Flag modules and the hit problem for the Steenrod algebra

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Abstract

The 'hit problem' of F. P. Peterson in algebraic topology asks for a minimal generating set for the polynomial algebra $P(n) = \mathbb{F}_2[x_1, \ldots, x_n]$ as a module over the Steenrod algebra $\mathcal{A}_2$. An equivalent problem is to find a $\mathbb{F}_2$-basis for the subring $K(n)$ of elements f in the dual Hopf algebra $D(n)$, a divided power algebra, such that $Sq_k(f) = 0$ for all $k > 0$. The Steenrod kernel $K(n)$ is a $\mathbb{F}_2 GL(n, \mathbb{F}_2)$-module dual to the quotient $Q(n)$ of $P(n)$ by the hit elements $\mathcal{A}_2^+ P(n)$. A submodule $S(n)$ of $K(n)$ is obtained as the image of a family of maps from the permutation module $Fl(n)$ of $GL(n, \mathbb{F}_2)$ on complete flags in an n-dimensional vector space V over $\mathbb{F}_2$. We use the Schubert cell decomposition of the flags to calculate $S(n)$ in degrees $d = \sum_{i=1}^{n}(2^{\lambda_i} - 1)$, where $\lambda_1 > \lambda_2 > \cdots > \lambda_n \geq 0$. When $\lambda_n = 0$, we define a $\mathbb{F}_2 GL(n, \mathbb{F}_2)$-module map $\delta : Q^d(n) \to Q^{2d+n-1}(n)$ analogous to the well-known isomorphism $Q^d(n) \to Q^{2d+n}(n)$ of M. Kameko. When $\lambda_{n-1} \geq 2$, we show that δ is surjective and $\delta^* : S^{2d+n-1}(n) \to S^d(n)$ is an isomorphism.

1 Introduction

The 'hit problem' of F. P. Peterson [15] asks for a minimal generating set for the polynomial algebra $P(n) = \mathbb{F}_2[x_1, \ldots, x_n]$ as a module over the Steenrod algebra $\mathcal{A}_2$. Thus we wish to calculate the quotient $Q(n)$ of $P(n)$ by the elements in the image of positively graded Steenrod operations $\mathcal{A}_2^+$, known informally as the *hit elements*. Since the left action of the Steenrod algebra $\mathcal{A}_2$ commutes with the right action of $GL(n) = GL(n, \mathbb{F}_2)$ by linear substitutions, $Q(n)$ is a $\mathbb{F}_2 GL(n)$-module. There are two parts to the problem, to find lower bounds on $Q(n)$, and to find upper bounds. For background and references, see [21, Section 7].

For the lower bound problem, our results include the following.

Theorem 1.1. *In degrees $d = \sum_{i=1}^{n}(2^{\lambda_i} - 1)$, where $\lambda_i - \lambda_{i+1} \geq 2$ for $1 \leq i \leq n-1$ and $\lambda_n \geq 0$, $Q^d(n)$ has a quotient isomorphic to $Fl(n)$, the permutation module of $GL(n)$ on the set of complete flags in a vector space of dimension n over $\mathbb{F}_2$. Thus* $\dim Q^d(n) \geq 1 \cdot 3 \cdot 7 \cdots (2^n - 1)$.

We call d 2-*dominant* if it satisfies these conditions. Thus the minimal 2-dominant degree is $d = \sum_{k=1}^{n-1}(4^k - 1) = (4^n - 1)/3 - n$. More generally, we obtain a lower bound for $Q^d(n)$ in terms of 2-multinomial coefficients when $\lambda_i - \lambda_{i+1} \geq 1$ for $1 \leq i \leq n-1$ and $\lambda_n \geq 0$, when we call d 1-*dominant*. We show that $Q^d(n)$ has a quotient which is isomorphic to a certain direct summand of $Fl(n)$, and give the following lower bound on its dimension. Given positive integers $c_1, \ldots, c_r$ with sum n, we define the corresponding 2-multinomial coefficient by

$$\binom{n}{c_1, \ldots, c_r}_2 = \frac{(n!)_2}{(c_1!)_2 \cdots (c_r!)_2},$$

where $(k!)_2 = 1 \cdot 3 \cdot 7 \cdots (2^k - 1)$ for $k \geq 0$.

Theorem 1.2. *Let $d = \sum_{i=1}^{n}(2^{\lambda_i} - 1)$, where $\lambda_i - \lambda_{i+1} \geq 1$ for $1 \leq i \leq n-1$ and $\lambda_n \geq 0$. Break $\lambda = (\lambda_1, \ldots, \lambda_n)$ into sections of lengths $c_1, \ldots, c_r$ so that λ_{i-1} and λ_i are in the same section if and only if $\lambda_{i-1} - \lambda_i = 1$. Then*

$$\dim Q^d(n) \geq \binom{n}{c_1, \ldots, c_r}_2 \cdot \prod_{k=1}^{r} 2^{c_k(c_k-1)/2}.$$

In the 2-dominant case, $(c_1, c_2, \ldots, c_n) = (1, 1, \ldots, 1)$ and the dimension is $(n!)_2$. In the minimal 1-dominant case $d = \sum_{k=1}^{n-1}(2^k - 1) = 2^n - 1 - n$, $c_1 = n$ and the dimension is $2^{n(n-1)/2}$. In this case, $Q^d(n) \cong St$, the Steinberg module for $GL(n)$.

Our method is based on the work of Crabb and Hubbuck [5]. By regarding the vector space V of dimension n over $\mathbb{F}_2$ as the group $(\mathbb{Z}/2)^n$, the polynomial algebra $P(n)$ is identified with the cohomology algebra $H^*(BV)$ over $\mathbb{F}_2$. The homology $H_*(BV)$ is the dual of $P(n)$ as a Hopf algebra. In characteristic 2, it carries the additional structure of an algebra with *divided powers* $v \mapsto v^{(r)}$ for $v \in V$ and $r \geq 0$, which reduce to the operations $v \mapsto v^r/r!$ in characteristic zero. We denote this algebra by $D(n)$, and treat a basis $v_1, \ldots, v_n$ of V as a set of variables dual to the variables $x_1, \ldots, x_n$ for $P(n)$. We use the same conventions to describe elements of $D(n)$ and $P(n)$, so that elements of $D(n)$ are 'polynomials' which are sums of 'monomials' $v_1^{(r_1)} \cdots v_n^{(r_n)}$ dual to $x_1^{r_1} \cdots x_n^{r_n}$. The Steenrod operation $Sq^k : P^d(n) \to P^{d+k}(n)$ gives by duality an operation $Sq_k : D^{d+k}(n) \to D^d(n)$, and the dual form of the hit problem is to calculate the *Steenrod kernel* $K(n)$, the $\mathbb{F}_2 GL(n)$-submodule of $D(n)$ given by all polynomials f such that $Sq_k(f) = 0$ for all $k > 0$.

The dual approach has the advantage that $K(n)$ is a subring of $D(n)$. This fact is exploited as follows in [5] and [16] to give a lower bound $S(n)$ for $K(n)$, and hence for $Q(n)$. It is easy to show that $K(1)$ is the $\mathbb{F}_2$-vector space spanned by $1, v, v^{(3)}, v^{(7)}, \ldots$, where v is the nonzero element in degree 1, and hence that $K(n)$ contains all monomials whose exponents are all of the form $2^k - 1$. It is easy to see that these are the only monomials which lie in $D(n)$. The corresponding monomials in $P(n)$ are characterised by the property that they do not occur in the image of any operation Sq^k for $k > 0$. Following Singer [18] we call these monomials *spikes*. We denote by $S(n)$ the $\mathbb{F}_2GL(n)$-submodule of $K(n)$ generated by the spikes. Since any product of more than n monomials of odd degree in $D(n)$ is zero, $S(n)$ is also a subring of $K(n)$. Since $S(n)$ is generated as a ring by the (2^k-1)th divided powers of elements of degree 1, it is called the 'ring of lines' in [5]. Our notation emphasises the relation to the spike monomials, and so we call $S(n)$ the *spike module*. There is a natural family of $\mathbb{F}_2GL(n)$-module maps $\phi^\omega : Fl(n) \to S(n)$, where $\omega = (\omega_1, \ldots, \omega_r)$ is a partition with $\omega_1 \leq n$. Given a flag W in V, we call $\phi^\omega(W)$ a *flag polynomial*. We prove Theorems 1.1 and 1.2 by finding linearly independent sets of flag polynomials in 2- or 1-dominant degrees respectively. Tran Ngoc Nam [13] has extended the results of [5] in another direction, and in particular has given a different proof of Theorem 1.1.

The more difficult direction of the hit problem is that of finding upper bounds for $Q(n)$. Our main result on the upper bound problem is as follows.

Theorem 1.3. *For $d = \sum_{i=1}^{n-1}(2^{\lambda_i}-1)$, where $\lambda_i - \lambda_{i+1} \geq 1$ for $1 \leq i < n-1$ and $\lambda_{n-1} \geq 2$, there is an epimorphism $\delta : Q^d(n) \to Q^{2d+n-1}(n)$ of $\mathbb{F}_2GL(n)$-modules.*

Recall [20] that $Q^d(n) \neq 0$ if and only if d is sum of at most n integers of the form 2^k-1, so that $P^d(n)$ contains a spike. Kameko's map $P^d(n) \to P^{2d+n}(n)$, defined on monomials by $f \mapsto x_1 \cdots x_n f^2$, induces an isomorphism $Q^d(n) \cong Q^{2d+n}(n)$ when the target degree $2d+n$ is the sum of a *minimum* of n such integers. This reduces the hit problem to the case of degrees d of the form $d = \sum_{i=1}^{n-1}(2^{\lambda_i}-1)$, where $\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq 0$. Our hypothesis on d in Theorem 1.3 is stronger than 1-dominance, but weaker than 2-dominance. The *duplication map* δ is analogous to Kameko's map: if the square-free part of f is the product $\overline{x_i} = x_1 \cdots \widehat{x}_i \cdots x_n$ of $n-1$ of the n variables, then $\delta f = \overline{x_i} f^2$, and otherwise $\delta f = 0$. Thus δ is nonzero only if d and n have opposite parity. In contrast, after one iteration of Kameko's map, d and n have the same parity.

To put Theorem 1.3 in context, we summarise some known results on the upper bound problem. The 1-variable case is elementary, and the 2-variable case was solved by Peterson in his original paper [15]. The pioneering work of Kameko solved the 3-variable case [9, 10]. In 'generic' degrees d, i.e. $d = \sum_{i=1}^{n}(2^{\lambda_i}-1)$, where $\lambda_i - \lambda_{i+1} \geq i+1$, $1 \leq i \leq n-1$, Nam has proved [12] that the lower bound of [5] is exact, so that $\dim Q^d(n) = 1 \cdot 3 \cdot 7 \cdots (2^n-1)$. The 4-variable case has recently been completely solved by Nguyen Sum [19], following computations of Kameko [11]. Computations for small n suggest that δ is an isomorphism under the given conditions on d, but we have been unable to prove this. The hypothesis of Theorem 1.3 cannot be weakened to 1-dominance, since $\delta : Q^8(3) \to Q^{18}(3)$ is neither injective nor surjective. The polynomial $f = xyz^6 + xy^6z + x^6yz$ represents a nonzero $GL(3)$-invariant in the 15-dimensional space $Q^8(3)$, but $\delta f = 0$. In fact the image of δ in this case is a 14-dimensional subspace of the 21-dimensional space $Q^{18}(3)$. Interestingly, $\dim S^8(3) = 14$ and $\delta^* : S^{18}(3) \to S^8(3)$ is surjective.

The rest of this paper is organised as follows. In Section 2, we associate certain monomials in $D(n)$ and $P(n)$ with the Schubert cell decomposition of the flags in V, and reduce the proof of Theorem 2.8, the computation of $S^d(n)$ in the 2-dominant case, to the combinatorial results of Section 3. In Section 4, we split $Fl(n)$ into indecomposable summands, and use this to prove Theorem 4.12, the computation of $S^d(n)$ in the 1-dominant case. Theorem 1.1 follows from Theorem 2.8, and Theorem 1.2 from Theorem 4.12, by dualizing back from $D(n)$ to $P(n)$. In Section 5, we introduce the duplication map δ , and prove Theorem 1.3, which is restated as Theorem 5.14.

2 Schubert cells and S and Z blocks

2.1 The divided power algebra $D(n)$

Let V be a vector space of dimension n over $\mathbb{F}_2$, with basis $v_1, \ldots, v_n$, and let $x_1, \ldots, x_n$ be the dual basis for V^*. Using matrix transposition $\gamma \mapsto \gamma^{\text{tr}}$ for $\gamma \in GL(n)$, the group $GL(n)$ acts on the right of both V and V^* by linear substitutions: if $\gamma = (a_{i,j})$, then $v_i \cdot \gamma = \sum_{j=1}^n a_{i,j} v_j$ and $x_i \cdot \gamma = \sum_{j=1}^n a_{i,j} x_j$, so that $\langle x \cdot \gamma, v \rangle = \langle x, v \cdot \gamma^{\text{tr}} \rangle$, where $v \in V$ and $x \in V^*$.

The action of $GL(n)$ on V^* extends to an action on the polynomial algebra $P(n) = \mathbb{F}_2[x_1, \ldots, x_n]$ by linear substitution of the variables. Let $P(n) = \sum_{d \geq 0} P^d(n)$ be the usual grading by degree, so that $P^1(n) = V^*$. As a Hopf algebra over $\mathbb{F}_2$, with the usual coproduct $x^k \mapsto \sum_{i+j=k} \binom{k}{i} x^i \otimes x^j$ for $k \geq 0$, the elements $x_i^{2^j}$ with $1 \leq i \leq n$ and $j \geq 0$ form a basis for the primitive elements of $P(n)$. Let $D(n)$ be the graded dual of $P(n)$, so that $D^1(n)$ is identified with V. The action of $GL(n)$ on V extends to an action on $D(n)$ by linear substitution, i.e. $\langle f \cdot \gamma, g \rangle = \langle f, g \cdot \gamma^{\text{tr}} \rangle$, where $f \in P(n)$, $g \in D(n)$. Thus $D^d(n)$ is the transpose dual of $P^d(n)$ as a right $\mathbb{F}_2 GL(n)$-module.

The algebra $D(n)$ is a **divided power algebra** over $\mathbb{F}_2$, i.e. for each $k \geq 0$ there is a k th divided power map $V \to D^k(n)$, denoted by $v \mapsto v^{(k)}$, such that, for all $r, s \geq 0$ and $v \in V$, $v^{(r)} v^{(s)} = \binom{r+s}{r} v^{(r+s)}$. In the 1-variable case $D(1) = D[v]$, $v^{(r)}$ denotes the element dual to x^r, so that $\langle x^r, v^{(r)} \rangle = 1$. Thus $v^{(0)} = 1$ and $v^{(1)} = v$. Since $D(n) \cong D[v_1] \otimes \cdots \otimes D[v_n]$, the products $v_1^{(r_1)} \cdots v_n^{(r_n)}$ form a $\mathbb{F}_2$-basis for $D(n)$, dual to the monomial basis in $x_1, \ldots, x_n$ for $P(n)$, and we call such a product a **monomial** in $D(n)$. As an algebra, $D(n)$ is the tensor product of the exterior algebras generated by the indecomposable elements $v_i^{(2^j)}$, $1 \leq i \leq n$, $j \geq 0$, dual to the primitive elements of $P(n)$. We define divided powers on V so that the 'divided binomial theorem'

$$(u + v)^{(r)} = \sum_{s+t=r} u^{(s)} v^{(t)} \tag{1}$$

holds for $r \geq 0$ and $u, v \in V$. The map $v \mapsto v^{(r)}$ then commutes with the action of $GL(n)$, and the coproduct in $D(n)$ is given by $v^{(r)} \mapsto \sum_{s+t=r} v^{(s)} \otimes v^{(t)}$, where $r \geq 0$ and $v \in V$.

2.2 The spike module $S(n)$

When f is either of the dual monomials $x_1^{r_1} \cdots x_n^{r_n} \in P(n)$ or $v_1^{(r_1)} \cdots v_n^{(r_n)} \in D(n)$, let $\omega_j(f)$ be the number of exponents r_i whose jth base 2 digit is 1. The **ω-vector** of f is $\omega(f) = (\omega_1(f), \ldots, \omega_c(f))$, and has **degree** $\deg \omega = \deg f = \sum_{j=1}^{c} 2^{j-1}\omega_j$. We also write $\omega(f) = \omega$ when f is a polynomial whose terms all have the same ω-vector ω. For $d \geq 0$, we order ω-vectors of degree d lexicographically. The set of monomials $f \in P^d(n)$ such that $\omega(f) \leq \omega$ spans a $\mathbb{F}_2GL(n)$-submodule of $P^d(n)$, and the set of monomials $f \in D^d(n)$ such that $\omega(f) \geq \omega$ spans a $\mathbb{F}_2GL(n)$-submodule of $D^d(n)$.

Following [18], we call a monomial s in $P(n)$ or $D(n)$ with all exponents of the form $2^k - 1$ a **spike**. Spikes play a fundamental role in the hit problem. In $P(n)$, they are the monomials which can not appear in a polynomial which is in the image of a Steenrod operation of positive degree, while in $D(n)$ they are the monomials which are in the kernel of all such operations. The $\mathbb{F}_2GL(n)$-submodule $S(n)$ of $D(n)$ generated by the set of all spikes has been studied by M. C. Crabb and J. R. Hubbuck [5] and by J. Repka and P. S. Selick [16]. Here we call $S(n)$ the **spike module**.

When $\omega_j \geq \omega_{j+1}$ for all j, then there is a spike s with $\omega(s) = \omega$, and so we say that ω is of **spike type**. In this case, ω can also be regarded as a partition, with conjugate partition λ, which we denote here by ω^{tr}, and $\omega(s) = \omega$ where $s = \prod_{i=1}^{n} x_i^{2^{\lambda_i}-1}$. Thus $\deg \omega = \sum_{i=1}^{n}(2^{\lambda_i} - 1)$. If $\lambda_i - \lambda_{i+1} \geq 1$ for $1 \leq i \leq n-1$ and $\lambda_n \geq 0$, then $\omega = \lambda^{\mathrm{tr}}$ is the unique ω-vector of spike type with degree d, and we call d, and ω, **1-dominant**. When d is 1-dominant, $d' = 2d + n$ is also 1-dominant, with $\omega' = (n, \omega)$, and Kameko's map defined on monomials by $f \mapsto x_1 \cdots x_n f^2$ induces an isomorphism $Q^d(n) \cong Q^{d'}(n)$ [10, 11]. For 1-dominant degrees, we may therefore restrict attention to the case $\lambda_n = 0$.

Following Crabb and Hubbuck [5], let $W : W_1 \subset W_2 \subset \cdots \subset W_{n-1}$ be a complete flag in V. A basis $w_1, \ldots, w_n$ for V is **adapted to** W if $w_1, \ldots, w_i$ is a basis for W_i for $1 \leq i \leq n-1$. We denote by $Fl(n)$ the $\mathbb{F}_2GL(n)$-module given by permutations of the complete flags W in V. Then $\phi^\omega : Fl(n) \longrightarrow D^d(n)$, where $d = \sum_{i=1}^{n}(2^{\lambda_i} - 1)$, is defined by

$$\phi^\omega(W) = \prod_{i=1}^{n} w_i^{(2^{\lambda_i}-1)}.$$

The same product is obtained with any basis of V which is adapted to W, and so ϕ^ω is well-defined. We call $\phi^\omega(W)$ a **flag polynomial**. In particular, every spike is a flag polynomial, and when d is 1-dominant with $\omega(s) = \omega$ for all spikes s of degree d, $S^d(n) = \mathrm{Im}(\phi^\omega)$.

2.3 Schubert cells

We identify a complete flag W with a right coset $L(n)\gamma$ of the lower triangular subgroup $L(n)$ of $GL(n)$, as follows. Given $\gamma = (a_{i,j}) \in GL(n)$, let $w_i = \sum_{j=1}^{n} a_{i,j} v_j \in V$, so that the coordinate vector of w_i is the ith row of γ, and let $w_1, \ldots, w_i$ be a basis for the subspace W_i. We associate to γ the flag $W: W_1 \subset W_2 \subset \cdots \subset W_{n-1}$ in V, which depends only on $L(n)\gamma$. In this way, we obtain a bijection between complete flags in V and right cosets of $L(n)$, which we use to identify $Fl(n)$ with the permutation module $\mathbb{F}_2(GL(n)/L(n))$ induced from the trivial 1-dimensional module for $L(n)$. Thus $\dim Fl(n) = [GL(n) : L(n)] = 1 \cdot 3 \cdot 7 \cdots (2^n - 1)$.

By choosing a basis for V, the complete flags in V may be divided into Schubert cells $\mathrm{Sch}(\rho)$ indexed by permutations ρ of $\{1, 2, \ldots, n\}$ [6, Section 10.2]. We write permutations in 'one-line' notation, i.e. $\rho = (\rho(1), \ldots, \rho(n))$. Each coset $L(n)\gamma$ contains a unique matrix γ_W which is **row-reduced**, in the sense that every entry below the last 1 in each row is 0. Thus we have a bijection between complete flags W and row-reduced matrices γ_W, and it is convenient to use these matrices in calculations with flags. The **Schubert cell** $\mathrm{Sch}(\rho)$ is the set of flags W such that for $1 \leq i \leq n$ the last 1 in row i of γ_W is in column $\rho(i)$, so that $GL(n)/L(n) = \coprod_{\rho \in \Sigma(n)} \mathrm{Sch}(\rho)$.

Example 2.1. For $n = 4$, $\rho = (2, 4, 3, 1)$, flags $W \in \mathrm{Sch}(\rho)$ correspond to the following matrices γ_W, where the stars stand for either 0 or 1 in $\mathbb{F}_2$.

$$\begin{pmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \\ * & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

The Schubert cells have a natural partial order, the **Bruhat order** [6]. Each permutation ρ of $\{1, 2, \ldots, n\}$ can be expressed as a product of the transpositions $\sigma_1, \ldots, \sigma_{n-1}$, where σ_i exchanges i with $i+1$. In the Bruhat order, ρ covers σ if deletion of some σ_i from a minimal word for ρ gives a minimal word for σ. The **length** $\mathrm{len}(\rho)$ is the minimum length of such an expression, and can alternatively be defined as the number of pairs (i, j) which are reversed by ρ, i.e. $i < j$ and $\rho(i) > \rho(j)$. For $W \in \mathrm{Sch}(\rho)$, the $(i, \rho(j))$th entry of γ_W is arbitrary for these pairs (i, j). Thus $\mathrm{Sch}(\rho)$ contains $2^{\mathrm{len}(\rho)}$ flags. For the identity permutation $\iota = (1, 2, \ldots, n)$, $\mathrm{Sch}(\iota)$ contains only the reference flag, while for the reversal $\rho_0 = (n, n-1, \ldots, 2, 1)$, $\mathrm{Sch}(\rho_0)$ contains $2^{n(n-1)/2}$ flags.

The set $\phi^\omega(\mathrm{Sch}(\rho))$ can be described as follows. There is a unique $W \in \mathrm{Sch}(\rho)$ such that γ_W is a permutation matrix. Let $\pi(\rho)$ be the matrix obtained by applying the permutation ρ to the rows of the identity matrix, so that $\rho \mapsto \pi(\rho)$ gives an isomorphism of the symmetric group $\Sigma(n)$ with the permutation matrices in $GL(n)$. Then $\gamma_W = \pi(\rho^{-1})$ is a row-reduced matrix giving a flag $W \in \mathrm{Sch}(\rho)$, and the flags in $\mathrm{Sch}(\rho)$ are the cosets $L(n)\gamma$ where $\gamma \in \gamma_W L(n)$.

The corresponding flag polynomial $\phi^\omega(W)$ is the spike $s_\omega(\rho) = v_1^{(r_1)} \cdots v_n^{(r_n)}$ with exponents $r_{\rho(i)} = 2^{\lambda_i} - 1$, where $\lambda = \omega^{\rm tr}$. The set $\phi^\omega(\mathrm{Sch}(\rho))$ is the orbit of $s_\omega(\rho)$ under the subgroup $L(\rho)$ of $L(n)$ obtained by permuting the rows of the row-reduced matrices γ_W, where $W \in \mathrm{Sch}(\rho)$. We also denote by $U(\rho)$ the transpose of $L(\rho)$ in the upper triangular group $U(n)$. The (i,j)th entry of matrices in $U(\rho)$ is arbitrary when $i < j$ and i appears later than j in $(\rho(1), \ldots, \rho(n))$.

Example 2.2. For $n = 4$, $\rho = (2,4,3,1)$, matrices in $L(\rho)$ and $U(\rho)$ have the forms

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \\ * & 0 & * & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

2.4 Spike and zip monomials

We use the notation of $(0,1)$**-blocks** to represent monomials f in $P(n)$ and $D(n)$. In this notation, the reverse binary expansion of the exponent in f of the ith variable x_i or v_i forms the ith row of the block F representing f. Thus $\omega(f)$ is the vector of column sums of the block F. We use matching pairs of lower and upper case Roman letters to denote monomials and the corresponding blocks. We right justify blocks by filling rows with zeros, and sometimes we add extra columns of zeros on the right.

When ω is of spike type in degree d with $\lambda = \omega^{\rm tr}$, spikes $s^\omega(\rho) = x_1^{r_1} \cdots x_n^{r_n} \in P^d(n)$ and $s_\omega(\rho) = v_1^{(r_1)} \cdots v_n^{(r_n)} \in D^d(n)$ are defined for each permutation ρ, where $r_{\rho(i)} = 2^{\lambda_i} - 1$. The spikes $s^\omega(\rho)$, $\rho \in \Sigma(n)$, are all distinct if and only if ω is 1-dominant, so that $\lambda = \omega^{\rm tr}$ has no repeated parts.

Example 2.3. Let $n = 4$ and $\omega = (3,2,1)$. For the identity permutation $\iota = (1,2,3,4)$, $s^\omega(\iota) = x_1^7 x_2^3 x_3$ and $s_\omega(\iota) = v_1^{(7)} v_2^{(3)} v_3$, and for the reversal permutation $\rho_0 = (4,3,2,1)$, $s^\omega(\rho_0) = x_2 x_3^3 x_4^7$ and $s_\omega(\rho_0) = v_2 v_3^{(3)} v_4^{(7)}$. In block notation

$$S(\iota) = \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}, \quad S(\rho_0) = \begin{matrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{matrix}.$$

When ω is 1-dominant of degree d and $\rho \in \Sigma(n)$, we define **zip monomials** $z^\omega(\rho) \in P^d(n)$ and $z_\omega(\rho) \in D^d(n)$ as follows. Let ω' be the subvector of ω whose components are the first occurrences in ω of $n-1, n-2, \ldots, 1$, and let ω'' be the complementary subvector. We call the subblocks F' and F'' formed by the corresponding columns of a block F the **leading** and **trailing** parts of F respectively. To obtain the zip block $Z(\rho)$, we modify the spike block $S(\rho)$ by replacing its leading part $S'(\rho)$ by $S'(\iota)$, leaving its trailing part $S''(\rho)$ unchanged.

Since ω'' is 1-dominant if and only if ω is 2-dominant, we obtain a full complement of $n!$ distinct zip monomials $z^\omega(\rho)$ if and only if ω is 2-dominant.

Example 2.4. Let $n = 4$, $\omega = (3,3,2,2,1,1)$ and $\rho = (4,3,2,1)$. Then $z^\omega(\rho) = x_1^{21}x_2^7x_3^{11}x_4^{42}$, $z_\omega(\rho) = v_1^{(21)}v_2^{(7)}v_3^{(11)}v_4^{(42)}$, and

$$S(\rho) = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{matrix}, \quad Z(\rho) = \begin{matrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{matrix}.$$

If ω is not 2-dominant, then the flag polynomials $\phi^\omega(W)$ are linearly dependent. For this we need a calculation in $D(n)$.

Lemma 2.5. *For $u, v \in V$ and $k \geq 1$,*

$$u^{(2^{k+1}-1)}v^{(2^k-1)} = v^{(2^{k+1}-1)}u^{(2^k-1)} + (u+v)^{(2^{k+1}-1)}u^{(2^k-1)}.$$

Proof By (1), $(u+v)^{(2^{k+1}-1)} = \sum_{i+j=2^{k+1}-1} u^{(i)}v^{(j)}$. Now $u^{(i)}u^{(2^k-1)} = 0$ unless 2^k divides i, so that $i = 0$ or $i = 2^k$. The $i = 0$ term gives $v^{(2^{k+1}-1)}u^{(2^k-1)}$, and since $\binom{2^{k+1}-1}{2^k-1}$ is odd, the $i = 2^k$ term gives $u^{(2^{k+1}-1)}v^{(2^k-1)}$. □

This yields the following dependence relations between polynomials for flags in $\mathrm{Sch}(\rho)$ and $\mathrm{Sch}(\sigma)$, where σ covers ρ in the Bruhat order.

Proposition 2.6. *Let λ be a partition with $\lambda_i - \lambda_{i+1} = 1$ for some i, and let $\omega = \lambda^{\mathrm{tr}}$. Let $w_1, \ldots, w_i, w_{i+1}, \ldots, w_n$ be a basis for V adapted to the flag W, and let X, Y be the corresponding flags for the bases $w_1, \ldots, w_{i+1}, w_i, \ldots, w_n$ and $w_1, \ldots, w_{i+1} + w_i, w_i, \ldots, w_n$. Then $\phi^\omega(W) = \phi^\omega(X) + \phi^\omega(Y)$.* □

Example 2.7. Let $n = 4$, $\rho = (2,4,3,1)$ and $\sigma = (4,2,3,1)$. Writing matrices γ_W etc. in place of flags,

$$\phi^\omega\begin{pmatrix} a & 1 & 0 & 0 \\ b & 0 & c & 1 \\ d & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \phi^\omega\begin{pmatrix} b & 0 & c & 1 \\ a & 1 & 0 & 0 \\ d & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \phi^\omega\begin{pmatrix} a+b & 1 & c & 1 \\ a & 1 & 0 & 0 \\ d & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

2.5 $S(n)$ in 2-dominant degrees

The following is our main theorem for 2-dominant ω. Recall that for $\mathrm{Sch}(\rho)$ flag polynomials in $D(n)$ have the form $s_\omega(\rho) \cdot \gamma$, where $\gamma \in L(\rho)$. We call the polynomials $z^\omega(\rho) \cdot \gamma$, where $\gamma \in U(\rho)$, **dual flag polynomials** for $\mathrm{Sch}(\rho)$.

Theorem 2.8. *Let ω be 2-dominant, with* $\deg \omega = d$. *Then*

(i) *$\phi^\omega : Fl(n) \longrightarrow D^d(n)$ is injective, so that $S^d(n) \cong Fl(n)$. Equivalently, the flag polynomials $s_\omega(\rho) \cdot \gamma$ are linearly independent for $\rho \in \Sigma(n)$, $\gamma \in L(\rho)$.*

(ii) *The dual flag polynomials $z^\omega(\rho) \cdot \gamma$, for $\rho \in \Sigma(n)$ and $\gamma \in U(\rho)$, are linearly independent in $Q^d(n)$, i.e. no nonzero linear combination of them is hit.*

In the remainder of this section, we reduce the proof of Theorem 2.8 to a number of combinatorial results, which are proved in Section 3. The proof is by recursion over the Schubert cells, using any total order which refines the Bruhat order. For $D(n)$, we begin with the largest Schubert cell $\mathrm{Sch}(\rho_0)$ and end with $\mathrm{Sch}(\iota)$. For $P(n)$, the recursion proceeds in the opposite direction, starting with the smallest Schubert cell $\mathrm{Sch}(\iota)$ and ending with $\mathrm{Sch}(\rho_0)$. At each step, we simultaneously remove from a possible linear dependence relation all terms arising from one of the Schubert cells.

Proof of Theorem 2.8 We begin with the case of $D(n)$. The flag polynomials for $\mathrm{Sch}(\rho)$ form the orbit of the spike $s_\omega(\rho)$ under the right action of the group $L(\rho)$. The vector space $D(\rho)$ spanned by these flag polynomials is the cyclic $\mathbb{F}_2 L(\rho)$-module generated by $s_\omega(\rho)$, and so is isomorphic to a quotient module of the regular representation of $L(\rho)$. Hence the set of flag polynomials arising from $\mathrm{Sch}(\rho)$ is linearly independent if and only if $D(\rho) \cong \mathbb{F}_2 L(\rho)$.

Since $L(\rho)$ is a finite 2-group, its only irreducible representation over $\mathbb{F}_2$ is the trivial 1-dimensional representation. The socle of the regular representation $\mathbb{F}_2 L(\rho)$ is therefore given by the $L(\rho)$-invariant elements. But the only nonzero invariant is $\overline{\gamma} = \sum_{\gamma \in L(\rho)} \gamma$, so the socle is 1-dimensional, generated by $\overline{\gamma}$. We define a $\mathbb{F}_2 L(\rho)$-map $\theta : \mathbb{F}_2 L(\rho) \to D(\rho)$ by $\theta(I_n) = s_\omega(\rho)$, where I_n is the identity matrix. Then $\theta(\overline{\gamma}) = \sum_{\gamma \in L(\rho)} s_\omega(\rho) \cdot \gamma = f(\rho)$, say. We shall prove that $f(\rho) \neq 0$. It then follows that $\theta : \mathbb{F}_2 L(\rho) \cong D(\rho)$, so that the socle of $D(\rho)$ is simple and is generated by $f(\rho)$. Thus $f(\rho)$ 'co-generates' $D(\rho)$, in the sense that given a polynomial $F \in D(\rho)$, there exists $u \in \mathbb{F}_2 L(\rho)$ such that $F \cdot u = f(\rho)$.

The following combinatorial results will be proved in Section 3.

Proposition 2.9. *Let ω be 1-dominant, let $\rho \in \Sigma(n)$ be a permutation, and let $f(\rho) = \sum_{\gamma \in L(\rho)} s_\omega(\rho) \cdot \gamma$ be the sum of the flag polynomials for* $\mathrm{Sch}(\rho)$. *Then when written irredundantly as a sum of monomials, $f(\rho)$ contains the zip monomial $z_\omega(\rho)$. In particular, $f(\rho) \neq 0$.*

Proposition 2.10. *Let ω be 2-dominant. If a flag polynomial $\phi^\omega(W)$ contains the zip monomial $z_\omega(\rho)$ when it is written irredundantly as a sum of monomials, then $W \in$* $\mathrm{Sch}(\sigma)$, *where $\sigma \geq \rho$ in the Bruhat order. More generally, $z_\omega(\rho)$ appears in a polynomial in the $L(n)$-orbit of the spike $s_\omega(\sigma)$ if and only if $\sigma \geq \rho$.*

Assuming these two results, the recursive argument in $D(n)$ goes as follows. Suppose given a linear dependence relation between the flag polynomials $\phi^\omega(W)$, where W runs through the set of complete flags. We may write it as $F + G = 0$, where F is the sum of the terms involving flags W in the top Schubert cell $\mathrm{Sch}(\rho_0)$, and G is the sum of the remaining terms. If $F \neq 0$, then by applying a suitable element $u \in \mathbb{F}_2 L(\rho_0)$ to the relation, by Proposition 2.9 we obtain $f(\rho_0) + G \cdot u = 0$, where the monomial $z_\omega(\rho_0)$ appears in $f(\rho_0)$ and where $G \cdot u$ is a linear combination of polynomials in the $L(n)$-orbits of the set of flag polynomials for flags in lower Schubert cells. By Proposition 2.10, the monomial $z_\omega(\rho_0)$ does not appear in $G \cdot u$, and so we obtain a contradiction. It follows that $F = 0$. Now we continue by writing $G = H + K$ where H is the sum of the terms involving flags in a maximal remaining Schubert cell $\mathrm{Sch}(\rho)$, and so on.

The argument in $P(n)$ is similar, but at each stage we obtain a contradiction in the form that a polynomial containing a spike is hit. The module $D(\rho)$ is replaced by the module $P(\rho)$ spanned by the orbit of the zip monomial $z^\omega(\rho)$ under the action of $U(\rho)$, so that $P(\rho)$ is the cyclic right $\mathbb{F}_2 U(\rho)$-module generated by $z^\omega(\rho)$. As above, we call these polynomials 'dual flag polynomials'. The argument using the regular representation of the finite 2-group $U(\rho)$ is as before for $L(\rho)$, but we work *upwards* in the Bruhat order, starting at the identity permutation ι, and ending at the reversal ρ_0. The analogues of Propositions 2.9 and 2.10 are as follows, and are also proved in Section 3.

Proposition 2.11. *Let ω be 1-dominant, let $\rho \in \Sigma(n)$ be a permutation, and let $h(\rho) = \sum_{\gamma \in U(\rho)} z^\omega(\rho) \cdot \gamma$ be the sum of the dual flag polynomials for* $\mathrm{Sch}(\rho)$. *Then when written irredundantly as a sum of monomials, $h(\rho)$ contains the spike $s^\omega(\rho)$. In particular, $h(\rho)$ is not hit.*

Proposition 2.12. *Let ω be 2-dominant. If a dual flag polynomial for* $\mathrm{Sch}(\sigma)$ *contains the spike $s^\omega(\rho)$ when it is written irredundantly as a sum of monomials, then $\sigma \leq \rho$ in the Bruhat order. More generally, $s^\omega(\rho)$ appears in a polynomial in the $U(n)$-orbit of the zip monomial $z^\omega(\sigma)$ if and only if $\sigma \leq \rho$.*

Both in $P(n)$ and $D(n)$, this recursive argument can be carried out over any subset of the Schubert cells for which the spike and zip monomials associated to ω are distinct. This allows us to extend Theorem 2.8 to obtain a corresponding result for the 1-dominant case, Theorem 4.3.

3 Combinatorial results

In this section, we give the proofs of the four combinatorial results stated in Section 2, so completing the proof of Theorem 2.8.

3.1 Proof of Proposition 2.9

The flag polynomial $\phi^\omega(W) = s_\omega(\rho) \cdot \gamma$, $\gamma \in L(\rho)$, is a product

$$\phi^\omega(W) = \prod_{i=1}^{n} w_i^{(2^{\lambda_i}-1)}$$

of divided powers of linear forms w_i in $v_1, \ldots, v_n$. Each occurrence of the zip monomial $z_\omega(\rho)$ in this product corresponds to a factorisation $z_\omega(\rho) = f_1 \cdots f_n$ where, for $1 \leq i \leq n$, f_i is a monomial of degree $2^{\lambda_i} - 1$ in the variables which occur in w_i. By the divided binomial theorem (1), f_i can be any monomial of this form. However, the possibilities are substantially reduced by the superposition rule for multiplication of monomials in $D(n)$: the product of two monomials is zero if they are represented by blocks F and G which have a 1 in the same position, so that $F_{i,j} = G_{i,j} = 1$ for some i and j . This ensures that the 1's appearing in the binary blocks representing $f_1, \ldots, f_n$ form a set partition of the 1's appearing in the binary block representing $z_\omega(\rho)$. We shall prove that all but one of the possible factorisations $f_1 \cdots f_n$ of $z_\omega(\rho)$ occur in an even number of flag polynomials $\phi^\omega(W)$, $W \in \mathrm{Sch}(\rho)$, while the remaining factorisation occurs only in $\phi^\omega(W)$ when W is the 'maximal' flag in $\mathrm{Sch}(\rho)$, defined by choosing all the arbitrary elements of $\mathbb{F}_2$ in γ_W as 1's.

For $1 \leq i \leq n$, $v_{\rho(i)}$ occurs in the linear form w_i for all $W \in \mathrm{Sch}(\rho)$, and for $k \neq \rho(i)$, v_k can occur in w_i for some $W \in \mathrm{Sch}(\rho)$ if and only if $k = \rho(j)$, where $i < j$ and $\rho(i) > \rho(j)$. Let $z_\omega(\rho) = f_1 \cdots f_n$ be a factorisationas above, where, for some i, f_i does not involve every such variable v_k. Then we may assign either coefficient 0 or 1 to v_k, and so the factorisation $f_1 \cdots f_n$ contributes to the coefficient of $z_\omega(\rho)$ in an even number of flag polynomials. In this way, we can pair off all occurrences of $z_\omega(\rho)$ which arise when a flag polynomial is multiplied out in full, except those in which, for each i, f_i involves all v_k which can arise in w_i. These are a subset of the occurrences of $z_\omega(\rho)$ which arise when the 'maximal' flag polynomial is multiplied out. To complete the proof, we shall prove that there is a unique set partition of the 1's in the binary block representing $z_\omega(\rho)$ which satisfies the required conditions.

Example 3.1. Let $n = 4$, $\omega = (3, 3, 2, 2, 1, 1)$ and $\rho = (4, 3, 2, 1)$, so that $Z(\rho)$ is the zip block of degree 81 given in Example 2.4. We wish to determine all factorisations $z_\omega(\rho) = f_1 f_2 f_3$, ($f_4 = 1$ since $\omega_1 = 3$) where $\deg f_1 = 63$ and f_1 involves v_1, v_2, v_3, v_4, $\deg f_2 = 15$ and f_2 involves v_1, v_2, v_3, $\deg f_3 = 3$ and f_3 involves v_1, v_2, and where the 1's in the blocks for the monomials f_1, f_2 and f_3 form a set partition of the 1's in the block for $z_\omega(\rho)$. There is a unique solution

$$F_1 = \begin{matrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{matrix}, \quad F_2 = \begin{matrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}, \quad F_3 = \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}.$$

For the general case, we argue by induction on n. The variable $v_{\rho(1)}$ can appear only in f_1, and so all the 1's in row $\rho(1)$ of $Z(\rho)$ must be assigned to f_1. By definition of $Z(\rho)$, there is a 1 in each of the trailing columns of this row, and in each of the first $n - \rho(1)$ leading columns. Since f_1 must also involve all the variables v_k for $1 \leq k < \rho(1)$, we must assign to f_1 at least one 1 from each row above row $\rho(1)$. As there remain only $\rho(1)-1$ 1's to be assigned to f_1, we must choose exactly one of these from each row. Again by construction of $Z(\rho)$, working along the block from right to left we see that these 1's can only be chosen as (part of) a NW to SE diagonal in the spike block $S'(\iota)=Z'(\rho)$, the leading part of $Z(\rho)$. Thus there is a unique choice for f_1. The result now follows by induction, since ρ determines a bijection from $\{2,3,\ldots,n\}$ to $\{1,2,\ldots,n\}\setminus\{\rho(1)\}$, and this can be regarded as a permutation of $\{1,2,\ldots,n-1\}$ by reindexing the variables. $\square$

Example 3.2. Let $n=4$, $\omega=(3,3,2,2,1,1)$, $\rho=(2,4,3,1)$. Then the block $Z(\rho)$ and its factorisation are shown below, where the integer $\rho(i)$ indicates the 1's to be assigned to f_i, for $1\le i\le 4$.

$$Z(\rho)=\begin{array}{cccccc} 1&0&1&0&1&0\\ 1&1&1&1&0&1\\ 1&1&0&0&0&0\\ 0&1&0&1&0&0 \end{array}, \qquad \begin{array}{cccccc} 3&0&4&0&2&0\\ 2&2&2&2&0&2\\ 4&3&0&0&0&0\\ 0&4&0&4&0&0 \end{array}$$

This notation clarifies the structure of the factorisation. Splitting this (integer) block into its leading and trailing parts, and omitting 0's at the ends of the rows, we obtain the arrays

$$\begin{array}{ccc} 3&4&2\\ 2&2&\\ 4&&\\ \cdot&& \end{array} \quad \text{and} \quad \begin{array}{ccc} \cdot&&\\ 2&2&2\\ 3&&\\ 4&4& \end{array},$$

from which we see that the leading part ω' of ω plays the active role and the trailing part ω'' a passive role in the factorisation of $z_\omega(\rho)$.

3.2 Spike diagrams and tableaux

The example above suggests the use of tableaux, analogous to Young tableaux for partitions, which are associated to permutations. Throughout this section, we fix ω 1-dominant of degree d. The **spike diagram** $[S^\omega(\rho)]$ associated to ρ is obtained by replacing each 1 in the block for $s^\omega(\rho)$ by a 'box' or 'node'. For example, with $\omega=(4,3,2,1)$,

$$[S^\omega(2,4,3,1)]=\begin{array}{l} \square\\ \square\square\square\square\\ \square\square\\ \square\square\square \end{array}.$$

Remark 3.3. Other authors (see e.g. [25]) use a different convention, in which the ith row of the diagram associated to ρ contains $\rho(i)$ boxes. Our choice is made to fit our use of spike and zip monomials.

For ω 1-dominant and $\rho \in \Sigma(n)$, we define a **ρ-tableau** $T = T^\omega(\rho)$ as a filling of each box in the diagram $[S^\omega(\rho)]$ by an integer i with $1 \le i \le n$. For example, we may fill all the boxes in the ith row by the integer i for $1 \le i \le n$, and we call this the **spike ρ-tableau** $S = ST^\omega(\rho)$. A ρ-tableau T has **content** σ if it is filled by the entries of $ST^\omega(\sigma)$. A ρ-tableau is **upward** if the entries in each column are distinct, and if each i appears only in rows $j \le i$.

Example 3.4. For $\omega = (4,3,2,1)$, let $\rho = (2,3,1,4)$, $\sigma = (2,4,3,1) \in \Sigma(4)$. Then the spike σ-tableau $S = ST^\omega(\sigma)$ and an upward ρ-tableau $T = T^\omega(\rho)$ of content σ are shown below. We can regard T as the result of an upward shift of some of the entries of S.

$$S = \begin{matrix} 1 & & & \\ 2 & 2 & 2 & 2 \\ 3 & 3 & & \\ 4 & 4 & 4 & \end{matrix}, \qquad T = \begin{matrix} 1 & 2 & & \\ 2 & 3 & 2 & 2 \\ 3 & 4 & 4 & \\ 4 & & & \end{matrix}.$$

Lemma 3.5. *For any 1-dominant ω-vector, there exists an upward ρ-tableau of content σ if and only if $\sigma \ge \rho$ in the Bruhat order.*

Before giving the proof, we recall that the Bruhat order is a partial order on permutations of $\{1, \ldots, n\}$ which has several equivalent definitions [6].

- $\rho \le \sigma$ if and only if some (equivalently, any) reduced word for σ in the transpositions σ_i, $1 \le i \le n-1$, contains a reduced word for ρ as a subword (where the letters omitted need not be consecutive).

- $\rho \le \sigma$ if and only if σ can be obtained from ρ by successive inversion of pairs $i < j$ which are not inverted in one-line notation, for example

$$\begin{matrix} (2,3,1,4) \overset{(1,4)}{\longleftarrow} (2,3,4,1) \overset{(3,4)}{\longleftarrow} (2,4,3,1) \\ (2,3,1,4) \overset{(3,4)}{\longleftarrow} (2,4,1,3) \overset{(1,3)}{\longleftarrow} (2,4,3,1) \end{matrix}$$

- A partial order on k-element subsets of $\{1, \ldots, n\}$ is defined by $I \le J$ if and only if $i_r \le j_r$ for $1 \le r \le k$ when I and J are listed in increasing order. Then $\rho \le \sigma$ if and only if $\{\rho(1), \ldots, \rho(k)\} \le \{\sigma(1), \ldots, \sigma(k)\}$ for $1 \le k \le n-1$.

- $\rho \le \sigma$ if and only if $r_{i,j}(\rho) \ge r_{i,j}(\sigma)$ for all i and j, where for $\tau \in \Sigma(n)$

$$r_{i,j}(\tau) = \#\{k \le i \mid \tau(k) \le j\}.$$

Proof of Lemma 3.5 ($\Rightarrow$) Let ω be 1-dominant, so that $\lambda = \omega^{\mathrm{tr}}$ is strictly decreasing. In terms of the spike ρ-tableau $ST^\omega(\rho)$, $r_{i,j}(\rho)$ is the number of rows among the first j which have length $\ge \lambda_i$. Then the entries in the spike σ-tableau

$S = ST^\omega(\sigma)$ which are $\leq j$ appear in the first j rows. Assume that a tableau T exists, where the entries of S are used to fill the boxes in $[S^\omega(\rho)]$. Then the entries just described have to appear in T in the same columns (since duplication of entries in a column is not allowed) and in the same or higher rows. Hence $r_{i,j}(\rho) \geq r_{i,j}(\sigma)$, and hence $\rho \leq \sigma$ in the Bruhat order.

($\Leftarrow$) For the converse, we use a chain in the Bruhat order from ρ to σ to convert S to a ρ-tableau T of content σ. Consider first the case where σ can be obtained from ρ by inversion of the pair i, j, i.e. $\sigma = \sigma_{i,j} \circ \rho$, where $\rho^{-1}(i) < \rho^{-1}(j)$ and the transposition $\sigma_{i,j} \in \Sigma(n)$ exchanges i and j. In decreasing order of length, the rows of the diagram $[S^\omega(\rho)]$ are $\rho(1), \rho(2), \ldots, \rho(n)$. Since $\rho^{-1}(i) < \rho^{-1}(j)$, row i is longer than row j. Now $\sigma^{-1}(i) = \rho^{-1}(j)$ and $\sigma^{-1}(j) = \rho^{-1}(i)$, and so row j is longer than row i in the diagram $[S^\omega(\sigma)]$. Thus we may move entries j from row j of S up to row i, so as to obtain a ρ-tableau T with content σ.

For the general case, σ can be obtained from ρ by a finite sequence of inversions of such pairs i, j. Corresponding to such a sequence, we can carry out a sequence of moves in which certain entries are moved up from the jth row to the ith row to give a tableau for the next lower permutation in the chain, with content σ. $\square$

Example 3.6. For $n = 4$, the two chains above from $(2, 3, 1, 4)$ to $(2, 4, 3, 1)$ give tableau moves as follows.

$$
\begin{array}{llll} 1 & & & \\ 2 & 2 & 2 & 2 \\ 3 & 3 & & \\ 4 & 4 & 4 & \end{array}
\quad \xrightarrow{(3,4)} \quad
\begin{array}{llll} 1 & & & \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 4 & \\ 4 & 4 & & \end{array}
\quad \xrightarrow{(1,4)} \quad
\begin{array}{llll} 1 & 4 & & \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 4 & \\ 4 & & & \end{array}
$$

$$
\begin{array}{llll} 1 & & & \\ 2 & 2 & 2 & 2 \\ 3 & 3 & & \\ 4 & 4 & 4 & \end{array}
\quad \xrightarrow{(1,3)} \quad
\begin{array}{llll} 1 & 3 & & \\ 2 & 2 & 2 & 2 \\ 3 & & & \\ 4 & 4 & 4 & \end{array}
\quad \xrightarrow{(3,4)} \quad
\begin{array}{llll} 1 & 3 & & \\ 2 & 2 & 2 & 2 \\ 3 & 4 & 4 & \\ 4 & & & \end{array}
$$

Proof of Proposition 2.10 Let ω be 2-dominant, with leading part ω' and trailing part ω''. Given $\rho \in \Sigma(n)$, we may split the zip block $Z(\rho)$ for ω into its leading part $Z'(\rho)$ and its trailing part $Z''(\rho)$. By replacing the 1's in $Z''(\rho)$ by boxes, we obtain the spike diagram $[S^{\omega''}(\rho)]$. As ω'' is 1-dominant, by Lemma 3.5 there exists an upward ρ-tableau $T^{\omega''}(\rho)$ whose content is σ if and only if $\sigma \geq \rho$ in the Bruhat order.

If we apply any linear substitution $\gamma \in L(n)$ to the spike $s_\omega(\sigma)$ and encode the factorisations of the monomials in the resulting product as above, we obtain a collection of tableaux in which the entries in each column are distinct, and in which the entries of the spike tableau $ST^\omega(\sigma)$ appear in the same or higher rows. Thus, on restriction to the leading part of the corresponding block, an occurrence of $z_\omega(\rho)$ will yield an upward ρ-tableau of content σ. $\square$

3.3 Proof of Proposition 2.11

The proof is similar to that of Proposition 2.9. Let $z^\omega(\rho) = x_1^{r_1} \cdots x_n^{r_n}$ be the zip monomial, so that a dual flag polynomial for $\mathrm{Sch}(\rho)$ has the form $z^\omega(\rho) \cdot \gamma$, $\gamma \in U(\rho)$. Writing $z^\omega(\rho) \cdot \gamma = \prod_{i=1}^n y_i^{r_i}$, where $y_1, \ldots, y_n \in V^*$, each occurrence of the spike $s^\omega(\rho)$ in this multiplication corresponds to a factorisation $s^\omega(\rho) = f_1 \cdots f_n$, where f_i is a monomial of degree r_i in the variables appearing in the linear form y_i. Because of the mod 2 binomial theorem, the 1's appearing in the blocks $F_1, \ldots, F_n$ for $f_1, \ldots, f_n$ form a set partition of the 1's appearing in the spike block $S(\rho)$. We wish to count all these occurrences of $s^\omega(\rho)$ mod 2, taking all dual flag polynomials $z^\omega(\rho) \cdot \gamma$ into account.

The variables which can occur in y_i for some $\gamma \in U(\rho)$ is determined by ρ: for $j \neq i$, x_j can occur in y_i if and only if the pair (i, j) is reversed by ρ^{-1}, i.e. $i < j$ and $\rho^{-1}(i) > \rho^{-1}(j)$, so that i appears later than j in $(\rho(1), \ldots, \rho(n))$. Consider a factorisation $s^\omega(\rho) = f_1 \cdots f_n$ where some monomial f_i does not involve a variable x_k which can occur in y_i for some flag in $\mathrm{Sch}(\rho)$. Since we may assign either coefficient 0 or 1 to x_k, this factorisation must arise in an even number of dual flag polynomials.

In this way, we can pair off all occurrences of $s^\omega(\rho)$ which arise when a dual flag polynomial $z^\omega(\rho) \cdot \gamma$ is multiplied out in full, except those in which, for each i, all the variables x_k which can arise in y_i are involved in f_i. These are a subset of the occurrences of $s^\omega(\rho)$ which arise when the 'maximal' dual flag polynomial is multiplied out. To complete the proof, we shall prove that there is a unique set partition of the 1's in the spike block $S(\rho)$ which satisfies the required conditions.

Example 3.7. Let $n = 4$, $\omega = (3, 3, 2, 2, 1, 1)$ and $\rho = (4, 3, 2, 1)$, so that $S(\rho)$ is the spike block of degree 81 given in Example 2.4. We wish to determine all factorisations $s^\omega(\rho) = f_1 f_2 f_3 f_4$, $(f_4 = x_4^{42})$ where $\deg f_1 = 21$ and f_1 involves x_2, x_3, x_4 and possibly x_1, $\deg f_2 = 7$ and f_2 involves x_3, x_4 and possibly x_2, $\deg f_3 = 11$ and f_3 involves x_4 and possibly x_3, and where the 1's in the corresponding blocks F_1, F_2, F_3 and F_4 form a set partition of the 1's in $S(\rho)$. There is a unique solution

$$F_1 = \begin{matrix} 0&0&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&1&0 \end{matrix}, \quad F_2 = \begin{matrix} 0&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&1&0&0&0 \end{matrix}, \quad F_3 = \begin{matrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&1&0&1&0&0 \\ 1&0&0&0&0&0 \end{matrix}.$$

For the general case, we argue by induction on n. First consider $x_{\rho(1)}$. Since $f_{\rho(1)}$ is a power of $x_{\rho(1)}$ given by the 1's in row $\rho(1)$ of $Z(\rho)$, the corresponding 1's in row $\rho(1)$ of $S(\rho)$ must be assigned to $F_{\rho(1)}$. These occur in all the trailing columns and in $n - \rho(1)$ of the leading columns. By definition of $S(\rho)$, there is a 1 in every column of this row. The variable $x_{\rho(1)}$ must also occur in f_k for $1 \leq k < \rho(1)$, so we must select at least one 1 in row $\rho(1)$ of $S(\rho)$ to assign to F_k.

As there are only $\rho(1)-1$ more 1's in row $\rho(1)$ of $S(\rho)$ to be assigned, we must assign exactly one of these to each F_k for $1 \leq k < \rho(1)$. Again by construction of $Z(\rho)$, working along the block from right to left we see that these 1's must correspond to (part of) a NW to SE diagonal in the leading part $S'(\rho) = Z'(\rho)$ of $Z(\rho)$. Thus there is a unique choice for the selection of powers of the variable $x_{\rho(1)}$. The result now follows by induction, since ρ determines a bijection from $\{2,3,\ldots,n\}$ to $\{1,2,\ldots,n\} \setminus \{\rho(1)\}$, and this can be regarded as a permutation of $\{1,2,\ldots,n-1\}$ by reindexing the variables. $\square$

Example 3.8. Let $n = 4$, $\omega = (3,3,2,2,1,1)$, $\rho = (2,4,3,1)$. Then the block $S^\omega(\rho)$ and its factorisation are shown below, where the integer i indicates the 1's to be assigned to f_i, for $1 \leq i \leq 4$.

$$S^\omega(\rho) = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{matrix}, \qquad \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 1 & 2 \\ 1 & 3 & 0 & 0 & 0 & 0 \\ 3 & 4 & 1 & 4 & 0 & 0 \end{matrix}.$$

Again the notation clarifies the structure of the factorisation. Splitting this (integer) block into its leading and trailing parts and omitting final 0's in the rows, we obtain the arrays

$$\begin{matrix} \cdot & & \\ 2 & 2 & 1 \\ 1 & & \\ 3 & 1 & \end{matrix} \qquad \begin{matrix} \cdot & & \\ 2 & 2 & 2 \\ 3 & & \\ 4 & 4 & \end{matrix}$$

showing that the leading part ω' of ω plays the active role and the trailing part ω'' a passive role in the factorisation of the spike $s^\omega(\rho)$.

3.4 Proof of Proposition 2.12

The proof is similar to that of Proposition 2.10. We call a ρ-tableau **downward** if the entries in each column are distinct, and if each i appears only in rows $j \geq i$.

Example 3.9. Let $\omega = (4,3,2,1)$, and let $\rho = (2,3,1,4)$, $\sigma = (2,4,3,1) \in \Sigma(4)$. Then the spike tableau $S = ST^\omega(\rho)$ and a downward σ-tableau $T = T^\omega(\sigma)$ of content ρ are shown below. We can regard T as the result of an downward shift of some of the entries of S.

$$S = \begin{matrix} 1 & 1 & & \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & \\ 4 & & & \end{matrix}, \qquad T = \begin{matrix} 1 & & & \\ 2 & 1 & 2 & 2 \\ 3 & 2 & & \\ 4 & 3 & 3 & \end{matrix}.$$

Lemma 3.10. *For any 1-dominant ω-vector, there exists a downward ρ-tableau of content σ if and only if $\sigma \leq \rho$ in the Bruhat order.*

Proof This follows from Lemma 3.5 using the reversal $\rho_0 = (n, n-1, \ldots, 1)$. The diagrams of $\rho_0 \circ \rho$ and $\rho_0 \circ \sigma$ are obtained by reversing the rows of the diagrams of ρ and σ. These row reversals interchange upward and downward tableaux, while composition with ρ_0 reverses the Bruhat order. □

Example 3.11. The two chains above from $(2,3,1,4)$ to $(2,4,3,1)$ give the tableau moves shown below.

$$\begin{array}{llll} 1 & & & \\ 2 & 2 & 2 & 2 \\ 3 & 3 & & \\ 4 & 1 & 3 & \end{array} \quad \xleftarrow{(3,4)} \quad \begin{array}{llll} 1 & & & \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & \\ 4 & 1 & & \end{array} \quad \xleftarrow{(1,4)} \quad \begin{array}{llll} 1 & 1 & & \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & \\ 4 & & & \end{array}$$

$$\begin{array}{llll} 1 & & & \\ 2 & 2 & 2 & 2 \\ 3 & 1 & & \\ 4 & 3 & 3 & \end{array} \quad \xleftarrow{(1,3)} \quad \begin{array}{llll} 1 & 1 & & \\ 2 & 2 & 2 & 2 \\ 3 & & & \\ 4 & 3 & 3 & \end{array} \quad \xleftarrow{(3,4)} \quad \begin{array}{llll} 1 & 1 & & \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & \\ 4 & & & \end{array}$$

Proof of Proposition 2.12 Let ω be 2-dominant and let $\rho \in \Sigma(n)$. Split the spike block $S(\rho)$ into its leading and trailing parts $S'(\rho)$, $S''(\rho)$. By replacing the 1's in $S''(\rho)$ by boxes, we obtain the spike diagram $[S^{\omega''}(\rho)]$. As ω'' is 1-dominant, by Lemma 3.10 there exists a downward ρ-tableau T of content σ if and only if $\sigma \le \rho$ in the Bruhat order.

If we apply any linear substitution $\gamma \in U(n)$ to the zip $z^\omega(\sigma)$ and encode the factorisations of the monomials in the resulting product as above, we obtain a collection of tableaux in which the entries in each column are distinct and in which the entries of the spike tableau $ST^\omega(\sigma)$ appear in the same or lower rows. Thus, on restriction to the leading part of the corresponding block, an occurrence of $s^\omega(\rho)$ will yield a downward ρ-tableau of content σ. □

4 The flag representation $Fl(n)$

4.1 Partial flag modules

In this section we split the $\mathbb{F}_2GL(n)$-module $Fl(n)$ as a direct sum of indecomposable submodules, and use this to extend our results on $S^d(n)$ from 2-dominant to 1-dominant degrees d.

There are 2^{n-1} vectors $c = (c_1, \cdots, c_r)$ of positive integers whose sum is n, often called *compositions* of n. These correspond to subsets $I = \{i_1, i_2, \ldots, i_{r-1}\} \subseteq \{1, 2, \ldots, n-1\}$, where $1 \le i_1 \le \cdots \le i_{r-1} \le n$, by writing $i_j = c_1 + \cdots + c_j$ for $1 \le j \le r-1$, so that $c \mapsto I$ where $I = \{i_1, i_2, \ldots, i_{r-1}\}$, and $I \mapsto c$ where $c_j = i_j - i_{j-1}$ and $i_0 = 0$, $i_r = n$. For each I, a $\mathbb{F}_2GL(n)$-module $Fl^I(n)$ is given by permutations of the **partial flags** W of **type** I in V, i.e. increasing sequences of subspaces of V with dimensions in I. The module $Fl^I(n)$ is induced from the trivial 1-dimensional module for the parabolic subgroup $L^I(n)$ given by block lower triangular matrices with diagonal blocks in $GL(c_1) \times \cdots \times GL(c_r)$, where $c = c(I)$. The number of partial flags of type I is the 2-multinomial coefficient

$$\dim Fl^I(n) = [GL(n) : L^I(n)] = \binom{n}{c(I)}_2, \tag{2}$$

derived from the usual multinomial coefficient $n!/c_1! \cdots c_r!$ by replacing each factor k by $2^k - 1$. In the case $I = \{1, 2, \ldots, n-1\}$, $Fl^I(n) = Fl(n)$, and in the case $I = \emptyset$, $Fl^I(n)$ is the trivial 1-dimensional representation of $GL(n)$.

As in Section 2.3, we identify a partial flag W of type I with a right coset $L^I(n)\gamma$, and $Fl^I(n)$ with $\mathbb{F}_2(GL(n)/L^I(n))$, as follows. For $\gamma = (a_{i,j}) \in GL(n)$, let $w_i = \sum_{j=1}^n a_{i,j} v_j$ for $1 \le i \le n$, and let $W : W_1 \subset W_2 \subset \cdots \subset W_{r-1}$, where $w_1, \ldots, w_{i_j}$ is a basis for W_j for $1 \le j < r$. The row-reduction process for matrices in the same coset $L^I(n)\gamma$ allows us to divide the rows into blocks of size $c_1, \ldots, c_r$ and to permute rows which lie in the same block. We do this so that the columns containing the last 1's in rows of the same block are in increasing order.

Given a permutation $\rho \in \Sigma(n)$, the **descent set** $\operatorname{des}(\rho)$ is the set of all i such that $\rho(i+1) < \rho(i)$. Thus $\operatorname{des}(\rho) \subseteq I = \{i_i, \ldots, i_{r-1}\}$ if and only if ρ is increasing, for $1 \le j \le r$, on the interval $C_j = \{i_{j-1}+1, \ldots, i_j\}$ of length c_j. These permutations parametrize the Schubert cell decomposition of $GL(n)/L^I(n)$. As the number of partial flags in $\operatorname{Sch}(\rho)$ is $2^{\operatorname{len}(\rho)}$,

$$\binom{n}{c(I)}_2 = \dim Fl^I(n) = \sum_{\operatorname{des}(\rho) \subseteq I} 2^{\operatorname{len}(\rho)}. \tag{3}$$

For $I \subseteq \{1, 2, \ldots, n-1\}$, the Young subgroup $\Sigma^I(n)$ associated to I is given by permutation matrices with diagonal blocks in $\Sigma(c_1) \times \cdots \times \Sigma(c_r)$. The permutations with descent set in I form a transversal T of $\Sigma^I(n)$ in $\Sigma(n)$.

Example 4.1. The Bruhat order on T for $n = 4$ and $I = \{2\}$ is shown below. This gives the Schubert cell decomposition of the 35 2-planes in V.

$$\begin{array}{ccccc}
 & & (3,4,1,2) & & \\
 & & \downarrow & & \\
 & & (2,4,1,3) & & \\
 & \swarrow & & \searrow & \\
(1,4,2,3) & & & & (2,3,1,4) \\
 & \searrow & & \swarrow & \\
 & & (1,3,2,4) & & \\
 & & \downarrow & & \\
 & & (1,2,3,4) & &
\end{array}$$

We can now extend Theorem 2.8 to the 1-dominant case.

Definition 4.2. *For a 1-dominant ω-vector ω with conjugate $\lambda = \omega^{\mathrm{tr}}$, let $I(\omega) \subseteq \{1, 2, \ldots, n-1\}$ be the set of all i such that $\omega_j = i$ for at least two values of j, i.e. $\lambda_i - \lambda_{i+1} \geq 2$.*

Theorem 4.3. *Let ω is 1-dominant, and let T be a transversal for $\Sigma^I(n)$, where $I = I(\omega)$. Then*

(i) *the flag polynomials $s_\omega(\rho) \cdot \gamma$, for $\rho \in T$, $\gamma \in L(\rho)$, are linearly independent;*

(ii) *the dual flag polynomials in $z^\omega(\rho) \cdot \gamma$ for $\rho \in T$, $\gamma \in U(\rho)$, are linearly independent in $Q^d(n)$, i.e. no nonzero linear combination of them is hit.*

Proof Since ω be 1-dominant, its trailing part ω'' is of spike type. For ρ and $\sigma \in \Sigma(n)$, the spikes $s^{\omega''}(\rho)$ and $s^{\omega''}(\sigma)$ are equal if and only if ρ and σ are in the same coset of $\Sigma^I(n)$, and so the zips $z^\omega(\rho)$ and $z^\omega(\sigma)$ are equal under the same conditions. The result now follows by recursion on the Schubert cells, using the method of Theorem 2.8. $\square$

Here W is a complete flag, but $\phi^\omega(W)$ depends only on the partial flag given by restriction to subspaces with dimensions in $I(\omega)$. As a special case of Theorem 4.3, the flag polynomials for flags in the Schubert cells $\mathrm{Sch}(\rho)$ such that $\mathrm{des}(\rho) \subseteq I(\omega)$ are linearly independent. However, the $\mathbb{F}_2$-subspace spanned by these polynomials is not a $\mathbb{F}_2 GL(n)$-submodule of $Fl(n)$: for example, it contains $s^\omega(\iota)$ but not $s^\omega(\rho_0)$. We can obtain a better result by using a transversal for $\Sigma^I(n)$ which is *maximal*, rather than minimal, for the Bruhat order, given by the permutations σ which are decreasing on the intervals C_j i.e $\mathrm{des}(\sigma) \supseteq I^*$ where $I^* = \{1, 2, \ldots, n-1\} \setminus I$. Thus the maximal transversal of $\Sigma^I(n)$ is obtained from the minimal one by reversing C_j, for $1 \leq j \leq r$. It follows that

$$\sum_{\mathrm{des}(\rho) \supseteq I^*} 2^{\mathrm{len}(\rho)} = \binom{n}{c(I)}_2 \cdot \prod_{i=1}^{r+1} 2^{c_i(c_i-1)/2}. \tag{4}$$

Example 4.4. For $n = 4$, let ω be 1-dominant with $(3, 2, 2, 1)$ as a subvector. The Schubert cells indicated below give 140 linearly independent flag polynomials.

$$\begin{array}{ccc} & (4,3,2,1) & \\ & \downarrow & \\ & (4,2,3,1) & \\ \swarrow & & \searrow \\ (4,1,3,2) & & (3,2,4,1) \\ \searrow & & \swarrow \\ & (3,1,4,2) & \\ & \downarrow & \\ & (2,1,4,3) & \end{array}$$

Corollary 4.5. *For ω 1-dominant of degree d, let $c(I) = (c_1, \ldots, c_r)$ where $I = I(\omega)$. Then*

$$\dim K^d(n) = \dim Q^d(n) \geq \binom{n}{c(I)}_2 \cdot \prod_{i=1}^{r+1} 2^{c_i(c_i-1)/2}. \qquad \square$$

We show in Theorem 4.12 that this lower bound is $\dim S^d(n)$.

4.2 Indecomposable summands of $Fl(n)$

In this section we summarise information we use about the structure of the representation $Fl(n)$. The arguments are elementary and the results are well-known in the more general context of finite groups of Lie type (see [7, Chapter 7]), but we have not been able to find a reference for the details we present.

Proposition 4.6. *For each $I \subseteq \{1, 2, \ldots, n-1\}$, the partial flag module $Fl^I(n)$ can be embedded as a direct summand in $Fl(n)$. More generally, if $I \subset J \subseteq \{1, 2, \ldots, n-1\}$, then $Fl^I(n)$ can be embedded as a direct summand in $Fl^J(n)$.*

Proof Consider the $\mathbb{F}_2 GL(n)$-maps $Fl^I(n) \xrightarrow{i} Fl(n) \xrightarrow{j} Fl^I(n)$ which associate to a partial flag W of type I the sum $i(W) \in Fl(n)$ of the complete flags containing W, and to a complete flag W' its restriction $j(W')$ to a partial flag of type I. The composition $j \circ i$ maps a partial flag W to $k \cdot W$, where k is the number of complete flags containg W. But $k = [L^I(n) : L(n)]$ is odd, since $L(n)$ is a 2-Sylow subgroup of $GL(n)$, and so $j \circ i$ is the identity. The same argument applies to the general case $J \subset I$, since $[L^I(n) : L^J(n)]$ is odd. $\square$

For $I \subseteq \{1, 2, \ldots, n-1\}$, we identify $Fl^I(n)$ with its image in $Fl(n)$ under the above embedding. More generally, for $I \subseteq J \subseteq \{1, 2, \ldots, n-1\}$, we identify $Fl^I(n)$ with its image in $Fl^J(n)$, by the embedding which associates to each partial flag of type I the sum of all partial flags of type J containing it. It is clear that these embeddings are consistent when $K \subseteq J \subseteq I$.

In order to express $Fl(n)$ as a direct sum of indecomposable submodules, we need to understand how the modules $Fl^I(n)$ fit together in $Fl(n)$.

Proposition 4.7. *For $I, J \subseteq \{1, 2, \ldots, n-1\}$, $Fl^I(n) \cap Fl^J(n) = Fl^{I \cap J}(n)$.*

Proof The inclusion $Fl^{I\cap J}(n) \subseteq Fl^I(n) \cap Fl^J(n)$ follows from Proposition 4.6. By replacing $Fl(n)$ by $Fl^{I\cup J}(n)$, we may assume that $I \cup J = \{1, 2, \ldots, n-1\}$.

Consider first the case $I \cap J = \emptyset$, $I \cup J = \{1, 2, \ldots, n-1\}$. Let $\mathcal{F}$ be a nonempty set of complete flags in V which is closed with respect to the operation of replacing $W : W_1 \subset W_2 \subset \cdots \subset W_{n-1}$ by $X : X_1 \subset X_2 \subset \cdots \subset X_{n-1}$ if $W_i = X_i$ for all $i \in I$ or for all $j \in J$. We must show that $\mathcal{F}$ contains all complete flags.

With notation as above, let $W \in \mathcal{F}$ and let X be any complete flag. Let $L = X_1$. If $L = W_1$, we may work in the quotient space V/L and the result will follow by induction on n. Hence we may assume that $L \neq W_1$. Define r such that $L \subseteq W_r$, $L \not\subseteq W_{r-1}$, so that r is uniquely defined and $2 \leq r \leq n$. For $1 \leq j \leq r-1$, define $Y_j = W_{j-1} \oplus L$. Thus $L = Y_1$ and $Y : Y_1 \subset Y_2 \subset \cdots \subset Y_{r-1} \subset W_r \subset \cdots \subset W_{n-1}$ is a complete flag. Using the inclusions

$$\begin{array}{ccc} W_{j-1} & \subset & W_j \\ \cap & & \cap \\ Y_j & \subset & W_{j+1} \end{array}$$

we may successively replace W_j by Y_j for $j = r-1, r-2, \ldots, 1$, obtaining at each stage a complete flag in $\mathcal{F}$. Thus $Y \in \mathcal{F}$. Since $L = Y_1$, the result follows as before by induction on $\dim V$.

For the general case, let W and X be complete flags in V which restrict to the same partial flag in $Fl^{I \cap J}(n)$. Since we are assuming that $I \cup J = \{1, 2, \ldots, n-1\}$, we may divide the interval from 1 to $n-1$ into subintervals whose end points are in $I \cap J$ and whose interior points are in I or J but not both. The general case follows by applying the argument for the case $I \cap J = \emptyset$ to each subinterval. $\square$

We are now ready to define the indecomposable summands $Fl_J(n)$ of $Fl(n)$.

Proposition 4.8. *For $I \subseteq \{1, 2, \ldots, n-1\}$, let $Fl_I(n) = Fl^I(n) / \sum_{J \subset I} Fl^J(n)$. Then $Fl^I(n) \cong \bigoplus_{J \subseteq I} Fl_J(n)$, and $\dim Fl_J(n) = \sum_{\mathrm{des}(\rho)=J} 2^{\mathrm{len}(\rho)}$ is the number of flags in Schubert cells for permutations with descent set J.*

Proof We fix n and argue by induction on $|I|$. For $I = \emptyset$, $Fl_\emptyset(n) = Fl^\emptyset(n)$ is the trivial 1-dimensional $\mathbb{F}_2 GL(n)$-submodule of $Fl(n)$ generated by the sum of all the flags. Let $I = \{i_1, \ldots, i_{r-1}\}$, where $r \geq 2$. Then $Fl^I(n)$ has $r-1$ direct summands $Fl^{I_k}(n)$ where $I_k = I \setminus \{i_k\}$. By the induction hypothesis, each of these is the direct sum of 2^{r-2} submodules of the form $Fl_J(n)$ with $J \subset I$. We collect a maximal subset of $Fl_J(n)$'s such that any two have zero intersection, as follows. We select all the 2^{r-2} summands $Fl_J(n)$ of $Fl^{I_1}(n)$. By Proposition 4.7, $Fl^{I_1}(n) \cap Fl^{I_2}(n) = Fl^{I_1 \cap I_2}(n)$ contains 2^{r-3} of these summands $Fl_J(n)$, and so we select the remaining 2^{r-3} summands. Continuing in this way, we can select 2^{r-4} summands of $Fl^{I_3}(n)$ which are not in $Fl^{I_1}(n) + Fl^{I_2}(n)$, and so on until we have $2^{r-2} + 2^{r-3} + \cdots + 1 = 2^{|I|} - 1$ summands of $Fl^I(n)$, and these are isomorphic to the modules $Fl_J(n)$ for $J \subset I$. The complementary summand in $Fl^I(n)$ is then isomorphic to $Fl_I(n)$. The formula for $\dim Fl_J(n)$ follows by recursion using equation (3). $\square$

Remark 4.9. It follows from the structure of the Hecke algebra of $\mathbb{F}_2 GL(n)$ endomorphisms of $Fl(n)$ [14, 4] that the 2^{n-1} modules $Fl_J(n)$ are indecomposable. Hence we have obtained a maximal direct sum decomposition of the partial flag module $Fl^I(n)$, and in particular of $Fl(n)$.

Example 4.10. The diagram below shows the four summands for $n = 3$. The trivial module $Fl_\emptyset$ has dimension 1, the module V and its contragredient dual

V^* have dimension 3, and the Steinberg module St has dimension 8. The 7 lines in V give a basis for $Fl^{\{1\}} = Fl_{\{1\}} \oplus Fl_\emptyset$, and the 7 planes give a basis for $Fl^{\{2\}} = Fl_{\{2\}} \oplus Fl_\emptyset$. (Boardman [2] discusses this case in detail.)

$$\begin{array}{ccccc} & & St = Fl_{\{1,2\}} & & \\ & & \oplus & & \\ & V & & V^* & \\ Fl_{\{1\}} = & | & \oplus & | & = Fl_{\{2\}} \\ & V^* & & V & \\ & & \oplus & & \\ & & Fl_\emptyset & & \end{array}$$

As it is a permutation representation, $Fl^J(n)$ is self-dual for *contragredient* duality, where the action of $GL(n)$ on M^* is given by $\langle f \cdot \gamma, x\rangle = \langle f, x \cdot \gamma^{-1}\rangle$, where $x \in M$, $f \in M^* = \mathrm{Hom}(M, \mathbb{F}_2)$ and $\gamma \in GL(n)$. It follows that $Fl_J(n)$ is also self-dual for contragredient duality . However, $Q^d(n)$ and $K^d(n)$ are *transpose* duals [23, Section 4]. The transpose dual M^{tr} of M is defined by $\langle f \cdot \gamma, x\rangle = \langle f, x \cdot \gamma^{\mathrm{tr}}\rangle$. Transpose duality preserves simple modules, and reverses composition series. In Example 4.10, it exchanges $Fl_{\{1\}}$ and $Fl_{\{2\}}$.

Proposition 4.11. *For $I \subseteq \{1, 2, \ldots, n-1\}$, let $j \in J$ if and only if $n - j \in I$. Then transpose duality maps $Fl^I(n)$ to $Fl^J(n)$ and $Fl_I(n)$ to $Fl_J(n)$. In particular, $Fl(n)$ itself is self-dual for transpose duality.*

Proof Let M be a finite-dimensionsl $\mathbb{F}_2GL(n)$-module, and let A be a vector subspace of M. Its annihilator $\mathrm{Ann}(A) \subseteq M^*$ is defined by $f \in \mathrm{Ann}(A)$ if and only if $\langle f, a\rangle = 0$ for all $a \in A$. Then $\dim A + \dim(\mathrm{Ann}(A)) = \dim M$. It follows from the definition of M^{tr} that for $\gamma \in GL(n)$, $f \cdot \gamma \in \mathrm{Ann}(A)$ if and only if $f \in \mathrm{Ann}(A \cdot \gamma^{\mathrm{tr}})$. Hence the relation $\mathrm{Ann}(A \cdot \gamma^{\mathrm{tr}}) = \mathrm{Ann}(A) \cdot \gamma^{-1}$ holds for the annihilators.

We shall define an isomorphism $\theta : Fl^I(V)^{\mathrm{tr}} \to Fl^J(V^{\mathrm{tr}})$. Since V is a simple module, V^{tr} is isomorphic to V, and hence $F^J(V^{\mathrm{tr}})$ is isomorphic to $F^J(V)$. First consider the case $I = \{d\}$, $J = \{n-d\}$. Then $F^I(V)^{\mathrm{tr}}$ has a basis given by the projections $p_A : F^I(V) \to \mathbb{F}_2$ defined by $p_A(B) = \delta_{A,B}$, where A and B are subspaces of V of dimension d. For each such subspace A of V, $\mathrm{Ann}(A)$ is a subspace of dimension $n - d$ of V^*, and so we may define $\theta(p_A) = \mathrm{Ann}(A)$. Clearly θ is bijective, and it follows from the annihilator relation above that $\theta(p_A \cdot \gamma) = \mathrm{Ann}(A) \cdot \gamma$, so that θ is a map of $\mathbb{F}_2GL(n)$-modules. The general case follows by noting that the annihilator of a partial flag of type I in V is a partial flag of complementary type J. □

4.3 $S^d(n)$ for 1-dominant degrees d

In this section, we consider the spike module $S^d(n)$ as a $\mathbb{F}_2GL(n)$-module when d is 1-dominant, and relate it to the Steenrod kernel $K^d(n)$ and quotient $Q^d(n)$.

Let $I_j = \{1, 2, \ldots, n-1\} \setminus \{j\}$, so that $Fl^{I_j}(n)$ is the maximal partial flag module given by flags which omit subspaces of dimension j, for $1 \leq j \leq n-1$. Since $c(I_j) = (1, \ldots, 1, 2, 1, \ldots, 1)$, with jth component 2 and and all other components 1, the parabolic subgroup $L^{I_j}(n)$ is obtained from $L(n)$ by replacing $L(2)$ by $GL(2)$ in the 2×2 submatrix in rows and columns i and $i+1$. Hence each maximal partial flag is contained in exactly three complete flags.

Theorem 4.12. *Let ω be 1-dominant with $\deg \omega = d$, and let $I = I(\omega)$. Then as $\mathbb{F}_2 GL(n)$-modules,*

$$S^d(n) \;\cong\; Fl(n) \Big/ \sum_{j \notin I} Fl^{I_j}(n) \;\cong\; \bigoplus_{I \cup K = \{1, \ldots, n-1\}} Fl_K(n), \text{ and}$$

$$\dim S^d(n) = \binom{n}{c(I)}_2 \cdot \prod_{i=1}^{r+1} 2^{c_i(c_i-1)/2}.$$

Proof The lower bound for $\dim S^d(n)$ follows from Theorem 4.3. Thus it suffices to show that the partial flag modules $Fl^{I_j}(n)$, for $j \notin I$ are in the kernel of ϕ^ω. This follows from Proposition 2.6: the sum of the flag polynomials for the three complete flags which contain a maximal partial flag is zero when $j \notin I$. □

Example 4.13. For $n = 3$, each line $L \subset V$ is in a unique complete flag W which lies in a Schubert cell $\mathrm{Sch}(\rho)$ with $\rho(2) < \rho(3)$, and in two complete flags X, Y in the higher Schubert cell $\mathrm{Sch}(\rho \circ \sigma_2)$, where $\sigma_2 = (1, 3, 2)$. Similarly each plane $P \subset V$ is in a unique complete flag W which lies in a Schubert cell $\mathrm{Sch}(\rho)$ with $\rho(1) < \rho(2)$, and in two complete flags X, Y in $\mathrm{Sch}(\rho \circ \sigma_1)$, where $\sigma_1 = (2, 1, 3)$.

For $\omega = (2, 1, 1)$, the polynomials for the flags containing L satisfy the relation $\phi^\omega(W) = \phi^\omega(X) + \phi^\omega(Y)$, and for $\omega = (2, 2, 1)$, polynomials for the flags containing P satisfy this relation. For $\omega = (2, 1, 1)$, $I = \{1\}$ and $c = (1, 2)$, and so $S^8(3) \cong Fl/Fl^{\{1\}} \cong Fl_{\{2\}} \oplus Fl_{\{1,2\}}$. For $\omega = (2, 2, 1)$, $I = \{2\}$ and $c = (2, 1)$, and so $S^{10}(3) \cong Fl/Fl^{\{2\}} \cong Fl_{\{1\}} \oplus Fl_{\{1,2\}}$.

Example 4.14. The results of Theorem 4.12 are tabulated below for $n = 4$, using the minimum ω for each $I = I(\omega)$. The last column, obtained by computer, agrees with the values of $\dim Q^d(n)$ given by Kameko [11] or Sum [19].

d	ω	$I(\omega)$	$S^d(4)$	$\dim S^d(4)$	$\dim K^d(4)$
11	$(3,2,1)$	$\emptyset$	$Fl_{\{1,2,3\}}$	64	64
19	$(3,2,1,1)$	$\{1\}$	$Fl_{\{1,2,3\}} \oplus Fl_{\{2,3\}}$	120	140
23	$(3,2,2,1)$	$\{2\}$	$Fl_{\{1,2,3\}} \oplus Fl_{\{1,3\}}$	140	155
25	$(3,3,2,1)$	$\{3\}$	$Fl_{\{1,2,3\}} \oplus Fl_{\{1,2\}}$	120	120
39	$(3,2,2,1,1)$	$\{1,2\}$	$Fl_{\{1,2,3\}} \oplus Fl_{\{2,3\}} \oplus Fl_{\{1,3\}} \oplus Fl_{\{3\}}$	210	225
41	$(3,3,2,1,1)$	$\{1,3\}$	$Fl_{\{1,2,3\}} \oplus Fl_{\{2,3\}} \oplus Fl_{\{1,2\}} \oplus Fl_{\{2\}}$	210	225
49	$(3,3,2,2,1)$	$\{2,3\}$	$Fl_{\{1,2,3\}} \oplus Fl_{\{1,3\}} \oplus Fl_{\{1,2\}} \oplus Fl_{\{1\}}$	210	210
81	$(3,3,2,2,1,1)$	$\{1,2,3\}$	$Fl(4)$	315	315

A more detailed comparison with [11] or [19] shows the effect of replacing the minimal ω with a 1-dominant ω-vector of higher degree with the same I, so that $S^d(4)$ is as in the table. We now obtain $S^d(4) = K^d(4)$ except for $\omega = (3, 2, 2, 1, \ldots, 1)$ (with at least three 1's) or $(3, \ldots, 3, 2, 1, 1)$ (with at least three 3's), when the discrepancy remains as in the fifth and sixth lines of the table.

By Proposition 4.11, the structure of the quotient of $Q^d(n)$ dual to the submodule $S^d(n)$ of $K^d(n)$ is obtained by exchanging I and J, where $j \in J$ if and only if $n - j \in I$. Thus the structure of the dual module can be read from the table above by exchanging the entries in the second and fourth rows and those in the fifth and seventh rows in the column for $S^d(4)$. Dualizing Theorem 4.12, we have the following result.

Theorem 4.15. *Let ω be 1-dominant, and let $j \in J$ if and only if $n - j \in I(\omega)$. Let $(\phi^\omega)^* : P^d(n) \to Fl(n)$ be the dual of $\phi^\omega : D^d(n) \to Fl(n)$. Then as $\mathbb{F}_2GL(n)$-modules*

$$(\phi^\omega)^* P^d(n) \;\cong\; Fl(n) \Big/ \sum_{j \notin J} Fl^{I_j}(n) \;\cong\; \bigoplus_{J \cup K = \{1, \ldots, n-1\}} Fl_K(n). \qquad \square$$

Remark 4.16. The map $(\phi^\omega)^*$ is essentially the same as the map θ of [16]. By duality the hit polynomials are in $\ker(\phi^\omega)^*$, so that $\mathrm{Im}(\phi^\omega)^*$ is a quotient module of $Q^d(n)$. The first discrepancy between $S^d(n)$ and $K^d(n)$ occurs for $n = 3$ and $d = 8$ (see [1, 2, 16]), when $\dim K^8(3) = 15$ and $\dim S^8(3) = 14$. By Theorem 4.3 or [2], $S^8(3) \cong St(3) \oplus Fl_{\{2\}}$. Dually, the polynomial $x^6yz + xy^6z + xyz^6$ of Singer [17] lies in $\ker(\phi^\omega)^*$ but is not hit: it is an invariant mod hit polynomials, and so represents a 1-dimensional submodule of $Q^8(3)$.

Proposition 4.17. *If d is 1-dominant, $K^d(n)$ and $Q^d(n)$ have a direct summand isomorphic to the Steinberg module $St(n)$ for $GL(n)$.*

Proof For $\omega = (n-1, n-2, \ldots, 1)$, $Q^d(n) \cong St(n)$ [24]. Thus $Fl_I(n) \cong St(n)$ for $I = \{1, 2, \ldots, n-1\}$. By Theorem 4.12, this summand of $Fl(n)$ occurs in $S^d(n)$ for all 1-dominant degrees d. $\square$

5 The duplication map $\delta : Q^d(n) \to Q^{2d+n-1}(n)$

Recall that a monomial $f \in P(n)$ is represented by a $(0,1)$-block F, an array whose entry $F_{i,j}$ in row i and column j is the jth digit in the reversed binary expansion of the exponent of x_i in f, $1 \le i \le n$. We right justify blocks by adding zeros at the ends of rows as required by the context, so that for example any zero block with n rows represents the constant monomial $1 \in P(n)$. A polynomial is represented by a formal sum of blocks. In this section, we interpret some properties of the action of the Steenrod algebra $\mathcal{A}_2$ on polynomials in terms of combinatorial processes on blocks. Recall that $f \in P(n)$ is **hit** if it is in the image of $\mathcal{A}_2^+$, the positive degree part of $\mathcal{A}_2$. We write $f \sim g$ if f and g are homogeneous of degree d and $f - g$ is hit, so that f and g represent the same element in the Steenrod quotient $Q^d(n) = P^d(n)/(P^d(n) \cap \mathcal{A}_2^+ P(n))$.

We write $F = ML$ to denote the partition of a block F into subblocks M and L between columns s and $s+1$ for some s. The corresponding monomial f has the form $f = ml^{2^s}$, where all exponents of m are $\leq 2^s - 1$. In particular, if $s = 1$, the partition corresponds to the square free factorization $f = ml^2$, where m, the **square free part** of f, is a product of distinct variables x_i or 1. The duplication of a monomial simply repeats the first column of its representing block, and so assigns to f the monomial mf^2.

The Kameko map from $P^d(n)$ to $P^{2d+n}(n)$ [9, 10] is defined by $f \mapsto \overline{x} f^2$, where $\overline{x} = x_1 \cdots x_n$ is the product of all the variables, and so duplicates a monomial f all of whose exponents are odd, so that its square free part is $\overline{x}$. In this case, d and n have the same parity. When d is 1-dominant, the Kameko map induces an isomorphism $Q^d(n) \cong Q^{2d+n}(n)$ of $\mathbb{F}_2 GL(n)$-modules. We shall show that duplication also has useful properties when $n > 1$ and the square free part $\overline{x_i} = x_1 \cdots \widehat{x_i} \cdots x_n$ of f contains all except one of the variables.

Definition 5.1. *The* **duplication map** $\delta : P^d(n) \to P^{2d+n-1}(n)$ *is the linear map defined on monomials by* $\delta f = \overline{x_i} f^2$ *if the square free part of* f *is* $\overline{x_i}$*, and* $\delta f = 0$ *otherwise.*

Recall that d is 1-dominant when $d = \sum_{i=1}^n (2^{\lambda_i} - 1)$, where $\lambda = (\lambda_1, \ldots, \lambda_n)$ is strictly decreasing and $\lambda_n \geq 0$. If $\lambda_n > 0$, then d and n have the same parity, and so $\delta = 0$. Hence we shall assume that $\lambda_n = 0$, so that d and n have opposite parity.

Proposition 5.2. *Let* $d = \sum_{i=1}^n (2^{\lambda_i} - 1)$ *be 1-dominant, with* $\lambda_n = 0$*. Then* δ *induces a map* $Q^d(n) \to Q^{2d+n-1}(n)$ *of* $\mathbb{F}_2 GL(n)$*-modules, which we also denote by* δ*.*

The map $\delta : P^d(n) \to P^{2d+n-1}(n)$ does not always send hit polynomials to hit polynomials, for example $f = xyz^2w^2 + x^2y^2zw$ is hit but $\delta f = x^3y^3z^4w^4 + x^4y^4z^3w^3$ is not. The Kameko map shares this defect, for example $x^2y^2z^2$ is hit, but $x^5y^5z^5$ is not.

We begin the proof of Proposition 5.2 by giving conditions for a polynomial $f \in P(n)$ which is a sum of terms with $\omega_1 = n - 1$ to be hit. By collecting terms with the same square free part $\overline{x_i}$, we can write $f = \sum_{i=1}^n \overline{x_i} g_i^2$.

Lemma 5.3. *Let* $n \geq 2$ *and let* $f = \sum_{i=1}^n \overline{x_i} g_i^2$ *be a polynomial in* $P^d(n)$*, where* $g_i \in P^{(d-n+1)/2}(n)$ *for* $1 \leq i \leq n$*. If* f *is hit, then there is a polynomial* $h \in P^{(d-n-1)/2}(n)$ *such that* $g_i \sim x_i h$ *for all* i*. Conversely, if all monomials* $m \in P^d(n)$ *with* $\omega_1(m) < n - 1$ *are hit and if* $g_i \sim x_i h$ *for all* i*, then* f *is hit.*

Proof Since f is hit and $\mathcal{A}_2$ is generated by Sq^{2^k} for $k \geq 0$, we may write

$$f = Sq^1 t + \sum_{j \geq 1} Sq^{2j} t_j \tag{5}$$

where $t \in P^{d-1}(n)$ and $t_j \in P^{d-2j}(n)$. Since $\deg t \equiv n \bmod 2$ and $\deg t_j \equiv n-1 \bmod 2$, we may write $t = \overline{x}h^2 + s$ and $t_j = \sum_{i=1}^n \overline{x_i} h_{i,j}^2 + s_j$, where $\overline{x} = x_1 \cdots x_n$ and where $\omega_1(m) < n-1$ for all monomials m which appear in s and s_j, $j \geq 1$. Hence all monomials m which appear in $Sq^1 s$ and $Sq^{2j} s_j$, $j \geq 1$ also have $\omega_1(m) < n-1$. The same is true for $Sq^k \overline{x_i}$, for all $k > 0$. Since $Sq^1 \overline{x} = \sum_{i=1}^n \overline{x_i} x_i^2$, $Sq^1(\overline{x}h^2) = \sum_{i=1}^n \overline{x_i}(x_i h)^2$, so equating terms in (5) with $\omega_1 = n-1$ gives

$$f = \sum_{i=1}^n \overline{x_i} g_i^2 = \sum_{i=1}^n \overline{x_i}(x_i h)^2 + \sum_{i=1}^n \overline{x_i} \left(\sum_{j \geq 1} Sq^j h_{i,j} \right)^2 .$$

Hence $g_i = x_i h + \sum_{j \geq 1} Sq^j h_{i,j}$, so that $g_i \sim x_i h$. The proof in the other direction follows by reversing the above argument, given that any monomial m in $P^d(n)$ with $\omega_1(m) < n-1$ is hit. □

Lemma 5.4. *Let d be 1-dominant and let $g \in P^{(d-n+1)/2}(n)$ be a polynomial such that $g \sim x_i h$ for some h. Then $\overline{x_i} g^2 \sim x_i(\overline{x}h^2)$, where $\overline{x} = x_1 \cdots x_n$.*

Proof Let $g = x_i h + \sum_{j \geq 1} Sq^j h_j$. Then

$$\overline{x_i} g^2 = \overline{x_i}(x_i^2 h^2) + \overline{x_i} \left(\sum_{j \geq 1} Sq^{2j} h_j^2 \right). \tag{6}$$

Since $\overline{x} = \overline{x_i} x_i$, the first term on the right of (6) is $x_i(\overline{x}h^2)$. By the Cartan formula, $Sq^{2j}(\overline{x_i} h_j^2) = \overline{x_i} Sq^{2j} h_j^2 + s$, where $\omega_1(m) < n-1$ for all monomials m which appear in s. Since d is 1-dominant, s is hit by Singer's theorem [18], and by linearity the second term on the right of (6) is hit. Hence $\overline{x_i} g^2 \sim x_i(\overline{x}h^2)$. □

Proof of Proposition 5.2 We prove first that if f is hit then δf is hit, so that δ induces a linear map $Q^d(n) \to Q^{2d+n-1}(n)$. Let $\omega = \lambda^{\mathrm{tr}}$ be the 1-dominant ω-vector in $P^d(n)$. Write $f \in P^d(n)$ as $f = f' + s$, where the terms in f' have $\omega_1 = n-1$ and the terms in s have $\omega_1 < n-1$. Since ω is 1-dominant, s is hit. Thus f is hit if and only if f' is hit. Since $\delta f = \delta f'$, it is sufficient to prove that if f' is hit then $\delta f'$ is hit.

We may write $f' = \sum_{i=1}^n f_i$ where $f_i = \overline{x_i} g_i^2$, so that $\delta f = \sum_{i=1}^n \overline{x_i} f_i^2$. By Lemma 5.3, there is a polynomial h such that $g_i \sim x_i h$ for $1 \leq i \leq n$. Since $f_i = \overline{x_i} g_i^2$, Lemma 5.4 gives $f_i \sim x_i(\overline{x}h^2)$. Applying Lemma 5.3 again, it follows that δf_1 is hit.

To show that $\delta : Q^d(n) \to Q^{2d+n-1}(n)$ is a map of $\mathbb{F}_2 GL(n)$-modules, it is sufficient to prove that $(\delta f) \cdot \gamma \sim \delta(f \cdot \gamma)$ for all $f \in P^d(n)$ and all γ in a generating set for $GL(n)$. It is clear that $(\delta f)\cdot\gamma = \delta(f\cdot\gamma)$ when γ is a permutation matrix, so the problem can be reduced to the case where γ is the transvection which acts on $P(n)$ by $x_1\cdot\gamma = x_1 + x_2$ and $x_j\cdot\gamma = x_j$ for $2\le j\le n$. Writing $f = f' + s$ as above, we have $\delta s = 0$ and $\delta(s\cdot\gamma) = 0$ since $\omega(s\cdot\gamma)\le\omega(s)$. By linearity, it is thus sufficient to prove that $(\delta f)\cdot\gamma \sim \delta(f\cdot\gamma)$, where $f = \overline{x_i}m^2$ is a monomial with square-free part $\overline{x_i}$, for $1\le i\le n$. Then $f\cdot\gamma = (\overline{x_i}\cdot\gamma)(m\cdot\gamma)^2$, and since $\delta f = \overline{x_i}f^2$, $(\delta f)\cdot\gamma = (\overline{x_i}\cdot\gamma)(f\cdot\gamma)^2 = (\overline{x_i}\cdot\gamma)^3(m\cdot\gamma)^4$. We separate the cases $i=1$, $i>2$ and $i=2$.

If $i=1$, then $\overline{x_1}\cdot\gamma = \overline{x_1}$, so $f\cdot\gamma = \overline{x_1}(m\cdot\gamma)^2$ and $\delta(f\cdot\gamma) = \overline{x_1}^3(m\cdot\gamma)^4 = (\delta f)\cdot\gamma$. If $i>2$, then $\overline{x_1}\cdot\gamma = (x_1+x_2)x_2\cdots\widehat{x_i}\cdots x_n$, and so $f\cdot\gamma = \overline{x_i}(m\cdot\gamma)^2 + s$ where $\omega_1(m) < n-1$ for all monomials m which appear in s. Hence $\delta(f\cdot\gamma) = \delta(\overline{x_i}(m\cdot\gamma)^2) = \overline{x_i}^3(m\cdot\gamma)^4$. Since $(\delta f)\cdot\gamma = ((x_1+x_2)x_2\cdots\hat{x_i}\cdots x_n)^3(m\cdot\gamma)^4$, $\delta(f\cdot\gamma)+(\delta f)\cdot\gamma$ is a sum of monomials m with $\omega_1(m) < n-1$, and so $\delta(f\cdot\gamma)\sim(\delta f)\cdot\gamma$.

In the remaining case $i=2$, $\overline{x_2}\cdot\gamma = \overline{x_1}+\overline{x_2}$, and so $f\cdot\gamma = (\overline{x_1}+\overline{x_2})(m\cdot\gamma)^2$. Hence $\delta(f\cdot\gamma) = (\overline{x_1}^3+\overline{x_2}^3)(m\cdot\gamma)^4$. Since $(\delta f)\cdot\gamma = (\overline{x_1}+\overline{x_2})^3(m\cdot\gamma)^4$, $\delta(f\cdot\gamma)+(\delta f)\cdot\gamma = (x_1^2x_2 + x_1x_2^2)w = Sq^1(x_1x_2)w$, where $w = x_3^3\cdots x_n^3(m\cdot\gamma)^4$. Now $Sq^1(x_1x_2w) = Sq^1(x_1x_2)w + x_1x_2Sq^1w$ and $Sq^1w = \sum_{i=3}^n x_iw$ is a sum of monomials with $\omega_1=\omega_2=n-3$. Hence $x_1x_2Sq^1w$ is a sum of monomials with ω-vector lower than $(n-1,\omega)$, and so it is hit. Thus $(\delta f)\cdot\gamma\sim\delta(f\cdot\gamma)$. □

5.1 The elements $P_t^s \in \mathcal{A}_2$ and 1-back splicing

Recall that the Milnor basis of the Steenrod algebra $\mathcal{A}_2$ consists of elements $Sq(K)$ indexed by finite sequences $K=(k_1,\dots,k_t)$ of integers ≥ 0. In the case $t=1$, $Sq(k)=Sq^k$. The action of $Sq(K)$ on $P(n)$ is determined by the formula $Sq(K)x_i = x_i^{2^t}$, $1\le i\le n$, when $K=(0,\dots,0,1)$ and the Cartan formula (see for example [22, Lemma 5.3])

$$Sq(K)(f\cdot g) = \sum_{I+J=K} Sq(I)f\cdot Sq(J)g, \text{ for all } f,g\in P(n). \tag{7}$$

It follows that, for all $k\ge 0$ and $f\in P(n)$,

$$Sq(I)f^{2^k} = \begin{cases} (Sq(J)f)^{2^k}, & \text{if } I = 2^kJ,\\ 0, & \text{otherwise.}\end{cases} \tag{8}$$

When $K=(0,\dots,0,2^s) = 2^s(0,\dots,0,1)$, the element $Sq(K)$ is commonly denoted by P_t^s. We begin by studying the effect of applying P_t^0 to a block.

Lemma 5.5. *Let $f = ml^2$ be a monomial in $P(n)$ with square-free part m. Then $P_t^0(f) = m'l^2$ where $m' = \sum_i (m/x_i)x_i^{2^t}$, where the sum is over the variables x_i dividing m. Thus the result of applying P_t^0 to a block F is the sum of the blocks formed by removing a digit 1 in the first column of F and adding 2^t, in binary arithmetic, to the same row.*

Proof By (7), $P^0_t(l^2) = 0$ and $P^0_t(f) = P^0_t(m) \cdot l^2$. Since $P^0_t(x_i) = x_i^{2^t}$, $P^0_t(m) = m'$. □

For a monomial $f = x_1^{a_1} \cdots x_n^{a_n} \in P(n)$, we denote by $\alpha_i(f)$ the number of 1's in the binary expansion of a_i, for $1 \leq i \leq n$, and we define the **α-vector** $\alpha(f) = (\alpha_1(f), \ldots, \alpha_n(f))$. Thus the α-vector $\alpha(F)$ of a $(0,1)$-block F is the vector of its row sums. We order α-vectors lexicographically. The Cartan formula $Sq^k(f \cdot g) = \sum_{i+j=k} Sq^i f \cdot Sq^j g$ shows that if f is a monomial and h is a term in the expansion of $Sq^k f$, for $k > 0$, then $\alpha(h) \leq \alpha(f)$.

The blocks given by Lemma 5.5 are of two types. If $F_{k,t+1} = 0$, then the process of applying P^0_t moves the digit 1 from position $(k,1)$ to position $(k, t+1)$ to form a block $B(k)$. All such blocks have the same ω-vector ω and the same α-vector $\alpha(B(k)) = \alpha(F)$. On the other hand, if $F_{k,t+1} = 1$, then the resulting block $E(k)$ has $\omega(E(k)) < \omega$ and $\alpha(E(k)) < \alpha(F)$, because binary addition lowers the number of 1's in row i.

We next consider the effect of applying P^s_t to a block.

Lemma 5.6. *Let $f = ml^{2^s}$ be a monomial in $P(n)$. Then*

$$P^s_t(f) = P^s_t(m) \cdot l^{2^s} + m \cdot (P^0_t(f))^{2^s}.$$

Consequently the result of applying P^s_t to a block $F = ML$, partitioned between columns s and $s+1$, is a sum of blocks of the form ML', where L' is a block in the expansion of P^0_t applied to L, and blocks of the form $M'L''$, where $\omega(M') < \omega(M)$ and $\alpha(M') < \alpha(M)$.

Proof Again, this follows easily from (7) and (8). □

Combining Lemmas 5.5 and 5.6, we obtain the following result.

Lemma 5.7. *The result of applying P^s_t to a block F is the sum of blocks of two types, $B(k)$ and $E(k)$. All blocks $B(k)$ have the same ω-vector ω, and are formed by moving a digit 1 from column $s+1$ of F to column $s+t+1$ of the same row, when F has a 0 in this position. Blocks $E(k)$ have ω-vector $< \omega$.* □

We now explain the combinatorial process of **1-back splicing** a block B in row i and columns $s+1$ and $s+t+1$. We assume that $B_{i,s+t+1} = 1$ and $B_{i,s+1} = 0$. Let F be the block formed from B by defining $F_{i,s+t+1} = 0$ and $F_{i,s+1} = 1$, leaving all other entries fixed. For $1 \leq k \leq n$ and $k \neq i$, let $k \in K$ if and only if $F_{k,s+1} = 1$ and $F_{k,s+t+1} = 0$. For each $k \in K$, let $B(k)$ be the block formed from F by putting $B(k)_{k,s+1} = 0$ and $B(k)_{k,s+t+1} = 1$, leaving all other entries fixed. The operation of **1-back splicing** replaces B by $\sum_{k \in K} B(k)$. When $K = \emptyset$, the operation deletes the block B.

Lemma 5.8. *Let d be 1-dominant, so that there is a unique ω-vector ω of spike type in $P^d(n)$. Let B be an $n \times c$ block representing a monomial in $P^d(n)$. Then for $1 \leq i \leq n$ and $1 \leq s+1 < s+t+1 \leq c$, $B \sim \sum_{k \in K} B(k)$, the result of 1-back splicing of B in row i and columns $s+1, s+t+1$. When $K = \emptyset$, B is hit.*

Proof By Lemma 5.7, $P^s_t(F) = B + \sum_{k\in K} B(k) + E$, where E is a sum of blocks with ω-vectors $< \omega$, and therefore hit, by the hypothesis on ω. □

By including reversed and iterated 1-back splicing operations, we obtain an equivalence relation on polynomials in $P^d(n)$ with the same α- and ω-vectors, which we also call 1-back splicing. When $\omega(f) = \omega(g)$ is the minimum ω-vector of spike type in its degree, $f \sim g$ when f and g are equivalent under 1-back splicing [18].

Definition 5.9. *Let ω_0 be the minimum ω-vector of spike type in $P^d(n)$. We write $f \approx g$ if $f, g \in P^d(n)$ and $f \sim g$ by a finite sequence of 1-back splicing operations, modulo terms in $P^d(n)$ with ω-vectors $< \omega_0$.*

The example $Sq^1(x^3y^3) = x^4y^3 + x^3y^4$ shows that $x^4y^3 \sim x^3y^4$, but $x^4y^3 \not\approx x^3y^4$ since the α-vectors are different. Note that $f \approx g$ includes the case when no splicing operations occur, but f and g differ by terms of ω-vector $< \omega_0$.

5.2 The Steinberg module

We recall that an element $\gamma \in GL(n)$ acts on $P(n)$ by matrix substitution. In particular, the transvections $\gamma(i,j) \in U(n)$ act on $P(n)$ by $x_i \cdot \gamma(i,j) = x_i + x_j$, where $i < j$, and $x_k \cdot \gamma(i,j) = x_k$ for $k \neq i$. The effect of $\gamma(i,j)$ on a block B is to produce the sum of all blocks B' which correspond to subvectors of row i of B in the following way. Each such subvector b is a binary number. The 1's in b are changed to 0's and b is added to the lower row j of B in binary arithmetic to produce a new block B'. If row j has a 0 in every column where b has a 1, then B' is formed simply by moving the 1's in b from row i to row j. Otherwise $\alpha(B') < \alpha(B)$ and $\omega(B') < \omega(B)$.

In [24] we showed that for $d = 2^n - n - 1$, $Q^d(n) \cong St(n)$, the Steinberg representation of $GL(n)$. It is well known that the restriction of $St(n)$ to $U(n)$ can be identified with the group algebra of $U(n)$ [7]. In particular, $Q^d(n)$ is a cyclic $\mathbb{F}_2U(n)$-module. The next result makes this more precise.

Proposition 5.10. *For $m \leq n$ let $s_m = \prod_{i=1}^m x_i^{2^{m-i}-1} \in P^d(n)$, a spike of degree $d = 2^m - m - 1$ with $\omega(s_m) = \nu_{m-1} = (m-1, m-2, \ldots, 1)$. Then for any polynomial $f \in P^d(n)$ such that $\omega(f) = \nu_{m-1}$, there is an element $u \in \mathbb{F}_2U(n)$ such that $f \approx s_m u$.*

Proof It is sufficient to prove the result when f is a monomial, since the general case follows by adding elements $u \in \mathbb{F}_2U(n)$ corresponding to the terms of f. The proof is in two parts; the cases $m = 1, 2$ being trivial, part **(i)** deals with the case $m = 3$, and part **(ii)** gives an inductive procedure for $m \geq 3$ which iteratively replaces a monomial by an equivalent polynomial which is 'closer' to s_m modulo the joint action of 1-back splicing operations and elements of $U(n)$ acting on s_m.

(i) Let $m = 3$, so that $\nu_{m-1} = (2,1)$. There are two types of block to consider, those of type A with a 1 in three different rows, and those of type B with a 1 in one row and two 1's in another row. Then $A_{i,2} = 1$ for some i and $A_{j,1} = A_{k,1} = 1$ for some $j < k$. Now 1-back splicing in row i gives $A \approx A' + A''$, where $A'_{i,1} = A'_{j,1} = 1, A'_{k,2} = 1$ and $A''_{i,1} = A''_{k,1} = 1, A''_{j,2} = 1$, as shown below for $n = 4$ and $i = 4$, $j = 1$, $k = 2$.

$$A = \begin{matrix} 1\,0 \\ 1\,0 \\ 0\,0 \\ 0\,1 \end{matrix}, \quad A' = \begin{matrix} 1\,0 \\ 0\,1 \\ 0\,0 \\ 1\,0 \end{matrix}, \quad A'' = \begin{matrix} 0\,1 \\ 1\,0 \\ 0\,0 \\ 1\,0 \end{matrix}, \quad B(1,4) = \begin{matrix} 1\,1 \\ 0\,0 \\ 0\,0 \\ 1\,0 \end{matrix}, \quad B(2,4) = \begin{matrix} 0\,0 \\ 1\,1 \\ 0\,0 \\ 1\,0 \end{matrix}.$$

Let $B(j,i)$ denote the block of the second type defined by $B_{j,1} = B_{j,2} = 1$, $B_{i,1} = 1$ and other entries zero, as illustrated above. Then $B(j,i) \cdot (\gamma(j,k) - 1) = B(k,i) + A' + A''$. Hence $A \approx A' + A'' = B(j,i) \cdot (\gamma(j,k) - 1) + B(k,i)$, and so the problem is reduced to blocks of type B.

If $j = 1$, $i > 2$ then $B = S_2 \cdot (1 - \gamma(2,i))$ and the argument is complete. Otherwise, when $j > 1$ and $i > 1$ we have $B(j,i) \approx B(j,1) \cdot (1 - \gamma(1,i))$, reducing the problem to the case $i = 1$. Then $B(1,j) \cdot (1 - \gamma(1,j)) = B(j,1) + E$, where $\omega(E) < \nu_2$. Since $\nu_2 = (2,1)$ is the lowest ω-vector of spike type in degree 4, we have $B(1,j) \cdot (1 - \gamma(1,j)) \approx B(j,1)$, reducing the problem to the previous case of $B(1,j)$. This completes the argument for $m = 3$.

(ii) Let $m \geq 4$, and assume that the result is proved for all $k < m$. Let $J = ML$ be a $n \times (m-1)$ block with $\omega(J) = \nu_{m-1}$, where M is the first column of J and L is a $n \times (m-2)$ block with $\omega(L) = \nu_{m-2}$. By the inductive hypothesis, $L \approx S_{m-1}u$ where $u \in \mathbb{F}_2U(n)$. Thus we may replace L by S_{m-1} by using transvections $\gamma(i,j)$ and 1-back splicing. We must consider the effect of these operations on the first column M of J. If a 1-back splicing operation affects M, this gives blocks with ω-vector $< \nu_{m-1}$. The action on J of each element $\gamma \in U(n)$ appearing in u is to produce blocks of two types, those of the type ML' which sum to give MS_{m-1}, and those of the type $M'L''$ where $\alpha(M') < \alpha(M)$, i.e. $M' < M$ considered as a $(0,1)$-vector with lexicographic order. Hence by induction on the (α-vector of) the first column, $J = ML \approx \sum_{M'} M'S_{m-1}$, a formal sum of blocks which are identical except for the first column, which is some vector $M' \leq M$. The problem is thus reduced to the case of blocks J of the form $J = ML$ where $L = S_{m-1}$.

If $J_{(1,1)} = 1$, then every digit in the first row of J is 1. In this case we may ignore any block J' arising from the first row of J by a 1-back splicing of J, because $\omega(J') < \nu_{m-1}$. Hence we may concentrate the action of $\mathcal{A}_2$ on the $(n-1) \times (m-1)$ block L' obtained from J by removing its first row. Now $\omega(L') = \nu_{m-2}$, and so by the inductive hypothesis $L' \approx S_{m-1}u'$ where $u' \in \mathbb{F}_2U(n)$ is a sum of elements of $U(n)$ which fix the first variable. Thus $J \approx S_m u'$, and the inductive step is established.

If, on the other hand, $J_{1,1} = 0$, then, by iterated 1-back splicing in the first row, we obtain a sum of blocks J' such that $J'_{1,j} = 1$ for $1 \leq j < m-1$, and again it suffices to deal with one block J' of this type. Then the $(n-1) \times (m-2)$ subblock L'' of J' obtained by deleting the first row and last column of J' again has ω-vector ν_{m-2}. Reasoning as in the case $J_{(1,1)} = 1$, we may assume $L'' \approx S_{m-1}u''$ for some $u'' \in \mathbb{F}_2U(n)$ consisting of elements of $U(n)$ which fix the first variable. Let J'' be the block obtained from J' by replacing L'' by S_{m-1}. Then the first $m-2$ columns of J'' and S_m are the same, but $J''_{i,m-1} = 1$ for some $i \geq 1$. The following example illustrates this for $n = m = i = 4$;

$$J'' = \begin{matrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{matrix}, \quad A' = \begin{matrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}, \quad A'' = \begin{matrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}.$$

If the 1 in the last column of J'' is not in the last row, then any 1-back splicing or transvection affecting the first $m-3$ columns of J'' produces blocks with ω-vector $< \omega(J'') = \nu_{m-1}$. However, in the remaining case (as in the example) transvections $\gamma(i,n)$ will affect the first $m-3$ columns of J'' without changing the ω-vector, and a different procedure is required.

In this case, we repeat the initial procedure of type A, as in the case $m = 3$, applied to the last two columns of J. This produces the equivalence $J'' \approx A' + A''$, where A' and A'' have the same first $m-3$ columns as J'' and are defined in the last two columns as in the case $m = 3$. Similarly we define the block $B(j,i)$ with the same first $m-3$ columns as J'' and the last two columns as in the case $m = 3$. Then $A' + A'' \approx B(1,n) \cdot (\gamma(1,2) - 1) + B(2,n)$. Every digit in the first row of $B(1,n)$ is 1, and so we can complete the inductive step for $B(1,n)$ as for the case above where $J_{(1,1)} = 1$. Since every digit in the second row of $B(2,n)$ is 1, we can similarly complete the inductive step for $B(1,n)$ by applying the inductive hypothesis to the subbblock obtained by omitting the second row and the last column, and by using a final transvection $\gamma(1,2)$ to move the 1 in the last column from row 2 to row 1. This completes the induction. □

We have been unable to prove that δ is an isomorphism when d is 2-dominant, although computer calculations for small cases suggest that this is true, so that $Q^d(n) \cong Fl(n)$. The following result gives a weaker upper bound for the dimension of $Q^d(n)$ in the minimal 2-dominant case.

Theorem 5.11. *Let* $d = (4^n - 1)/3 - n$. *Then* $Q^d(n)$ *is a cyclic* G*-module and* $\dim Q^d(n) \leq 2^{n(n-1)}$.

Proof Recall that d is the minimal 2-dominant degree in $P(n)$, with spikes of type $\omega = (n-1, n-1, n-2, n-2, \ldots, 2, 2, 1, 1)$. Let F be a block with $\omega(F) = \omega$, and let F' denote the subblock of odd columns of F and F'' the subblock of even columns. In particular, for the zip block $Z(\rho_0)$ for the reversal permutation ρ_0,

Z' is the spike block in the Steinberg degree $2^n - n - 1$ associated with the identity permutation ι, and Z'' is the spike block associated with ρ_0. We shall demonstrate that Z is a generator of $Q^d(n)$ in a special way, namely that any element $F \in Q^d(n)$ has the form $Z\gamma$ for some element γ in $\mathbb{F}_2 L(n) \otimes \mathbb{F}_2 U(n)$. This establishes the upper bound for the dimension since $\dim \mathbb{F}_2 U(n) = \dim \mathbb{F}_2 L(n) = 2^{n(n-1)/2}$. The argument is completed by the following lemmas, which exhibit the symmetrical roles played by $U(n)$ and $L(n)$.

Lemma 5.12. *For $f \in P^d(n)$, $d = (4^n - 1)/3 - n$, there exist monomials h_i whose blocks H_i are such that $H_i' = Z'$ and elements $u_i \in \mathbb{F}_2 U(n)$ such that $f \approx \sum_i h_i \cdot u_i$.*

Proof By linearity we may assume that f is a monomial, with block F. Let H denote the block defined by $H' = Z'$ and $H'' = F''$, as illustrated below in the case $n = 3$, $d = 18$;

$$F = \begin{matrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{matrix}, \quad H = \begin{matrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}, \quad Z = \begin{matrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{matrix}.$$

By Proposition 5.10 there exists an element $u \in \mathbb{F}_2 U(n)$ such that $Z' \cdot u \approx F'$. Then two types of blocks occur in the expansion of $H \cdot u$. In the first, linear substitution by a matrix of u leaves H'' unaltered. In the second, H'' (and possibly H') is affected. For a block G of the second type, we have $\alpha(G'') < \alpha(F'')$, whereas the sum of the blocks of the first type yields F. Hence we can write $F \approx H \cdot u + \sum_j G_j$, where $\alpha(G_j'') < \alpha(F'')$ for all blocks G_j. Now the minimum value of $\alpha(G'')$ occurs in the case $G'' = Z''$. Since $U(n)$ fixes Z'', the lemma is true in this case. By induction on $\alpha(F'')$, it is true in general. □

Since $L(n)$ fixes Z', we obtain a similar result interchanging $U(n)$ with $L(n)$ and Z' with Z''. In particular the following is true.

Lemma 5.13. *For $d = (4^n - 1)/3 - n$, let $h \in P^d(n)$ be a monomial with block H such that $H' = Z'$. Then there exists $u' \in \mathbb{F}_2 L(n)$ such that $h \approx Z \cdot u'$.* □

Combining Lemmas 5.12 and 5.13, a formal sum of blocks F representing a polynomial $f \in P^d(n)$ can be written

$$F \approx \sum_i H_i \cdot u_i \approx Z \cdot \sum_i u_i' u_i = Z \cdot \sum_{i,j,k} \gamma_{i,j}' \gamma_{i,k}$$

where $\gamma_{i,j}' \in L(n)$ and $\gamma_{i,k} \in U(n)$. This completes the proof of Theorem 5.11.

5.3 Surjectivity of δ

We restate Theorem 1.3, and prove it in this section.

Theorem 5.14. *Let* $d = \sum_{i=1}^{n-1}(2^{\lambda_i} - 1)$*, where* $\lambda_i - \lambda_{i+1} \geq 1$ *for* $1 \leq i < n-1$ *and* $\lambda_{n-1} \geq 2$*. Then* $\delta : Q^d(n) \to Q^{2d+n-1}(n)$ *is a surjection.*

Since $Q^d(n) \cong St(n)$ when $d = 2^n - n - 1$ [24], Theorem 4.15 shows that δ is not surjective in general when $\lambda_{n-1} = 1$. We first prove the result in the minimal case $\lambda = (n, n-1, \dots, 2, 0)$.

Proposition 5.15. *For* $n \geq 2$ *and* $d = 2^{n+1} - n - 3$*, the duplication map* $\delta : Q^d(n) \to Q^{2d+n-1}(n)$ *is a surjection.*

Proof The argument is by induction on n for $n \geq 2$. The unique ω-vector of spike type in $P^d(n)$ is $\omega = (n-1, n-1, n-2, \dots, 2, 1)$. For $n = 2$, each of the four blocks

$$C_1 = \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}, \quad C_2 = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{matrix}, \quad C_3 = \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{matrix}, \quad C_4 = \begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{matrix}$$

is equivalent by 1-back splicing to a duplicate block, and so $\delta \colon Q^3(2) \to Q^7(2)$ is surjective.

We assume the result for $k < n$. The idea is to show that $Q^{2d+n-1}(n)$ is generated by duplicated blocks up to the action of $U(n)$, and so, since $\delta \colon Q^d(n) \to Q^{2d+n-1}(n)$ is a $\mathbb{F}_2GL(n)$-module map, actually generated by duplicated blocks,

Let F be a block of degree $2d + n - 1$ with $\omega(F) = (n-1, \omega)$. Partition F in the form $F = AB$, where A is the first two columns of F and B has degree $2^n - n - 1$. By Proposition 5.10, we can write $B \approx S_n u$ for some $u \in \mathbb{F}_2U(n)$, where S_n is the block of the spike s_n. Then $F \approx (AS_n)u + E$, where E is a sum of partitioned blocks of the form $A'B'$ in which $\alpha(A') < \alpha(A)$. The lowest possible α-vector for A' is $(0, 2, \cdots, 2)$, in which case the corresponding block is a duplicate. Hence, by induction on the α-vector, we may assume that $Q^{2d+n-1}(n)$ is generated as a $\mathbb{F}_2U(n)$-module by blocks of the form AS_n. Blocks of this form which are not duplicates must have at least one digit 1 in the first row of A. Now the top row of the spike block S_n starts with $n-1$ 1's. By 1-back splicing in the first row, we can now assume that $Q^{2d+n-1}(n)$ is generated by duplicate blocks and blocks whose first row starts with n 1's. For such a block F, consider the subblock G obtained from F by taking the first n columns and deleting the first row of F. Since $\deg G = 2d' + n - 2$ where $d' = 2^n - n - 4$ and $\omega(G) = (n-2, n-2, n-2, n-3, \cdots, 2, 1)$, we are presented with the same problem for G as for F, but with $n-1$ instead of n. By the inductive hypothesis we may therefore assume that G is equivalent by 1-back splicing to a sum of duplicate blocks in $P(n-1)$. Since the action of the corresponding 1-back splicing in $P(n)$ on the first row gives blocks of lower ω-vector, this completes the induction. □

Proof of Theorem 5.14 By using the Milnor basis elements $P^s_t \in \mathcal{A}_2$, we can extend the above argument to the general case. Given a block $F \in Q^{2d+n-1}(n)$, form a subblock G containing the first two columns of F with $\omega(G) =$

$(n-1, n-1, n-2, \ldots, 1)$. By Lemma 5.7, 1-back splicing can be conducted on G without disturbing the other columns of F up to equivalence. Thus the proof of Proposition 5.15 can be generalized to show that $Q^{2d+n-1}(n)$ is generated by duplicated blocks. □

Finally we consider the dual map $\delta^* : D^{2d+n-1}(n) \to D^d(n)$. It follows from Theorem 5.14 that $\delta^* : K^{2d+n-1}(n) \to K^d(n)$ is a $\mathbb{F}_2GL(n)$-module when d is as in Proposition 5.2. To describe δ^* explicitly, recall that $\langle f, g\rangle$ for $f \in P^d(n)$ and $g \in D^d(n)$ is defined by $\langle m, m'\rangle = 1$ if $m' \in D^d(n)$ is the divided monomial with the same exponents as the monomial $m \in P^d(n)$, and $\langle m, m'\rangle = 0$ otherwise. Since $\langle m, \delta^*(m')\rangle = \langle \delta(m), m'\rangle$ for all m', $\delta^*(m') = 0$ unless m' is a duplicate, in which case it is the monomial obtained from m' by deleting the first column of its block. Since deleting the first column of a spike block in $D^{2d+n-1}(n)$ gives a spike block in $D^d(n)$, δ^* maps $S^{2d+n-1}(n)$ to $S^d(n)$.

Proposition 5.16. *When d is as in Theorem 5.14, $\delta^* : K^{2d+n-1}(n) \to K^d(n)$ is injective and maps $S^{2d+n-1}(n)$ isomorphically to $S^d(n)$.*

Proof The first statement is clear from duality, and the second is clear since we may prefix a suitable column to the block of a spike in $D^d(n)$ to give the block of a spike in $D^{2d+n-1}(n)$.

□

In the case $\lambda = (2n-2, 2n-4, \ldots, 2, 0)$, when $d = (4^n-1)/3-n$, $S^d(n) \cong Fl(n)$ by Theorem 2.8, and $Q^d(n)$ has a quotient isomorphic to $Fl(n)$. For $n \leq 4$, it is known that $Q^d(n) \cong Fl(n)$, but we are unable to prove this for $n > 4$.

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