

$$\delta(\lambda) = \prod_{(i,j)} \frac{n - i + j}{h(i, j)},$$

where the **hook-length** $h(i, j)$ of the box $(i, j)$ is the number of boxes $(i', j')$ such that $i' \geq i, j' = j$ or $i' = i, j' \geq j$, and the product is taken over all boxes $(i, j)$ in the diagram of $\lambda$.

The following theorem is our main 'upper bound' result, Theorem 1.2. The special case where there is only one spike type $\omega$ in degree $d$, so that $\mathbf{Q}^\omega = \mathbf{Q}^d$, is Theorem 2.15 of [29]. The generalization is based on the choice of an ordering adapted to a given $\omega$: recall that $|\omega|$ is the number of 1's in any $(0,1)$-block $B$ with $\omega(B) = \omega$. The proof consists in verifying that the details given in [29] are valid in this context. Since we use an ordering depending 'locally' on $\omega$, we cannot infer a 'global' upper bound on $\mathbf{Q}^d$, even if we were able to prove that no contribution to $\mathbf{Q}^d$ is made by non-decreasing $\omega$-vectors.

**Theorem 5.2** *Let $\omega$ be weakly decreasing and let $\leq$ be an ordering on the set of $\omega$-vectors $\rho$ with degree $\deg\omega$ such that $\rho \leq \omega$ if $|\rho| \leq |\omega|$. Then $\mathbf{Q}^\omega$ is spanned by (equivalence classes of) semi-standard blocks with diagram $\lambda = \omega'$ and content $\{1, 2, \ldots, n\}$. In particular, $\dim \mathbf{Q}^\omega \leq \delta(\lambda)$.*

We next explain the key combinatorial concept used to prove Theorem 5.2, which we call **splicing**. This is a two-step procedure for modifying a formal sum of blocks as follows. First, for two adjacent columns of some block $B$ and some $k \geq 1$, choose a set of $k$ 1's in the right hand column which are opposite 0's in the left hand column, and form an intermediate block $A$ by switching these 0's and 1's. Second, form all possible blocks $C_i$ which can be obtained from $A$ by switching $k$ 1's in the left hand column (but not any of those moved in the first step) with matching 0's in the right hand column. The $k$-splicing operation replaces $B$ by the formal sum $C$ of all the blocks $C_i$ which can be obtained in this way, which we call the **$k$-splice** of $B$ (at rows which are specified if necessary). If the block $A$ can be formed, but not any block $C_i$, then $C$ is the zero polynomial.

For example, the unique 2-splice of $B$ is $C$, where

$$B = \begin{matrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{matrix}\,, \qquad C = \begin{matrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{matrix} + \begin{matrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{matrix} + \begin{matrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{matrix}\,.$$

We may iterate these splicing operations for various choices of pairs of adjacent columns, various values of $k$ and various choices of the $k$ rows containing the 1's to be moved back to the preceding column, with the following result, which is analogous to the well-known 'straightening lemma' for tableaux [12].

**Proposition 5.3** *([29, Theorem 1.10]) By iterated splicing operations, any block with weakly decreasing $\omega$-vector can be replaced by a formal sum of semistandard blocks with the same $\omega$-vector.* □

The proof of Theorem 5.2 is carried out by showing how the splicing operations can be realised by **hit equations** of the form $h = \sum_{i>0} Sq^i(f_i)$ using Steenrod operations. Unfortunately this procedure can no longer be carried out within the set of blocks with a fixed $\omega$-vector: we must deal with blocks with other $\omega$-vectors which arise as 'error terms'. The following observation (cf. [5, Lemma 2.2]) is basic to our work.

**Lemma 5.4** *For $\theta \in \mathcal{A}_2^+$ and a monomial $m \in \mathbf{P}$, if $m'$ is a monomial appearing in $\theta(m)$ then $\omega(m') \leq_l \omega(m)$ and $\leq_r \omega(m)$, and $\alpha(m') \leq \alpha(m)$.*

**Proof** This follows by an elementary argument from the formula $Sq^k(x^a) = \binom{a}{k}x^{a+k}$. □

In the example above, the block $B$ represents the monomial $m = t_1^2 t_2 t_3^2 t_4 t_5 = u \cdot Sq^2(v) \in \mathbf{P}^7$, where $u = t_2 t_4 t_5$, $v = t_1 t_3$. By the $\chi$-trick, $m \equiv \chi(Sq^2)u \cdot v \bmod \mathbf{H}^7$, and this polynomial is represented in block notation by $C$. In this case, the 2-splicing operation is realised exactly by an equation in $\mathbf{Q}^{(3,2)}$.

However things are not usually so simple as this. For example, if the $(5,2)$ entry of $B$ is changed from 0 to 1, then 2-splicing produces only the first block in $C$ (with the $(5,2)$ entry changed to 1). Now $u = t_2 t_4 t_5^3$, and so $\chi(Sq^2)u = Sq^2(u) = t_2^2 t_4^2 t_5^3 + t_2^2 t_4 t_5^4 + t_2 t_3^2 t_5^4$ has additional terms, resulting in additional terms in $\chi(Sq^2)u \cdot v$ which are not represented by the splicing operation. These terms are represented by 3-column blocks, and so have $\omega$-vector right lower than that of any 2-column block. The following result is Proposition 2.10 of [29].

**Proposition 5.5** *Let $C$ be a 2-column block of degree $d$ and let $C'$ be a $k$-splice of $C$. Then $C \equiv C' + \sum_i R_i$ mod $\mathbf{H}^d$, where $\omega_3(R_i) > 0$ for all $i$.* □

Stated informally, this result means that we can realise $k$-splicing operations on any 2-column block $C$ by Steenrod operations if we work modulo blocks with $\omega$-vector right lower than that of any 2-column block.

The next step is to implant a 2-column block as a pair of adjacent columns in a larger block. In this situation, the attempt to realise a splicing operation using the $\chi$-trick can result in further terms which are lower in the *left* order than the original block. Specifically, the following result is Proposition 2.12 of [29].

**Proposition 5.6** *Let $B = FCG$ be a vertical splitting of a block $B$, where $C$ is a 2-column block, and let $C'$ be a $k$-splice of $C$. Then*

$$B \equiv FC'G + \sum_i F_i H + \sum_j F K_j \bmod \mathbf{H}^d,$$

*where $\omega(F_i) <_l \omega(F)$ for all $i$ and $\omega(K_j) <_r \omega(CG)$ for all $j$.* □

Stated informally, this result means that we can realise $k$-splicing operations on any block $B$ by Steenrod operations if we work modulo blocks with $\omega$-vectors which are either lower than $\omega(B)$ in the left order or lower than $\omega(B)$ in the right order.

**Proof of Theorem 5.2** We now combine Proposition 5.6 with Theorem 5.3. For this we require one further observation. The $\alpha$-count of a block is not changed by the splicing operation, since 0's and 1's are merely exchanged between adjacent columns. Thus, in the notation of Proposition 5.6, if $B = FCG$ then $FC'G$ is a sum of blocks with the same $\alpha$-count as $B$. Further, any intermediate block $A$ formed by the first stage of a splicing operation on $B$ also has the same $\alpha$-count as $B$, since $A$ also differs from $B$ only by exchanges of 0's and 1's between adjacent columns. By Lemma 5.4 the $\alpha$-count of each 'left error term' $F'H$ or 'right error term' $FK$ which arises in Proposition 5.6 is $\leq \alpha(B)$. It follows that if we choose a total ordering $\leq$ on the set of $\omega$-vectors in degree $d = \deg(\omega)$ which refines the $\alpha$-filtration, then any monomial $m \in \mathbf{P}^\omega$ can be written as a sum of semi-standard monomials, a hit polynomial, and monomials with $\alpha$-count $\leq \alpha(m)$ and which have $\omega$-vectors $\rho$ which are either $\leq_l \omega$ or $\leq_r \omega$. Thus if we further assume that $\omega \geq \rho$ when $|\omega| = |\rho|$, $\mathbf{Q}^\omega$ is spanned by semi-standard monomials for the partition $\lambda = \omega'$ conjugate to $\omega$. □

Using Proposition 4.2 to rephrase the description of $\mathbf{K}^\omega$, we may combine Theorem 4.3 with the upper bound on $\mathbf{Q}^\omega$ given by Theorem 5.2, as follows.

**Theorem 5.7** *Let $\omega$ be strictly decreasing, with $\lambda = \omega'$, and let $\leq$ be an ordering as in Theorem 5.2. Then $\mathbf{Q}^\omega \cong \nabla(\lambda)$ as a $\overline{\mathbb{F}}_2\mathbf{G}$-module, with a $\overline{\mathbb{F}}_2$-basis given by the (equivalence classes of) semi-standard monomials. Dually, $\mathbf{K}^\omega \cong \Delta(\lambda)$ as a $\overline{\mathbb{F}}_2\mathbf{G}$-module, and $\mathbf{K}^\omega$ can be identified with the cyclic submodule of $\mathbf{D}^\omega$ generated by $s^*(\lambda) = u_1^{(2^{\lambda_1}-1)} \cdots u_n^{(2^{\lambda_n}-1)}$. In particular, $\dim \mathbf{Q}^\omega = \dim \mathbf{K}^\omega = \delta(\lambda)$.* □

We can say a little more if $\omega = \omega^{\min}$ or $\omega = \omega^{\max}$. In these cases, the left or right orderings can be used to obtain the same result $\mathbf{Q}^\omega \cong \nabla(\lambda)$. For $\omega = \omega^{\min}$, the left and right error terms arising in Proposition 5.6 are all hit, by results quoted above. For $\omega = \omega^{\max}$, for the left order we can use Proposition 5.9 to replace the right error terms arising in Proposition 5.6 by terms with $\omega$-vectors $\leq_l \omega^{\max}$. Thus we can realise the splicing operations by the Steenrod algebra action modulo left error terms only.

If $\omega(B)$ is non-decreasing, so that $\omega_j(B) < \omega_{j+1}(B)$ for some $j$, then the splicing operation whose first step switches all 1's in column $j+1$ which are opposite 0's in column $j$ replaces $B$ by zero. In this case only the left and right error terms will arise when we realise the splicing operation by the action of $\mathcal{A}_2$, with the following result [29, Proposition 2.13]. As in the proof of Theorem 4.3, these error terms can not have a higher $\alpha$-count than $B$. This leads to the following result for non-decreasing $\omega$-vectors.

**Proposition 5.8** *Let $B$ be a block such that $\omega = \omega(B)$ is non-decreasing. Then $B \equiv \sum_i F_i H + \sum_j F K_j$ mod $\mathbf{H}^d$, where $\omega(F_i) <_l \omega(F)$ for all $i$ and $\omega(K_j) <_r \omega(CG)$ for all $j$. If an ordering is used on $\omega$-vectors such that $\rho \leq \omega$ if $|\rho| \leq |\omega|$, then $\mathbf{Q}^\omega = 0$.* □

We use this to prove the following theorem of [21].

**Proposition 5.9** (M. F. Mothebe) *For all $n$, $d$ and $f \in \mathbf{P}^d$, $f \equiv \sum_i g_i$ mod $\mathbf{H}^d$, a sum of monomials with $\omega(g_i) \leq_l \omega^{\max}$ for each $i$.*

**Proof** We may assume that $\omega(f) >_l \omega^{\max}$. If the splicing operation in Proposition 5.8 is carried out for the first position $j$ where $\omega$ is non-decreasing, then $\rho = (\omega_1(f), \ldots, \omega_{j-1}(f))$ is weakly decreasing. The hypothesis on $f$ then implies that this initial part $\rho$ of $\omega(f)$ agrees with the corresponding part of $\omega^{\max}$. For the left error terms $F_i H$, it follows that $\omega(F_i H) <_l \omega^{\max}$. Thus $f$ can be expressed as the sum of a hit polynomial, monomials $g_i$ with $\omega(g_i) \leq_l \omega^{\max}$ and monomials $h_j$ with $\omega(h_j) <_r \omega(f)$. The result follows by induction on $<_r$. □

We remind the reader that the corresponding results hold if the field of scalars $\overline{\mathbb{F}}_2$ is replaced by the prime field $\mathbb{F}_2$. This is because the representation theory of $\mathbf{GL}(n)$ is the same over both fields, since $\mathbb{F}_2$ is a splitting field for $\mathbf{GL}(n)$, and because the action of the Steenrod algebra on $\mathbf{P}$ was obtained by extending its standard action on $\mathbb{F}_2$ by making it act trivially on $\overline{\mathbb{F}}_2$.

# 6 The case $\omega = (u, v)$

In this section we determine $\mathbf{Q}^\omega$ for $\omega = (u, v)$ using the right order $\leq_r$. Thus $\omega$-vectors of length 2 are linearly ordered and are higher than all $\omega$-vectors of length $\geq 3$. If $u \geq v > 0$, so that $\omega' = \lambda = (2^v, 1^{u-v})$, then the $\overline{\mathbf{G}}$-composition factors of the Weyl module $\overline{\Delta}(\lambda)$ are known from the work of James [16] on the decomposition matrices of the symmetric groups. The submodule structure was determined by A. M. Adamovich [1] and can be described as follows [19, Section 3]. Let $u - v + 1$ have binary expansion $\sum_{i \geq 0} a(i)\, 2^i$, and consider all sets of the form

$$I = [i_1, i_2) \cup \ldots \cup [i_{2t-1}, i_{2t})$$

with $i_1 < i_2 < \ldots < i_{2t}$ and $a(i_{2j-1}) = 1$, $a(i_{2j}) = 0$ for $1 \leq j \leq t$. (The case $t = 0$, $I = \emptyset$ is included, and corresponds to $\nu_i = \lambda$.) For each set $I$, let $\gamma(I) = \sum_i 2^i$, where the sum is over all $i \in I$ for which $a(i) = 0$ or $i = 2j - 1$, $1 \leq j \leq t$. Let $\nu_I$ be the partition conjugate to $(u + \gamma(I), v - \gamma(I))$. Then

- the Weyl module $\overline{\Delta}(\lambda)$ is multiplicity-free, i.e. no two composition factors of $\overline{\Delta}(\lambda)$ are isomorphic;
- the composition factors of $\overline{\Delta}(\lambda)$ are the simple $\overline{\mathbf{G}}$-modules $\overline{\mathbf{L}}(\nu_I)$ such that $\gamma(I) \leq \min(v, n - u)$;
- if we write $\nu_I \succeq \nu_J$ to mean that every submodule containing $\overline{\mathbf{L}}(\nu_I)$ also contains $\overline{\mathbf{L}}(\nu_J)$, then $\nu_I \succeq \nu_J$ if and only if $I \subseteq J$.

**Proposition 6.1** *For the order $<_r$ on $\omega$-vectors*

$$\mathbf{Q}^{(u,v)} \cong \begin{cases} \nabla(2^v 1^{u-v}), & \text{if } u \geq v \\ 0, & \text{if } u < v. \end{cases}$$

**Proof** **(i)** If $u > v$, this is a special case of Theorem 5.7, so let $u = v$. In this case, Adamovich's theorem shows that $\overline{\Delta}(2^u)$ is uniserial as a $\overline{\mathbf{G}}$-module, with top composition factor $\overline{\mathbf{L}}(2^u) = (\Lambda^u)^{(1)}$, the Frobenius twist of the $u$th exterior power of $V$, and all other composition factors are of the form $\overline{\mathbf{L}}(\mu)$ with $\mu$ column 2-regular. Thus on restriction to $\mathbf{G}$, the top factor $\overline{\mathbf{L}}(2^u)$ becomes $\Lambda^u$, and the other factors $\overline{\mathbf{L}}(\mu)$ become simple $\overline{\mathbb{F}}_2\mathbf{G}$-modules $\mathbf{L}(\mu)$. We denote by $\overline{M}$ the maximal submodule of $\overline{\Delta}(2^u)$, so that its restriction $M$ to $\mathbf{G}$ is the maximal submodule of $\Delta(2^u)$. We consider $\overline{\Delta}(2^u)$ as the $\overline{\mathbf{G}}$-submodule of $\overline{\mathbf{P}}^{(u,u)} \cong \Lambda^u \otimes \Lambda^u$ generated by $s(2^u) = t_1^3 \cdots t_n^3$.

By Adamovich's theorem, $\overline{M}$ is an indecomposable $\overline{\mathbf{G}}$-module with unique simple quotient $\overline{\mathbf{L}}(2^{u-1}1^2)$. It follows from the universal property of Weyl modules that $\overline{M}$ is a quotient module of $\overline{\Delta}(2^{u-1}1^2)$. Since the partition $(2^{u-1}1^2)$ is column 2-regular, it follows from Wong's theorem (see Section 3) that $\Delta(2^{u-1}1^2)$ is indecomposable with unique simple quotient $\mathbf{L}(2^{u-1}1^2)$. As a quotient module of $\Delta(2^{u-1}1^2)$, $M$ also has these properties.

Thus we need only show that a generator of $M$ lies in the $\overline{\mathbb{F}}_2\mathbf{G}$-submodule of $P^{(u,u)}$ generated by $s(2^u)$. Let $f = t_1^3 \cdots t_{u-1}^3 (t_u^2 t_{u+1} + t_u t_{u+1}^2)$. Then $f$ is the image of $s(2^u)$ under the transvection which maps $t_u$ to $t_u + t_{u+1}$ and fixes $t_i$ for $i \neq u$. Since $Sq^1 s(2^{u-1}1^2) = f + g$, where $\omega(g) = (u, u-2, 1) <_r (u, u)$, the submodule of $\Delta(2^{u-1}1^2)$ generated by $f$ has top composition factor $\overline{\mathbf{L}}(2^{u-1}1^2)$. Hence $f$ generates $M$ as a $\overline{\mathbb{F}}_2\mathbf{G}$-module, and it follows that $s(2^u)$ generates $\Delta(2^u)$.

Using Proposition 4.2 to identify the Weyl module $\Delta(2^u)$ with the submodule of $\mathbf{D}^{(u,u)}$ generated by $s^*(2^u)$, we see that $\Delta(2^u)$ is in the Steenrod kernel $\mathbf{K}^{(u,u)}$. By duality, the dual Weyl module $\nabla(2^u)$ with socle generated by $s(2^u)$ is in $\mathbf{Q}^{(u,u)}$. Finally, by Proposition 5.5, $\mathbf{Q}^{(u,u)}$ is spanned by semistandard blocks, since we are using the right order on $\omega$-vectors. Hence $\mathbf{Q}^{(u,u)} = \nabla(2^u)$.

**(ii)** Let $u < v$, and let $k = v - u$. By Proposition 5.5, any $k$-splice operation at column 2 can be realised by a hit equation which can only have right errors, so lowering the $\omega$-vector below $(u, v)$ in the right order. If we choose $k = v - u$, the result of the splicing operation is zero, and hence any block with $\omega = (u, v)$ is hit modulo blocks whch are lower in the right order. □

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