Linking first occurrence polynomials over $\mathbb{F}_p$ by Steenrod operations

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Abstract

This paper provides analogues of the results of [16] for odd primes p. It is proved that for certain irreducible representations $L(\lambda)$ of the full matrix semigroup $M_n(\mathbb{F}_p)$, the first occurrence of $L(\lambda)$ as a composition factor in the polynomial algebra $\mathbf{P} = \mathbb{F}_p[x_1, \ldots, x_n]$ is linked by a Steenrod operation to the first occurrence of $L(\lambda)$ as a submodule in $\mathbf{P}$. This operation is given explicitly as the image of an admissible monomial in the Steenrod algebra $\mathcal{A}_p$ under the canonical anti-automorphism χ. The first occurrences of both kinds are also linked to higher degree occurrences of $L(\lambda)$ by elements of the Milnor basis of $\mathcal{A}_p$.

1 Introduction

Our aim is to obtain results corresponding to those of [16] for the case where the prime $p > 2$. In this we are only partly successful. The main theorem of [16] gives a Steenrod operation which links the first occurrence of each irreducible representation $L(\lambda)$ of the full matrix semigroup $M_n(\mathbb{F}_2)$ in the polynomial algebra $\mathbf{P} = \mathbb{F}_2[x_1, \ldots, x_n]$ with the first occurrence of $L(\lambda)$ as a submodule in $\mathbf{P}$. Here $M_n(\mathbb{F}_2)$ acts on $\mathbf{P}$ on the right by linear substitutions, which commute with the action of the Steenrod algebra $\mathcal{A}_2$ on $\mathbf{P}$ on the left. By 'first occurrence' we have in mind the decomposition $\mathbf{P} = \sum_{d \geq 0} \mathbf{P}^d$, where $\mathbf{P}^d$ is the module of homogeneous polynomials of total degree d, and the known facts that there are minimum degrees $d_c(\lambda)$ and $d_s(\lambda)$ in which $L(\lambda)$ occurs, uniquely in each case, as a composition factor and as a submodule respectively.

For an odd prime p, we have again the commuting actions of $M_n = M_n(\mathbb{F}_p)$ on the right of the polynomial algebra $\mathbf{P} = \mathbb{F}_p[x_1, \ldots, x_n]$ and the algebra $\mathcal{A}_p$ of Steenrod pth powers (no Bocksteins) on the left. We refer to $\mathcal{A}_p$, somewhat inaccurately, as the Steenrod algebra, and grade it so that P^r raises degree by $r(p-1)$. There are p^n isomorphism classes of irreducible $\mathbb{F}_p[M_n]$-modules $L(\lambda)$, indexed by partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, which are column p-regular, i.e. $0 \leq \lambda_i - \lambda_{i+1} \leq p-1$ for $1 \leq i \leq n$, where $\lambda_{n+1} = 0$ [8, 9, 10]. The problem solved in [16] is certainly more difficult in this context. The submodule degree

$d_s(\lambda)$ has recently been determined [12] for every irreducible representation $L(\lambda)$ of M_n, but $d_c(\lambda)$ is not known in general. In particular, the first occurrence problem appears to be difficult even for the 1-dimensional representations $\det^k$, $1 \leq k \leq p-3$, $p > 3$, see [2, 3], although it is solved for $\det^{p-2}$ [1]. (The partition indexing $\det^k$ is $(k, \ldots, k) = (k^n)$, i.e. k repeated n times.) Further, it is not known in general whether $\mathbf{P}^{d_c(\lambda)}$ has a unique composition factor isomorphic to $L(\lambda)$. Here we identify a class of irreducible representations $L(\lambda)$ which behave systematically. Since they arise naturally by considering tensor powers of the p-truncated polynomial algebra $\mathbf{T} = \mathbf{P}/(x_1^p, \ldots, x_n^p)$, we call them $\mathbf{T}$-*regular.*

Our main result, Theorem 5.7, gives a Steenrod operation $\theta(\lambda)$ which links the first occurrence and the first submodule occurrence in $\mathbf{P}$ of a $\mathbf{T}$-regular $L(\lambda)$. This determines $d_c(\lambda)$ in the $\mathbf{T}$-regular case. The operation $\theta(\lambda)$ is given explicitly as the image of an admissible monomial under the canonical anti-automorphism χ of $\mathcal{A}_p$. Calculations for $n \leq 3$ suggest that such an operation $\theta(\lambda)$ may exist for every irreducible representation $L(\lambda)$ of M_n, but we do not pursue this here. Tri [14] has given an 'algebraic' alternative to this 'topological' method of finding $d_c(\lambda)$, using coefficient functions of $\mathbb{F}_p[M_n]$-modules.

For $p = 2$, $\mathbf{T}$ may be identified with the exterior algebra $\Lambda(x_1, \ldots, x_n)$, and all the irreducible representations $L(\lambda)$ of M_n are $\mathbf{T}$-regular. For $p > 2$, the only irreducible 1-dimensional $\mathbf{T}$-regular representations of M_n are the 'trivial' representation, in which all matrices act as 1, and the $\det^{p-1}$ representation, in which non-singular matrices act as 1 and singular matrices as 0. The 'trivial' representation, for which $\lambda = (0)$, occurs in $\mathbf{P}$ only as $\mathbf{P}^0$, the constant polynomials. Our key example is the $\det^{p-1}$ representation. This occurs first as a composition factor as the top degree $\mathbf{T}^{n(p-1)}$ of $\mathbf{T}$, where it is generated by the monomial $(x_1 x_2 \cdots x_n)^{p-1}$ modulo pth powers, and first as a submodule in degree $p_n = (p^n - 1)/(p-1)$, where it is generated by the Vandermonde determinant

$$w(n) = \begin{vmatrix} x_1 & x_2 & \cdots & x_n \\ x_1^p & x_2^p & \cdots & x_n^p \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{p^{n-1}} & x_2^{p^{n-1}} & \cdots & x_n^{p^{n-1}} \end{vmatrix}.$$

Theorem 1.1 *Let χ be the canonical anti-isomorphism of $\mathcal{A}_p$. Then for $n \geq 1$,*

$$\chi(P^{p_n - n})(x_1 x_2 \cdots x_n)^{p-1} = w(n)^{p-1},$$

where $p_n = (p^n - 1)/(p-1)$.

This result is true for $p = 2$ if we interpret P^r as Sq^r [16]. The operation $\chi(P^{p_n - n})$ may be replaced by the admissible monomial $P^{p^{n-1}-1} \cdots P^{p^2-1} P^{p-1}$, which is identical to the Milnor basis element $P(p-1, \ldots, p-1)$ of length $n-1$ (see Proposition 3.2). In general the operation $\chi(P^{r_1} P^{r_2} \cdots P^{r_m})$ used in Theorem 5.7 can not be replaced by an admissible monomial or a Milnor basis element.

The structure of the paper is as follows. Section 2 contains basic facts about the action of $\chi(P^r)$ and Milnor basis elements on polynomials. Section 3 contains independent proofs of Theorem 1.1 using invariant theory and by direct computation. In Section 4 we introduce the class of $\mathbf{T}$-regular partitions to which our main results apply, and extend Theorem 1.1 to $\mathbf{T}^d$ for all d. The main results are stated in Section 5 and proved in Section 6. Section 7 relates these results to the $\mathbb{F}_p[M_n]$-module structure of $\mathbf{P}$. Section 8 gives Milnor basis elements which link the first occurrence and (in certain cases) the first submodule occurrence of a $\mathbf{T}$-regular representation of M_n with submodules in higher degrees.

The remarks which follow are intended to place our results in topological, combinatorial and algebraic contexts. As for topology, recall (e.g. [17]) that there is an $\mathcal{A}_p$-module decomposition $\mathbf{P} = \oplus_\lambda \delta(\lambda)\mathbf{P}(\lambda)$, where the λ-isotypical summand $\mathbf{P}(\lambda)$ is an indecomposable $\mathcal{A}_p$-module, and where $\delta(\lambda) = \dim L(\lambda)$, the dimension of $L(\lambda)$. Identifying $\mathbf{P}$ with the cohomology algebra $H^*(\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty; \mathbb{F}_p)$, this decomposition can be realized (after localization at p) by a homotopy equivalence $\Sigma(\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty) \sim \bigvee_\lambda \delta(\lambda) Y_\lambda$, which splits the suspension of the product of n copies of infinite complex projective space $\mathbb{C}P^\infty$ as a topological sum of spaces Y_λ such that $H^*(Y_\lambda; \mathbb{F}_p) = \Sigma\mathbf{P}(\lambda)$. The family of $\mathcal{A}_p$-modules $\mathbf{P}(\lambda)$ is of major interest in algebraic topology. From this point of view, we determine the connectivity of Y_λ for $\mathbf{T}$-regular λ (Corollary 5.8) and find a nonzero cohomology operation $\theta(\lambda)$ on its bottom class (Theorem 5.7).

As for combinatorics and algebra, our aim is to provide information relating the $\mathcal{A}_p$-module structure of $\mathbf{P}(\lambda)$ to combinatorial properties of λ and representation theoretic properties of $L(\lambda)$. The operation $\theta(\lambda)$ and its source and target polynomials are combinatorially determined by λ. The target polynomial is defined by $w(\lambda') = \prod_{j=1}^{\lambda_1} w(\lambda'_j)$, where λ' is the conjugate of λ, so that $w(\lambda')$ is a product of determinants corresponding to the columns of the diagram of λ. This polynomial has already appeared in various forms in the literature. In Green's description [8, (5.4d)] of the highest weight vector of the dual Weyl module $H^0(\lambda)$, $w(\lambda')$ appears as a 'bideterminant' in the coordinate ring of $M_n(K)$, where K is an infinite field of characteristic p. A proof that $w(\lambda')$ generates a submodule of $\mathbf{P}^{d_s(\lambda)}$ isomorphic to $L(\lambda)$ was given in [7, Proposition 1.3], and a proof that this is the first occurrence of $L(\lambda)$ as a submodule in $\mathbf{P}$ was given in [12].

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2 Preliminary results

In this section we use variants of the Cartan formula $P^r(fg) = \sum_{r=s+t} P^s f \cdot P^t g$ to study the action on polynomials of the elements $\chi(P^r)$ and Milnor basis elements

$P(R)$ in the Steenrod algebra $\mathcal{A}_p$. We begin with the standard formula

$$P^i(x^{p^b}) = \begin{cases} x^{p^{b+1}}, & \text{if } i = p^b, \\ 0, & \text{otherwise for } i > 0. \end{cases} \tag{1}$$

In particular, we wish to evaluate Steenrod operations on Vandermonde determinants of the form

$$[x_{i_1}^{s_1}, x_{i_2}^{s_2}, \ldots, x_{i_n}^{s_n}] = \begin{vmatrix} x_{i_1}^{s_1} & x_{i_2}^{s_1} & \ldots & x_{i_n}^{s_1} \\ x_{i_1}^{s_2} & x_{i_2}^{s_2} & \ldots & x_{i_n}^{s_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{i_1}^{s_n} & x_{i_2}^{s_n} & \ldots & x_{i_n}^{s_n} \end{vmatrix},$$

where the exponents $s_1, \ldots, s_n$ are powers of p. As above, we shall abbreviate such determinants by listing their diagonal entries in square brackets: in particular, $w(n) = [x_1, x_2^p, \ldots, x_n^{p^{n-1}}]$. As in Theorem 1.1, we write $p_n = (p^n - 1)/(p - 1)$, so that $p_0 = 0$ and $p_n - p_j = (p^n - p^j)/(p - 1)$. The following result is a straightforward calculation using the Cartan formula and (1).

Lemma 2.1 *If $r = p_n - p_j$, $0 \leq j \leq n$, then*

$$P^r w(n) = [x_1, x_2^p, \ldots, x_j^{p^{j-1}}, x_{j+1}^{p^{j+1}}, \ldots, x_n^{p^n}],$$

and $P^r w(n) = 0$ otherwise. In particular, $P^r w(n) = 0$ for $0 < r < p^{n-1}$. □

To simplify signs, we usually write $\widehat{P}^r$ for $(-1)^r \chi(P^r)$. Thus if v is one of the generators x_i of $\mathbf{P}$, or more generally any linear form $v = \sum_{i=1}^n a_i x_i$ in $\mathbf{P}^1$,

$$\widehat{P}^r v = \begin{cases} v^{p^b}, & \text{if } r = p_b,\ b \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Formula (2) follows from (1) by using the identity $\sum_{i+j=r} (-1)^i P^i \widehat{P}^j = 0$ in $\mathcal{A}_p$ and induction on r. Using the identity $\sum_{i+j=r} (-1)^i \widehat{P}^i P^j = 0$ and induction on k, (2) can be generalized to

$$\widehat{P}^r x^{p^k} = \begin{cases} x^{p^b}, & \text{if } r = p_b - p_k,\ b \geq k, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

This leads to the following generalization of [16, Lemma 2.2].

Lemma 2.2

$$\widehat{P}^r [x_1^{p^k}, x_2^{p^{k+1}}, \ldots, x_n^{p^{k+n-1}}] = \begin{cases} [x_1^{p^k}, \ldots, x_{n-1}^{p^{k+n-2}}, x_n^{p^b}], & \textit{if } r = p_b - p_{k+n-1}, \\ 0, & \textit{otherwise.} \end{cases}$$

The modifications required to the proof given in [16] are straightforward. $\square$

In evaluating the operations $\widehat{P}^r$, we shall frequently make use of the Cartan formula expansion for polynomials $f, g \in \mathbf{P}$:

$$\widehat{P}^r(fg) = \sum_{s+t=r} \widehat{P}^s f \cdot \widehat{P}^t g, \tag{4}$$

which holds because χ is a coalgebra homomorphism.

Lemma 2.3 *For all polynomials f, g in $\mathbf{P}$ and all $r \geq 0$,*

$$\widehat{P}^r(f^p g) = \sum_{r=ps+t} (\widehat{P}^s f)^p \widehat{P}^t g.$$

Proof By (4) it suffices to prove the case $g = 1$, i.e.

$$\widehat{P}^r f^p = \begin{cases} (\widehat{P}^s f)^p, & \text{if } r = ps, \\ 0, & \text{if } r \text{ is not divisible by } p. \end{cases}$$

In this case, the Cartan formula (4) gives $\widehat{P}^r f^p = \sum \widehat{P}^{r_1} f \cdots \widehat{P}^{r_p} f$, where the sum is over all ordered decompositions $r = \sum_{i=1}^p r_i$, $r_i \geq 0$. Except in the case where $r_1 = \ldots = r_p = s$, cyclic permutation of $r_1, \ldots, r_p$ gives p equal terms which cancel in the sum. $\square$

We write $\alpha(k)$ for the sum of the digits in the base p expansion of a positive integer k, i.e. if $k = \sum_{i \geq 0} a_i p^i$ where $0 \leq a_i \leq p-1$, then $\alpha(k) = \sum_{i \geq 0} a_i$. Thus $\alpha(k)$ is the minimum number of powers of p which have sum k, and $\alpha(k) \equiv k \mod p-1$. Formula (2) leads to the following simple sufficient condition for the vanishing of $\widehat{P}^r$ on a homogeneous polynomial of degree d.

Lemma 2.4 *If $\alpha(r(p-1)+d) > d$, then $\widehat{P}^r f = 0$ for all $f \in \mathbf{P}^d$.*

Proof Since the action of $\widehat{P}^r$ is linear and commutes with specialization of the variables, it is sufficient to prove this when $f = x_1 x_2 \cdots x_d$. By (4) $\widehat{P}^r f = \sum \widehat{P}^{r_1} x_1 \widehat{P}^{r_2} x_2 \cdots \widehat{P}^{r_d} x_d$, where the sum is over all ordered decompositions $r = r_1 + r_2 + \ldots + r_d$ with $r_1, r_2, \ldots, r_d \geq 0$. By (2), the only non-zero terms are those in which $r_i = p_{k_i}$ for some non-negative integers $k_1, k_2, \ldots, k_d$. But then $r(p-1) + d = \sum_i p^{k_i}$, and the result follows by definition of α. $\square$

Lemma 2.5 *Let $k \geq 0$ and let $v = \sum_{i=1}^n a_i x_i$ be a linear form in $\mathbf{P}^1$. Then*

$$\widehat{P}^{p^k-1} v^{p-1} = v^{p^k(p-1)}.$$

Proof There is a unique way to write $p^k - 1$ as the sum of $p-1$ integers of the form p_i for $i \geq 0$, namely $p^k - 1 = (p-1)p_k$. The result now follows from (2) and the Cartan formula (4). $\square$

Remark 2.6 The same method can be used to evaluate $\widehat{P}^r v^{p-1}$ for all r. The result is

$$\widehat{P}^r v^{p-1} = \begin{cases} c_r v^{(r+1)(p-1)}, & \text{if } \alpha((r+1)(p-1)) = p-1, \\ 0, & \text{otherwise,} \end{cases}$$

where if $(r+1)(p-1) = j_1 p^{a_1} + \ldots + j_s p^{a_s}$, with $a_1 > \ldots > a_s \geq 0$ and $\sum_{i=1}^{s} j_i = p-1$, then $c_r = (p-1)!/(j_1! j_2! \cdots j_s!)$.

The following result, the 'Cartan formula for Milnor basis elements' is well-known (cf. [16, Lemma 5.3]).

Lemma 2.7 *For a Milnor basis element $P(R) = P(r_1, \ldots, r_n)$ and polynomials $f, g \in \mathbf{P}$,*

$$P(R)(fg) = \sum_{R=S+T} P(S)f \cdot P(T)g,$$

where the sum is over all sequences $S = (s_1, \ldots, s_n)$ and $T = (t_1, \ldots, t_n)$ of non-negative integers such that $r_i = s_i + t_i$ for $1 \leq i \leq n$. □

In the same way as for Lemma 2.3, this gives the following result.

Lemma 2.8 *Let $P(R) = P(r_1, \ldots, r_n)$ be a Milnor basis element and let $f, g \in \mathbf{P}$ be polynomials. Then*

$$P(R)(f^p g) = \sum_{R=pS+T} (P(S)f)^p \cdot P(T)g. \quad \square$$

Here $R = pS + T$ means that $r_i = ps_i + t_i$ for $1 \leq i \leq n$.

3 The $\det^{p-1}$ representation

In this section we give three proofs of Theorem 1.1. The first uses the results of [12] on submodules, while the second is a variant of this which uses only classical invariant theory. The third proof is computational. The first two proofs use the following preliminary result, which shows that the operation $\widehat{P}^{p_n - n}$ maps to 0 all monomials of degree $n(p-1)$ other than the generating monomial $(x_1 x_2 \cdots x_n)^{p-1}$ for $\det^{p-1}$.

Lemma 3.1 *Let f be a polynomial in $\mathbf{P}^{n(p-1)}$ which is divisible by x^p for some variable $x = x_i$, $1 \leq i \leq n$. Then $\widehat{P}^{p_n - n} f = 0$.*

Proof Let $f = x^p g$, where $g \in \mathbf{P}$. Then by Lemma 2.3

$$\widehat{P}^{p_n - n} f = \sum_{p_n - n = ps + t} (\widehat{P}^s x)^p \widehat{P}^t g. \tag{5}$$

By (2), $\widehat{P}^s x = 0$ if $s \neq p_k$ for some k with $0 \leq k \leq n-2$. Thus it is sufficient to prove that $\widehat{P}^t g = 0$ for $t = p_n - n - p \cdot p_k$, where $g \in \mathbf{P}^{n(p-1)-p}$. By Lemma 2.4, this holds when $\alpha((t+n)(p-1)-p) > n(p-1)-p$. Now $(t+n)(p-1)-p = p_n(p-1) - p \cdot p_k(p-1) - p = p^n - p^{k+1} - 1$, hence $\alpha((t+n)(p-1)-p) = n(p-1)-1 > n(p-1)-p$ as required. Thus $\widehat{P}^t g = 0$ in all terms of (5) in which $\widehat{P}^s x \neq 0$, and so $\widehat{P}^{p_n - n} f = 0$. □

First Proof of Theorem 1.1 We first show that the monomial $m = (x_1 x_2^p \cdots x_n^{p^{n-1}})^{p-1}$ appears in $\widehat{P}^{p_n-n}(x_1 \cdots x_n)^{p-1}$ with coefficient 1. In the Cartan formula expansion (4), m can appear only in the term arising from the decomposition $p_n - n = r_1 + r_2 + \ldots + r_n$, where $r_k = p^{k-1} - 1$ for $1 \leq k \leq n$. By Lemma 2.5, m appears in this term with coefficient 1.

By Lemma 3.1, $\widehat{P}^{p_n-n}$ maps all other monomials in degree $n(p-1)$ to 0. Hence $\widehat{P}^{p_n-n}(x_1 \cdots x_n)^{p-1}$ generates a 1-dimensional $\mathbb{F}_p[M_n]$-submodule of $\mathbf{P}^{p^n-1}$. Since $(x_1 \cdots x_n)^{p-1}$ generates the 1-dimensional quotient $\mathbf{T}^{n(p-1)}$ of $\mathbf{P}^{n(p-1)}$ and since $\mathbf{T}^{n(p-1)} \cong \det^{p-1}$, this submodule of $\mathbf{P}^{p^n-1}$ is also isomorphic to $\det^{p-1}$.

It is known [12] that the first submodule occurrence of $\det^{p-1}$ for M_n in $\mathbf{P}$ is generated by $w(n)^{p-1}$, and that this is the unique submodule occurrence of the $\det^{p-1}$ module in degree $p^n - 1$. Since m is the product of the leading diagonal terms in $w(n)^{p-1} = [x_1, x_2^p, \ldots, x_n^{p^{n-1}}]^{p-1}$, m also has coefficient 1 in $w(n)^{p-1}$. □

Second Proof of Theorem 1.1 We recall that $D(n,p)$ is the ring of $GL_n(\mathbb{F}_p)$-invariants in $\mathbf{P}$, and that it is a polynomial algebra over $\mathbb{F}_p$ with generators $Q_{n,i}$ in degree $p^n - p^i$ for $0 \leq i \leq n-1$. We may identify $Q_{n,0}$ with $w(n)^{p-1}$. Since $\mathbf{T}^{n(p-1)}$ is isomorphic to the trivial $GL_n(\mathbb{F}_p)$-module, it follows as in our first proof that $\widehat{P}^{p_n-n}(x_1 \cdots x_n)^{p-1} \in D(n,p)$.

We shall prove that $w(n)$ divides $\widehat{P}^{p_n-n}(x_1 \cdots x_n)^{p-1}$. Recall that $w(n)$ is the product of linear factors $c_1 x_1 + \ldots + c_n x_n$, where $c_1, \ldots, c_n \in \mathbb{F}_p$. If we specialize the variables in $(x_1 \cdots x_n)^{p-1}$ by imposing the relation $c_1 x_1 + \ldots + c_n x_n = 0$, then every monomial in the resulting polynomial is divisible by x^p for some variable $x = x_i$. By Lemma 3.1, such a monomial is in the kernel of $\widehat{P}^{p_n-n}$. Thus $\widehat{P}^{p_n-n}(x_1 \cdots x_n)^{p-1}$ is divisible by $c_1 x_1 + \ldots + c_n x_n$, and so it is divisible by $w(n)$.

Now an element of $D(n,p)$ in degree $p^n - 1$ which is divisible by $w(n)$ must be a scalar multiple of $Q_{n,0} = w(n)^{p-1}$. For if a polynomial in the remaining generators $Q_{n,1}, \ldots, Q_{n,n-1}$ of $D(n,p)$ is divisible by $w(n)$, the quotient would be $SL_n(\mathbb{F}_p)$-invariant, giving a non-trivial polynomial relation between $Q_{n,1}, \ldots, Q_{n,n-1}$ and $w(n)$. This contradicts Dickson's theorem that these are algebraically independent generators of the polynomial algebra of $SL_n(\mathbb{F}_p)$-invariants in $\mathbf{P}$. □

Our third proof of Theorem 1.1 is by direct calculation. We shall evaluate the Milnor basis element $P(p-1, \ldots, p-1)$ of length $n-1$ on $(x_1 \cdots x_n)^{p-1}$. The following result relates the element $P(p-1, \ldots, p-1, b)$ of length n to admissible monomials and to the anti-automorphism χ. In particular, we show that $P(p-1, \ldots, p-1)$ and $\widehat{P}^{p_n-n}$ have the same action on $(x_1 \cdots x_n)^{p-1}$.

Proposition 3.2 *For $n \geq 1$ and $1 \leq b \leq p-1$,*

(i) $P(p-1,\ldots,p-1,b) = P^{(b+1)p^{n-1}-1} \cdots P^{(b+1)p-1} P^b$,

(ii) $\widehat{P}^{(b+1)p_n-n} g = P(p-1,\ldots,p-1,b)g$ *if* $\deg g \leq n(p-1)+b$,

(iii) $\widehat{P}^{(b+1)p_n-n} = P^{(b+1)p^{n-1}} \widehat{P}^{(b+1)p_{n-1}-n} + P(p-1,\ldots,p-1,b)$, *where* $\widehat{P}^{-1} = 0$.

Proof Statement **(i)**is a special case of [4, Theorem 1.1]. For **(ii)**, recall [11] that $\widehat{P}^d$ is the sum of all Milnor basis elements $P(R)$ in degree $d(p-1)$. Here $R = (r_1, r_2, \ldots)$ is a finite sequence of non-negative integers, and $P(R)$ has degree $|R| = \sum(p^i-1)r_i$ and excess $e(R) = \sum r_i$. In particular, $P^d = P(d)$ is the unique Milnor basis element of maximum excess d in degree $d(p-1)$, but in general there may be more than one element of minimum excess in a given degree.

We will show that $P(p-1,\ldots,p-1,b)$ is the unique element of minimum excess $e = (n-1)(p-1)+b$ in degree $d(p-1)$ when $d = (b+1)p_n - n$. By [11, Lemma 8] a bijection $P(r_1, r_2, \ldots, r_m) \leftrightarrow P^{t_1} P^{t_2} \cdots P^{t_m}$ between the Milnor basis and the admissible basis of $\mathcal{A}_p$ is defined by $t_m = r_m$ and $t_i = r_i + pt_{i+1}$ for $1 \leq i < m$. This preserves both the degree and the excess. Thus it is equivalent to prove that $m = P^{(b+1)p^{n-1}-1} \cdots P^{(b+1)p-1} P^b$ is the unique admissible monomial of minimum excess in degree $d(p-1)$. Now the excess of an admissible monomial $P^{t_1} P^{t_2} \cdots P^{t_m}$ is $pt_1 - d(p-1)$ where $d = \sum_i t_i$, and so it is minimal when t_1 is minimal. It is easy to verify that m is the unique admissible monomial in degree $d(p-1)$ for which $t_1 = (b+1)p^{n-1}-1$, and that this value of t_1 is minimal.

Note that p divides $|R|+e(R)$ for all R. Hence $\widehat{P}^{(b+1)p_n-n} - P(p-1,\ldots,p-1,b)$ has excess $> e+p-1 = n(p-1)+b$, and so $\widehat{P}^{(b+1)p_n-n} g = P(p-1,\ldots,p-1,b)g$ when g is a polynomial of degree $\leq n(p-1)+b$.

(iii) Recall Davis's formula [5]

$$P^u \widehat{P}^v = \sum_{|R|=(p-1)(u+v)} \binom{|R|+e(R)}{pu} P(R), \tag{6}$$

which we may apply in the case $u = (b+1)p^{n-1}$, $v = (b+1)p_{n-1} - n$ to show that $P^u \widehat{P}^v$ is the sum of all Milnor basis elements in degree $d(p-1)$ other than the element $P(p-1,\ldots,p-1,b)$ of minimal excess.

For $R = (p-1,\ldots,p-1,b)$ we have $|R|+e(R) = (b+1)p^n - p$, and since $pu = (b+1)p^n$ the coefficient in (6) is zero. Since p divides $|R|+e(R)$ for all R, $|R|+e(R) \geq (b+1)p^n$ for all other R with $|R| = d(p-1)$. As remarked above, the unique element of maximal excess is P^d itself, and so for all R we have $|R|+e(R) \leq pd = (b+1)(p+p^2+\ldots+p^n) - pn$. It is clear from this inequality that the coefficient in (6) is 1 for all $R \neq (p-1,\ldots,p-1,b)$. □

Third Proof of Theorem 1.1 Let $\theta_n = P^{p^n-1} \cdots P^{p^2-1} P^{p-1}$ for $n \geq 1$, and $\theta_0 = 1$. We assume that $\theta_{n-1}(x_1 \cdots x_n)^{p-1} = w(n)^{p-1}$ as induction hypothesis on n, the case $n = 1$ being trivial.

The cofactor expansion of $w(n+1) = [x_1, x_2^p, \ldots, x_{n+1}^{p^n}]$ by the top row gives $w(n+1) = \sum_{i=1}^{n+1}(-1)^i x_i \Delta_i^p$, where $\Delta_i = [x_1, \ldots, x_{i-1}^{p^{i-2}}, x_{i+1}^{p^{i-1}}, \ldots, x_{n+1}^{p^{n-1}}]$. Hence $w(n+1)\cdot(x_1\cdots x_{n+1})^{p-1} = \sum_{i=1}^{n+1}(-1)^i x_i^p \Delta_i^p (x_1\cdots x_{i-1}x_{i+1}\cdots x_{n+1})^{p-1}$.

By Proposition 3.2(**i**), $\theta_n = P(p-1,\ldots,p-1)$ of length n, and so by Lemma 2.8 $\theta_n(w(n+1)\cdot(x_1\cdots x_{n+1})^{p-1}) = \sum_{i=1}^{n+1}(-1)^i x_i^p \Delta_i^p \theta_n (x_1\cdots x_{i-1}x_{i+1}\cdots x_{n+1})^{p-1}$. Since $\theta_n = P^{p^{n-1}}\theta_{n-1}$, $\theta_n(x_1\cdots x_{i-1}x_{i+1}\cdots x_{n+1})^{p-1} = P^{p^{n-1}}\Delta_i^{p-1}$ by the induction hypothesis. Since Δ_i^{p-1} has degree p^n-1, $P^{p^{n-1}}\Delta_i^{p-1} = \Delta_i^{p(p-1)}$. Hence $\theta_n(w(n+1)\cdot(x_1\cdots x_{n+1})^{p-1}) = \sum_{i=1}^{n+1}(-1)^i x_i^p \Delta_i^{p^2} = w(n+1)^p$.

By Lemma 2.1, $P^r w(n+1) = 0$ for $0 < r < p^n$. As $\theta_n = P^{p^n-1}\cdots P^{p^2-1}P^{p-1}$, iterated application of the Cartan formula gives $\theta_n(w(n+1)\cdot(x_1\cdots x_{n+1})^{p-1}) = w(n+1)\cdot\theta_n(x_1\cdots x_{n+1})^{p-1}$. Hence $w(n+1)\cdot\theta_n(x_1\cdots x_{n+1})^{p-1} = w(n+1)^p$. Cancelling the factor $w(n+1)$, the inductive step is proved.

4 T-regular partitions

In this section we define the special class of $\mathbf{T}$-*regular* partitions, and extend Theorem 1.1 to give a Steenrod operation $\widehat{P}^r$ which links the first occurrence and first submodule occurrence of $\mathbf{T}^d$ for all d. In fact we prove a more general result which links the first occurrence to a family of higher degree occurrences.

The truncated polynomial module $\mathbf{T}^d = \mathbf{P}^d/(\mathbf{P}^d \cap (x_1^p, \ldots, x_n^p))$ has a $\mathbb{F}_p$-basis represented in $\mathbf{P}^d$ by the set of all monomials $x_1^{s_1}x_2^{s_2}\cdots x_n^{s_n}$ of total degree $d = \sum_i s_i$ with $s_i < p$ for $1 \le i \le n$. By [2, Theorem 6.1] $\mathbf{T}^d \cong L((p-1)^{n-1}b)$, where $d = (n-1)(p-1)+b$ and $1 \le b \le p-1$. We regard the corresponding diagram as a block of $p-1$ columns, in which the first b columns have length n and the remaining $p-b-1$ columns have length $n-1$. Given a partition λ, we can divide its diagram into m blocks of $p-1$ columns and compare the blocks with the diagrams corresponding to these. (The mth block may have $< p-1$ columns.) For $1 \le j \le m$, let $\lambda_{(j)}$ be the partition whose diagram is the jth block, and let $\gamma_j = \deg \lambda_{(j)}$ be the number of boxes in the jth block.

Definition 4.1 A column p-regular partition λ is $\mathbf{T}$-*regular* if $L(\lambda_{(j)}) \cong \mathbf{T}^{\gamma_j}$ for all j. Equivalently, for all $a \ge 1$, there is at most one value of i for which $(a-1)(p-1) < \lambda_i < a(p-1)$. If λ is $\mathbf{T}$-regular, we call γ the $\mathbf{T}$-*conjugate* of λ.

In the case $p=2$, all column 2-regular partitions are $\mathbf{T}$-regular, and $\gamma = \lambda'$, the conjugate of λ. If κ is column 2-regular, then the partition $\lambda = (p-1)\kappa$ obtained by multiplying each part of κ by $p-1$ is $\mathbf{T}$-regular. Since λ is column p-regular, $\gamma_j - \gamma_{j+1} \ge p-1$ for all j, and $m \le n$. Thus there is a bijection $\lambda \leftrightarrow \gamma$ between the set of $\mathbf{T}$-regular partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ and the set of partitions $\gamma = (\gamma_1, \ldots, \gamma_n)$ which satisfy $\gamma_1 \le n(p-1)$ and $\gamma_j - \gamma_{j+1} \ge p-1$ for $1 \le j \le n-1$. In terms of the Mullineux involution M on the set of all row p-regular partitions, λ and γ are related by $M(\gamma) = \lambda'$ [15, Proposition 3.13].

We next extend Theorem 1.1 to give linking formulae for the representations $\mathbf{T}^d$. It will be convenient to introduce abbreviated notation for some further Vandermonde determinants. Let $w(n,a) = [x_1, \ldots, x_a^{p^{a-1}}, x_{a+1}^{p^{a+1}}, \ldots, x_n^{p^n}]$ for $0 \le a \le n$, where the exponent p^a is omitted. In particular, $w(n,n) = w(n)$ and $w(n,0) = w(n)^p$.

Proposition 4.2 *For $n \ge 1$ and $1 \le i \le p-1$, let $i = i_1 + \cdots + i_s$ where $i_1, \ldots, i_s > 0$, and let $j = i_1 p_{a_1} + \cdots + i_s p_{a_s}$, where $a_1 > \ldots > a_s \ge 0$. Then*

$$\widehat{P}^{p_n - n - j}\left((x_1 x_2 \cdots x_{n-1})^{p-1} x_n^{p-i-1}\right) = (-1)^{i(n-1)-j} w(n)^{p-i-1} \cdot \prod_{r=1}^{s} w(n-1, a_r)^{i_r}.$$

Specializing to the case $s = 1$, $j = i p_{n-1}$ and putting $b = p-1-i$, we obtain an operation linking the first occurrence and the first submodule occurrence of the representation $\mathbf{T}^d$. Theorem 1.1 can be taken as the case $b = 0$ or as the case $b = p-1$; we choose $b = p-1$ to fit notation later.

Corollary 4.3 *For $n \ge 1$ and $1 \le b \le p-1$,*

$$\widehat{P}^{(b+1)p_{n-1}-(n-1)}\left((x_1 x_2 \cdots x_{n-1})^{p-1} x_n^b\right) = w(n)^b \cdot w(n-1)^{p-b-1}. \qquad \square$$

Proof of Proposition 4.2 We introduce a parameter into Theorem 1.1, by working in $\mathbb{F}_p[x_1, \ldots, x_{n+1}]$ and writing $x_{n+1} = t$ in order to distinguish this variable. Since the action of $\mathcal{A}_p$ commutes with the linear substitution which maps x_n to $x_n + t$ and fixes x_i for $i \ne n$, we obtain

$$\widehat{P}^{p_n - n}(x_1 \cdots x_{n-1}(x_n + t))^{p-1} = [x_1, x_2^p, \ldots, x_{n-1}^{p^{n-2}}, (x_n + t)^{p^{n-1}}]^{p-1}. \tag{7}$$

Expanding the left hand side of (7) by the binomial theorem, we obtain

$$\sum_{i=0}^{p-1} (-1)^i \widehat{P}^{p_n - n}((x_1 \cdots x_{n-1})^{p-1} x_n^{p-1-i} t^i).$$

The right hand side of (7) is

$$[x_1, x_2^p, \ldots, x_{n-1}^{p^{n-2}}, x_n^{p^{n-1}} + t^{p^{n-1}}]^{p-1} = \sum_{i=0}^{p-1} (-1)^i w(n)^{p-1-i} [x_1, x_2^p, \ldots, x_{n-1}^{p^{n-2}}, t^{p^{n-1}}]^i,$$

since $w(n) = [x_1, x_2^p, \ldots, x_n^{p^{n-1}}]$. The summands in (7) corresponding to $i = 0$ give the original result, Theorem 1.1, and so are equal. In fact we can equate the ith summands for all i. This happens because $\widehat{P}^r$ raises degree by $r(p-1)$, so that the powers t^k which occur in the ith summand on the left have $k \equiv i \bmod p-1$, while if t^k occurs in the ith summand on the right, then k is the sum of i powers of p, so that again $k \equiv i \bmod p-1$. Hence for $1 \le i \le p-1$ we have

$$\widehat{P}^{p_n - n}((x_1 \cdots x_{n-1})^{p-1} x_n^{p-1-i} t^i) = w(n)^{p-1-i} \cdot [x_1, x_2^p, \ldots, x_{n-1}^{p^{n-2}}, t^{p^{n-1}}]^i. \tag{8}$$

Since the powers t^k of t which can appear here are such that k is the sum of i powers of p, we can write $k = i_1 p^{a_1} + \ldots + i_s p^{a_s}$, where $a_1 > \ldots > a_s \geq 0$ and $i_1 + \ldots + i_s = i$. Using the expansion

$$[x_1, x_2^p, \ldots, x_{n-1}^{p^{n-2}}, t^{p^{n-1}}] = \sum_{a=0}^{n-1} (-1)^{n-1-a} w(n-1, a) t^{p^a}$$

we can evaluate the coefficient of t^k on the right hand side of (8) as

$$(-1)^{i(n-1)-j} \frac{i!}{i_1! \cdots i_s!} w(n)^{p-1-i} \cdot w(n-1, a_1)^{i_1} \cdots w(n-s, a_s)^{i_s},$$

where we have simplified the sign by noting that $a_1 i_1 + \ldots + a_s i_s \equiv j \bmod 2$ since $p_a \equiv a \bmod 2$,. By the Cartan formula (4), the left hand side of (8) is

$$\sum_{j=0}^{p_n - n} \widehat{P}^{p_n - n - j} \left((x_1 \cdots x_{n-1})^{p-1} x_n^{p-1-i} \right) \cdot \widehat{P}^j t^i$$

Here the term in t^k arises from $\widehat{P}^j t^i$ where $k = j(p-1) + i$, so that $j = i_1 p_{a_1} + \ldots + i_s p_{a_s}$, and since this decomposition of j as a sum of at most i powers of p is unique, formulas (2) and (4) give $\widehat{P}^j t^i = (i!/i_1! \cdots i_s!) t^k$. Thus equating coefficients of t^k in (8) gives the result. □

5 Linking for T-regular representations

In this section we state our main results. We fix an odd prime p and a positive integer n throughout. As in [16], our results will be statements about polynomials in n variables when λ has length n, i.e. λ has n nonzero parts. There is no loss of generality, since the projection in M_n which sends x_n to 0 and x_i to x_i for $i < n$ maps $L(\lambda)$ to zero if $\lambda_n > 0$ and on to the corresponding $\mathbb{F}_p[M_{n-1}]$-module $L(\lambda)$ if $\lambda_n = 0$ (cf. [2, Section 3]). Hence we shall always assume that $\lambda_n \neq 0$.

We first establish some notation. Given a $\mathbf{T}$-regular partition λ of length n, we define a polynomial $v(\lambda)$ whose degree $d_c(\lambda)$ is given by (9) and which 'represents' $L(\lambda)$, in the sense that the submodule of $\mathbf{P}^{d_c(\lambda)}$ generated by $v(\lambda)$ has a quotient module isomorphic to $L(\lambda)$. We index the diagram of λ using matrix coordinates (i, j), so that $1 \leq i \leq n$ and $1 \leq j \leq \lambda_i$.

Definition 5.1 *The kth* antidiagonal *of the diagram of λ is the set of boxes such that $j + i(p-1) = k + p - 1$. If the lowest box is in row i and the highest is in row $i - s + 1$, let $v_k(\lambda) = [x_{i-s+1}, x_{i-s+2}^p, \ldots, x_i^{p^{s-1}}]$, and let $v(\lambda) = \prod_{k=1}^{\gamma_1} v_k(\lambda)$.*

Thus an antidiagonal is the set of boxes which lie on a line of slope $1/(p-1)$ in the diagram, and $v(\lambda)$ is a product of corresponding Vandermonde determinants.

Indenting successive rows by $p-1$ columns, we obtain a shifted diagram whose columns correspond to these antidiagonals. The **T**-conjugate γ of λ records the number of antidiagonals γ_s of length $\geq s$ for all $s \geq 1$.

Example 5.2 Let $p=5$, $\lambda=(9,6,3)$, so that $\gamma=(11,6,1)$. The shifted diagram

```
* * * * * * * * *
        * * * * * *
                * * *
```

gives $v(\lambda) = x_1^4 \cdot [x_1, x_2^5]^4 \cdot [x_1, x_2^5, x_3^{25}] \cdot [x_2, x_3^5] \cdot x_3$.

Recall [12] that $w(\lambda') = \prod_{j=1}^{\lambda_1} w(\lambda'_j)$ generates the first occurrence of $L(\lambda)$ as a submodule in $\mathbf{P}$. Thus we can rewrite the linking theorem for $\mathbf{T}^d$, Corollary 4.3, as follows.

Theorem 5.3 *Let $d=(n-1)(p-1)+b$, where $n \geq 1$ and $1 \leq b \leq p-1$, so that $\mathbf{T}^d \cong L(\lambda)$ where $\lambda = ((p-1)^{n-1}b)$. Then $\widehat{P}^r v(\lambda) = w(\lambda')$, where $r=(b+1)p_{n-1}-(n-1)$ and $p_{n-1}=(p^{n-1}-1)/(p-1)$.* □

By the *leading monomial* of a polynomial we mean the monomial $\prod_{i=1}^n x_i^{s_i}$ occurring in it (ignoring the nonzero coefficient) whose exponents are highest in left lexicographic order. The leading monomial $s(\lambda)$ of $v(\lambda)$ is obtained by multiplying the principal antidiagonals in the determinants $v_k(\lambda)$, $1 \leq k \leq \gamma_1$. (In Example 5.2, $s(\lambda) = x_1^{49}x_2^{14}x_3^3$.) The base p expansion of every exponent in $s(\lambda)$ has the form $s_i = c_k p^k + (p-1)p^{k-1} + \ldots + (p-1)p + (p-1)$, i.e. $s_i \equiv -1 \bmod p^k$, where $p^k < s_i < p^{k+1}$. We adapt the terminology introduced by Singer [13], by calling such a monomial a 'spike'. In the case $p=2$, $s(\lambda) = x_1^{2^{\lambda_1}-1} \cdots x_n^{2^{\lambda_n}-1}$. A polynomial which contains such a spike can not be 'hit', i.e. it can not be the image of a polynomial of lower degree under a Steenrod operation. This is easily seen by considering the 1-variable case. Hence the polynomial $v(\lambda)$ is not hit.

Proposition 5.4 *Let λ be **T**-regular with **T**-conjugate γ.*

(i) *If $\lambda_i = a_i(p-1)+b_i$, $a_i \geq 0$, $1 \leq b_i \leq p-1$, then $s(\lambda) = \prod_{i=1}^n x_i^{(b_i+1)p^{a_i}-1}$.*

(ii) *With $\lambda_{(j)}$ as in Definition 4.1, $s(\lambda) = v(\lambda_{(1)}) \cdot v(\lambda_{(2)})^p \cdots v(\lambda_{(m)})^{p^{m-1}}$.*

(iii) *The coefficient of $s(\lambda)$ in $v(\lambda)$ is $(-1)^{\epsilon(\lambda)}$, where $\epsilon(\lambda) = \sum_{j=1}^{[m/2]}(-1)^{j-1}\gamma_{2j}$.*

Proof Formulae **(i)** and **(ii)** are easily read off from a tableau obtained by entering p^{j-1} in each box in the jth block of $p-1$ columns of the diagram of λ, and reading this according to rows and to blocks of columns. For **(iii)**, note that the sign of the term arising from the leading antidiagonal in the expansion of an

$s \times s$ determinant is $+1$ for $s \equiv 0, 1 \bmod 4$ and -1 for $s \equiv 2, 3 \bmod 4$, and that the diagram of λ has γ_j antidiagonals of length $\geq j$. □

In Theorem 5.5 we establish **(i)** a 'level 0 formula', which gives a sufficient condition for $\widehat{P}^r v(\lambda) = 0$, and **(ii)** a 'level 1 formula', which gives a sufficient condition for $\widehat{P}^r v(\lambda)$ to be a product related to the decomposition $\lambda = \lambda_{(1)} + \lambda^-$ which splits off the first $p-1$ columns of the diagram. Thus $\lambda_{(1)} = ((p-1)^{n-1}b)$, where $\gamma_1 = (n-1)(p-1)+b$ and $1 \leq b \leq p-1$, and λ^- is defined by $\lambda_i^- = \lambda_i - (p-1)$ if $\lambda_i \geq p-1$, and $\lambda_i^- = 0$ otherwise. Our main linking result, Theorem 5.7, follows from Theorem 5.5 by induction on m, the length of γ. The proofs of Theorems 5.5 and 5.7 are deferred to Section 6.

Theorem 5.5 *Let λ be $\mathbf{T}$-regular with $\mathbf{T}$-conjugate γ, let d_c be defined by* (9) *below, and let $R(r, \lambda) = r(p-1) + d_c(\lambda) - d_c(\lambda^-)$. Recall that $\alpha(k)$ is the sum of the digits in the base p expansion of k.*

(i) *If $\alpha(R(r, \lambda)) > \gamma_1$, then $\widehat{P}^r v(\lambda) = 0$.*

(ii) *If $\alpha(R(r, \lambda)) = \gamma_1$, then $\widehat{P}^r v(\lambda) = \widehat{P}^{r+d_c(\lambda^-)} v(\lambda_{(1)}) \cdot v(\lambda^-)$.*

Remark 5.6 Taking $p = 2$ and $P^r = Sq^r$, this reduces to [16, Theorem 2.1], since that theorem can be applied to $\lambda_{(1)} = (1^n)$ to obtain $\widehat{Sq}^{r+d_c(\lambda^-)} v(\lambda_{(1)}) = [x_1^{2^{a_1}}, \ldots, x_n^{2^{a_n}}]$, where $a_1 < \ldots < a_n$. The hypothesis on r is satisfied since $r + d_c(\lambda^-) + n = r + d_c(\lambda) - d_c(\lambda^-) = 2^{a_1} + \ldots + 2^{a_n}$.

Combining Theorem 5.3 with Theorem 5.5, we obtain our main theorem.

Theorem 5.7 *Let λ be $\mathbf{T}$-regular with $\mathbf{T}$-conjugate γ of length m. For $1 \leq k \leq m$, let $\gamma_k = (n_k - 1)(p-1) + b_k$, where $n_k \geq 1$ and $1 \leq b_k \leq p-1$. Then*

$$\widehat{P}^{r_m} \cdots \widehat{P}^{r_2} \widehat{P}^{r_1} v(\lambda) = w(\lambda'),$$

where $r_k = (b_k + 1)p_{n_k - 1} - (n_k - 1) - \sum_{j=k+1}^{m} p^{j-k-1} \gamma_j$.

This theorem determines the first occurrence degree $d_c(\lambda)$ when λ is $\mathbf{T}$-regular.

Corollary 5.8 *Let λ be $\mathbf{T}$-regular with $\mathbf{T}$-conjugate γ. Then the degree in which the irreducible module $L(\lambda)$ first occurs as a composition factor in the polynomial algebra $\mathbf{P}$ is given by*

$$d_c(\lambda) = \sum_{i=1}^{m} p^{i-1} \gamma_i, \qquad (9)$$

and the $\mathbb{F}_p[M_n]$-submodule of $\mathbf{P}^{d_c(\lambda)}$ generated by $v(\lambda)$ has a quotient module isomorphic to $L(\lambda)$.

Proof By [7] or [12] $w(\lambda')$ generates a submodule of $\mathbf{P}^{d_s(\lambda)}$ isomorphic to $L(\lambda)$. By Theorem 5.7, there is a Steenrod operation $\theta = \theta(\lambda)$ and a polynomial $v(\lambda) \in \mathbf{P}^d$, where d is given by (9), such that $\theta(v(\lambda)) = w(\lambda')$. Hence the quotient of the submodule generated by $v(\lambda)$ in $\mathbf{P}^d$ by the intersection of this submodule with the kernel of θ is a composition factor of $\mathbf{P}^d$ which is isomorphic to $L(\lambda)$. Hence the first occurrence degree $d_c(\lambda) \leq d$. But $d_c(\lambda) \geq d$ by [3, Proposition 2.13], and hence $d_c(\lambda) = d$. □

As an example, for $p = 3$ the partition $\lambda = (5, 3, 2)$ is **T**-regular with **T**-conjugate $\gamma = (6, 3, 1)$. The module $L(5, 3, 2)$ first occurs as a composition factor in degree $6 + 3 \cdot 3 + 1 \cdot 9 = 24$, and as a submodule in degree $5 + 3 \cdot 3 + 2 \cdot 9 = 32$. The calculations of [1] and [6] for $n \leq 3$ support the conjecture that the the first occurrence degree $d_c(\lambda)$ is given by the formula above if and only if λ is **T**-regular.

The integers r_i in Theorem 5.7 can be calculated from a tableau $\mathrm{Tab}(\lambda)$ obtained by entering integers into the diagram of λ as follows: if a box in row i is the highest box in its antidiagonal, write p_{i-1} in that box and continue down the antidiagonal, multiplying the number entered at each step by p.

Lemma 5.9 *The sum of the numbers entered in the kth block of $p-1$ columns using the above rule is r_k. The element $P^{r_1}P^{r_2}\cdots P^{r_m}$ is an admissible monomial in $\mathcal{A}_p$, i.e. $r_k \geq pr_{k+1}$ for $1 \leq k \leq m-1$.*

Example 5.10 For $p = 3$, $\lambda = (6, 5, 4, 3, 2)$, the tableau below gives $(r_1, r_2, r_3) = (100, 20, 1)$.

$\mathrm{Tab}(\lambda) =$

0	0	0	0	0	0
0	0	0	0	1	
0	0	3	4		
9	12	13			
39	40				

Noting that $\widehat{P}^r = (-1)^r\chi(P^r)$, in this case Theorem 5.7 states that in $\mathbf{P}^{300}$,

$$\chi(P^{100}P^{20}P^1)\left(x_1^2 \cdot [x_1, x_2^3]^2 \cdot [x_1, x_2^3, x_3^9]^2 \cdot [x_2, x_3^3, x_4^9] \cdot [x_3, x_4^3] \cdot [x_4, x_5^3] \cdot x_5\right)$$
$$= -[x_1, x_2^3, x_3^9, x_4^{27}, x_5^{81}]^2 \cdot [x_1, x_2^3, x_3^9, x_4^{27}] \cdot [x_1, x_2^3, x_3^9] \cdot [x_1, x_2^3] \cdot x_1.$$

Proof of Lemma 5.9 The inequality $r_k \geq pr_{k+1}$ for $1 \leq k \leq m-1$ is clear from the algorithm, and can also be checked directly from the definition of r_k. Since $r_2(\lambda) = r_1(\lambda^-)$, and so on, we need only check the algorithm for r_1.

To do this, we introduce a second tableau by entering p_{i-1} in the ith row of the first block of $p-1$ columns and $-p^{j-2}$ in all the boxes in the jth block of $p-1$ columns for $j > 1$. In Example 5.10 this is as follows.

0	0	−1	−1	−3	−3
1	1	−1	−1	−3	
4	4	−1	−1		
13	13	−1			
40	40				

The entries in a antidiagonal running from the (i,j) box for $1 \le j \le p-1$ are then $p_{i-1}, -1, -p, \ldots, -p^{s-2}$, and their sum $p_{i-1} - p_{s-1} = p^{s-1}p_{i-s}$ is the number entered in this box in $\mathrm{Tab}(\lambda)$.

It remains to check that the sum of all the entries in the second tableau is $r_1 = (b_1+1)p_{n-1} - (n-1) - d_c(\lambda^-)$. To see this, note that the entries in λ^- sum to $-d_c(\lambda^-)$, while the entries in the last row of $\lambda_{(1)}$ sum to bp_{n-1} and the entries in the first $n-1$ rows sum to $(p-1)(p_0+p_1+\ldots+p_{n-2}) = p_{n-1} - (n-1)$. $\square$

Since $w(n)$ is a product of linear factors, so also is $v(\lambda)$, and by Theorems 5.3 and 5.5 so also is $\widehat{P}^{r_1}v(\lambda)$. The following calculation shows that $v(\lambda)$ divides $\widehat{P}^{r_1}v(\lambda)$, and that the quotient can be read off from $\mathrm{Tab}(\lambda)$ as follows: replace the entry $p_{i-1} - p_{s-1}$ in the (i,j) box, $1 \le j \le p-1$, by the product of all linear polynomials of the form $x_i + \sum_{k<i} c_k x_k$, excluding those where $c_k = 0$ for $1 \le k \le i-s$.

Corollary 5.11 *Let λ be a $\mathbf{T}$-regular partition. Let the kth antidiagonal in the diagram of λ have length s_k and lowest box in row n_k. Then*

$$\frac{\widehat{P}^{r_1}v(\lambda)}{v(\lambda)} = \prod_{k=1}^{\gamma_1}\prod_{\mathbf{c}}(c_1x_1 + \ldots + c_{n_k-1}x_{n_k-1} + x_{n_k}),$$

(ii) *where the inner product is over all vectors $\mathbf{c} = (c_1, \ldots, c_{n_k-1}) \in \mathbb{F}_p^{n_k-1}$ such that $(c_1, \ldots, c_{n_k-s_k}) \ne (0, \ldots, 0)$.*

In Theorem 1.1, $\lambda = ((p-1)^n)$, $v(\lambda) = (x_1x_2\cdots x_n)^{p-1}$ and $\widehat{P}^{r_1}v(\lambda) = [x_1, x_2^p, \ldots, x_n^{p^{n-1}}]^{p-1}$. Since $s_k = 1$ for $1 \le k \le n(p-1)$, the quotient is the product of all linear polynomials in $x_1, \ldots, x_n$ which are not monomials.

Proof of Corollary 5.11 The proof is by induction on the number of antidiagonals γ_1. Let $\phi(\lambda) = \widehat{P}^{r_1}v(\lambda)/v(\lambda)$, where $r_1 = r_1(\lambda)$. Let s denote the length of the last antidiagonal in the diagram of λ, and let μ be the $\mathbf{T}$-regular partition obtained by removing this antidiagonal from the diagram of λ. Then by Theorems 5.3 and 5.5,

$$\frac{\phi(\lambda)}{\phi(\mu)} = \frac{[x_1, x_2^p, \ldots, x_n^{p^{n-1}}]}{[x_1, x_2^p, \ldots, x_{n-1}^{p^{n-2}}]} \cdot \frac{v(\lambda^-)}{v(\mu^-)} \cdot \frac{v(\mu)}{v(\lambda)}.$$

Note that $\lambda^- = \mu^-$ in the case $s=1$. Now $[x_1, x_2^p, \ldots, x_n^{p^{n-1}}]/[x_1, x_2^p, \ldots, x_{n-1}^{p^{n-2}}] = \prod_{\mathbf{c}}(c_1x_1 + \ldots + c_{n-1}x_{n-1} + x_n)$, where the product is taken over all vectors $\mathbf{c} = (c_1, \ldots, c_{n-1}) \in \mathbb{F}_p^{n-1}$. Also $v(\lambda)/v(\mu) = v_{\gamma_1}(\lambda) = [x_{n-s+1}, x_{n-s+2}^p, \ldots, x_n^{p^{s-1}}]$. Similarly $v(\lambda^-)/v(\mu^-) = [x_{n-s+1}, x_{n-s+2}^p, \ldots, x_{n-1}^{p^{s-2}}]$. The quotient of these determinants is the product of all p^{s-1} linear polynomials $c_{n-s+1}x_{n-s+1} + \ldots + c_{n-1}x_{n-1} + x_n$, so $\phi(\lambda)/\phi(\mu) = \prod_{\mathbf{c}}(c_1x_1 + \ldots + c_{n-1}x_{n-1} + x_n)$, where the product is over all $\mathbf{c} = (c_1, \ldots, c_{n-1}) \in \mathbb{F}_p^{n-1}$ with $c_i \ne 0$ for some i such that $1 \le i \le n-s$.

6 Proof of the linking theorem

In this section we prove Theorems 5.5 and 5.7. The following lemma will help in checking conditions on the numerical function α.

Lemma 6.1 **(i)** *Let $R \geq 1$ have base p expansion $R = j_1 p^{a_1} + \ldots + j_t p^{a_t}$, where $1 \leq j_1, \ldots, j_t \leq p-1$, $0 \leq a_1 < \ldots < a_t$, and let $k \geq 0$. Then $\alpha(R - p^k) \geq \alpha(R) - 1$, with equality if and only if $k = a_i$, $1 \leq i \leq t$.*

(ii) *With notation as in Theorem 5.5, and with μ and s as in the proof of Corollary 5.11, for $r \geq 1$ and $k \geq 0$ we have*

$$R(r - p_k + p_{s-1}, \mu) = R(r - p_k + d_c(\lambda^-), \mu_{(1)}) = R(r, \lambda) - p^k.$$

Proof If $k \neq a_i$ for $1 \leq i \leq t$, then subtraction of p^k must yield at least one new term $(p-1)p^a$ in the base p expansion. This proves **(i)**. For **(ii)**, since $d_c(\lambda) = d_c(\lambda_{(1)}) + p d_c(\lambda^-)$ and $d_c(\lambda_{(1)}) = \gamma_1$ we have $R = R(r, \lambda) = (p-1)(r + d_c(\lambda^-)) + \gamma_1$. Comparing the first occurrence degrees for $L(\lambda)$ and $L(\mu)$ given by (9),

$$d_c(\lambda) = d_c(\mu) + p_s, \quad d_c(\lambda^-) = d_c(\mu^-) + p_{s-1}, \quad d_c(\lambda_{(1)}) = d_c(\mu_{(1)}) + 1. \quad (10)$$

Hence $R(r - p_k + p_{s-1}, \mu) = (p-1)(r - p_k + p_{s-1} + d_c(\mu^-)) + d_c(\mu_{(1)}) = (p-1)(r - p_k + d_c(\lambda^-)) + d_c(\mu_{(1)}) = R(r - p_k + d_c(\lambda^-), \mu_{(1)}) = R - (p-1)p_k - 1 = R - p^k$.
□

Proof of Theorem 5.5(i) We argue by induction on γ_1, the number of antidiagonals of λ. With μ and s as above, $v(\lambda) = [x_{n-s+1}, x_{n-s+2}^p, \ldots, x_n^{p^{s-1}}] \cdot v(\mu)$. Using formula (4) and Lemma 2.2, for all $r \geq 1$ we have

$$\widehat{P}^r v(\lambda) = \sum_{k \geq s-1} [x_{n-s+1}, x_{n-s+2}^p, \ldots, x_{n-1}^{p^{s-2}}, x_n^{p^k}] \cdot \widehat{P}^{r - p_k + p_{s-1}} v(\mu). \quad (11)$$

By Lemma 6.1, if $\alpha(R(r, \lambda)) > \gamma_1$ then $\alpha(R(r - p_k + p_{s-1}, \mu)) > \gamma_1 - 1$ for all $k \geq 0$. Since μ has $\gamma_1 - 1$ antidiagonals, the second factor in each term of (11) is zero by the induction hypothesis. Hence $\widehat{P}^r v(\lambda) = 0$ if $\alpha(R(r, \lambda)) > \gamma_1$, completing the induction.

Proof of Theorem 5.5(ii) As in Lemma 6.1, let $R = R(r, \lambda)$ have base p expansion $R = j_1 p^{a_1} + \ldots + j_t p^{a_t}$, let $\alpha(R) = \gamma_1$ and let $R' = R(r - p_k + p_{s-1}, \mu)$. Then the lemma gives $\alpha(R') = \gamma_1 - 1$ if $k = a_i$, $1 \leq i \leq t$, and $\alpha(R') > \gamma_1 - 1$ otherwise. Hence, applying part **(i)** of the theorem to (11), we have

$$\widehat{P}^r v(\lambda) = \sum_{i=1}^{t} [x_{n-s+1}, x_{n-s+2}^p, \ldots, x_{n-1}^{p^{s-2}}, x_n^{p^{a_i}}] \cdot \widehat{P}^{r - p_{a_i} + p_{s-1}} v(\mu).$$

Since $\alpha(R(r - p_{a_i} + p_{s-1}, \mu) = \gamma_1 - 1 = d_c(\mu_{(1)})$ by the lemma, and $p_{s-1} + d_c(\mu^-) = d_c(\lambda^-)$, the inductive hypothesis on μ gives

$$\widehat{P}^{r - p_{a_i} + p_{s-1}} v(\mu) = \widehat{P}^{r - p_{a_i} + d_c(\lambda^-)} v(\mu_{(1)}) \cdot v(\mu^-), \quad 1 \leq i \leq t.$$

We can similarly use the lemma to simplify the right hand side of the required identity. Since $v(\lambda_{(1)}) = x_n v(\mu_{(1)})$, from (4) and (2) we have

$$\widehat{P}^{r+d_c(\lambda^-)} v(\lambda_{(1)}) = \sum_{k\geq 0} x_n^{p^k} \widehat{P}^{r+d_c(\lambda^-)-p_k} v(\mu_{(1)}).$$

By the lemma, $R(r+d_c(\lambda^-)-p_k, \mu_{(1)}) = R-p^k$, so that by **(i)** we can again reduce to the sum over $k = a_i$, $1 \leq i \leq t$. As $v(\lambda^-) = [x_{n-s+1}, x_{n-s+2}^p, \ldots, x_{n-1}^{p^{s-2}}] \cdot v(\mu^-)$, it remains after cancelling the factor $v(\mu^-)$ and rearranging terms to prove that

$$\sum_{i=1}^{t} \Big([x_{n-s+1}, x_{n-s+2}^p, \ldots, x_{n-1}^{p^{s-2}}, x_n^{p^{a_i}}] - [x_{n-s+1}, x_{n-s+2}^p, \ldots, x_{n-1}^{p^{s-2}}] x_n^{p^{a_i}} \Big) \cdot f_i = 0,$$

where $f_i = \widehat{P}^{r-p_{a_i}+d_c(\lambda^-)} v(\mu_{(1)})$. The expansion of the $s \times s$ determinant in the p^{a_i} powers of the variables is

$$\sum_{j=1}^{s} (-1)^{s-j} [x_{n-s+1}, \ldots, x_{n-s+j-1}^{p^{j-2}}, x_{n-s+j+1}^{p^{j-1}}, \ldots, x_n^{p^{s-2}}] x_{n-s+j}^{p^{a_i}}.$$

Thus the term with $j = s$ cancels, and interchanging the i and j summations, the required formula becomes

$$\sum_{j=1}^{s-1} (-1)^{s-j} [x_{n-s+1}, \ldots, x_{n-s+j-1}^{p^{j-2}}, x_{n-s+j+1}^{p^{j-1}}, \ldots, x_n^{p^{s-2}}] \cdot \sum_{i=1}^{t} x_{n-s+j}^{p^{a_i}} f_i = 0.$$

Since $\widehat{P}^{r+d_c(\lambda^-)}(x_{n-s+j} v(\mu_{(1)})) = \sum_{i=1}^{t} x_{n-s+j}^{p^{a_i}} f_i$ by a similar argument using (4), (1) and Lemma 6.1, it suffices to prove that the monomial $x_{n-s+j} v(\mu_{(1)})$ is in the kernel of $\widehat{P}^{r+d_c(\lambda^-)}$ for $1 \leq j \leq s-1$. This monomial is divisible by x_{n-s+j}^p. By permuting the variables, it suffices to consider the case where it is divisible by x_1^p. Hence the proof of Theorem 5.5 is completed by the following calculation.

Proposition 6.2 *Let $R = R(r, \lambda)$ and let $\alpha(R) = \gamma_1$, where $\gamma_1 = (n-1)(p-1)+b$ and $1 \leq b \leq p-1$. Then*

$$\widehat{P}^{r+d_c(\lambda^-)}(x_1^p (x_2 \cdots x_{n-1})^{p-1} \cdot x_n^{b-1}) = 0.$$

Proof By Lemma 2.3, with $f = x_1$ and $g = (x_2 \cdots x_{n-1})^{p-1} \cdot x_n^{b-1}$,

$$\widehat{P}^u (x_1^p \cdot g) = \sum_{u=pv+w} (\widehat{P}^v x_1)^p \cdot \widehat{P}^w(g).$$

Note that $g = v(\nu)$ where $\nu = ((p-1)^{n-2}(b-1))$. By (2), $\widehat{P}^v x_1 = 0$ for $v \neq p_k$, $k \geq 0$, so we may assume that $w = u - pv = r + d_c(\lambda^-) - p \cdot p_k$. Since $p \cdot p_k = p_{k+1} - 1$

and $d_c(\mu_{(1)}) = p - 1 + d_c(\nu)$, $R(w,\nu) = R(r - p_{k+1} + d_c(\lambda^-), \mu_{(1)}) = R - p^{k+1}$ by Lemma 6.1**(ii)**. Since $\alpha(R) = \gamma_1$, Lemma 6.1**(i)** gives $\alpha(R(w,\nu)) \geq \gamma_1 - 1 > \gamma_1 - p$. Since $d_c(\nu) = \gamma_1 - p$, $\widehat{P}^w g = 0$ by Theorem 5.5**(i)**. □

Proof of Theorem 5.7 This follows from Theorem 5.5 by induction on m. Let $\gamma_1 = (n-1)(p-1) + b$, $1 \leq b \leq p-1$. We wish to apply Theorem 5.5 with $r = r_1$, so we must check that $\alpha(R(r_1, \lambda)) = \gamma_1$. For this, note that (9) gives $d_c(\lambda^-) = \sum_{j=2}^m p^{j-2}\gamma_j$, so that $r_1 + d_c(\lambda^-) = (b+1)p_{n-1} - (n-1)$. Thus $R(r_1,\lambda) = (p-1)(r_1 + d_c(\lambda^-)) + \gamma_1 = (b+1)(p^{n-1}-1) - (p-1)(n-1) + \gamma_1 = bp^{n-1} + (p^{n-1} - 1)$. Hence r_1 satisfies the hypothesis of Theorem 5.5, so that $\widehat{P}^{r_1} v(\lambda) = \widehat{P}^{r_1 + d_c(\lambda^-)} v(\lambda_{(1)}) \cdot v(\lambda^-)$. By Theorem 5.3, $\widehat{P}^{r_1 + d_c(\lambda^-)} v(\lambda_{(1)}) = w(\lambda'_{(1)})$.

Now $r_i(\lambda) = r_{i-1}(\lambda^-)$ for $2 \leq i \leq m$, and so the inductive step reduces to showing that

$$\widehat{P}^{r_m} \cdots \widehat{P}^{r_2} \left(w(\lambda'_{(1)}) \cdot v(\lambda^-) \right) = w(\lambda'_{(1)}) \cdot \widehat{P}^{r_m} \cdots \widehat{P}^{r_2} v(\lambda^-). \tag{12}$$

Recall from Lemma 5.9 that $r_1, \ldots, r_m$ is an admissible sequence, i.e. $r_k \geq p r_{k+1}$ for $k \geq 1$. Since $r_1 \leq (b+1)p_{n-1}$, $r_1 < p^{n-1}$ if $b < p-1$ and $r_1 < p^n$ if $b = p-1$. Thus we can deduce (12) from Lemma 2.2 and the coproduct formula (4), as follows. We have $w(\lambda'_{(1)}) = w(n)^b w(n-1)^{p-1-b}$. Now $\widehat{P}^r w(n) = 0$ for $0 < r < p^{n-1}$ and $\widehat{P}^r w(n-1) = 0$ for $0 < r < p^{n-2}$. If there are any factors $w(n-1)$ in $w(\lambda'_{(1)})$, then $r_2 < p^{n-2}$, and otherwise it suffices to have $r_2 < p^{n-1}$.

7 First occurrence submodules

For a $\mathbf{T}$-regular partition λ, the $\mathbb{F}_p[M_n]$-submodule of $\mathbf{P}^{d_c(\lambda)}$ generated by the first occurrence polynomial $v(\lambda)$ is a 'representative polynomial' for $L(\lambda)$ in the sense that this module has a quotient isomorphic to $L(\lambda)$(see Corollary 5.8). In the case where $\lambda = (p-1)\kappa$ for a column 2-regular partition κ, the leading monomial $s(\lambda) = x_1^{p^{\kappa_1}-1} \cdots x_n^{p^{\kappa_n}-1}$ has the same property. This is implicit in the work of Carlisle and Kuhn [2], who identify a subquotient $\mathbf{T}^\gamma$ of $\mathbf{P}^{d_c(\lambda)}$ such that $\mathbf{T}^\gamma \cong \mathbf{T}^{\gamma_1} \otimes \ldots \otimes \mathbf{T}^{\gamma_m}$, where γ is the $\mathbf{T}$-conjugate of λ. Explicitly, if $v_i \in \mathbf{T}^{\gamma_i}$ corresponds to a monomial in $x_1, \ldots, x_n$ with all exponents $< p$, then $v_1 \otimes \ldots \otimes v_m \in \mathbf{T}^{\gamma_1} \otimes \ldots \otimes \mathbf{T}^{\gamma_m}$ corresponds to the equivalence class of $v_1 \cdot v_2^p \cdots v_m^{p^{m-1}}$ in the appropriate subquotient of $\mathbf{P}^{d_c(\lambda)}$. Proposition 5.4**(ii)** shows that, taking $v_j = v(\lambda_{(j)})$, this monomial is $s(\lambda)$. Tri [14] has recently proved that if λ is $\mathbf{T}$-regular, then $L(\lambda)$ is a composition factor in $\mathbf{T}^\gamma$.

We recall from [16, Section 4] the notion of a *base p ω-vector.*

Definition 7.1 Given a prime p, the *base p ω-vector* $\omega(s)$ of a sequence of non-negative integers $s = (s_1, \ldots, s_n)$ is defined as follows. Write each s_i in base p as $s_i = \sum_{j \geq 1} s_{i,j} p^{j-1}$, where $0 \leq s_{i,j} \leq p-1$, and let $\omega_j(s) = \sum_{i=1}^n s_{i,j}$, i.e. add the base p expansions without 'carries'. Then $\omega(s) = (\omega_1(s), \ldots, \omega_l(s))$, with *length* $l = \max\{j : \omega_j(s) > 0\}$ and *degree* $d = \sum_{i=1}^n s_i = \sum_{j=1}^l \omega_j(s) p^{j-1}$.

Given ω-vectors ρ and σ, we say that ρ *dominates* σ, and write $\rho \succeq \sigma$ or $\sigma \preceq \rho$, if and only if $\sum_{i=1}^{k} p^{i-1}\rho_i \geq \sum_{i=1}^{k} p^{i-1}\sigma_i$ for all $k \geq 1$. By the ω-vector of a monomial $\prod_{i=1}^{n} x_i^{s_i}$ we mean the ω-vector of its sequence of exponents $s = (s_1, \ldots, s_n)$. The dominance order on ω-vectors of the same degree is compatible with left lexicographic order.

Example 7.2 The lattice of base p ω-vectors of degree $1+p+p^2$ is shown below.

$$
\begin{array}{ccccc}
 & & (1+p+p^2) & & \\
 & & \downarrow & & \\
 & & (1+p^2,1) & & \\
 & & \downarrow & & \\
 & & \vdots & & \\
 & & \downarrow & & \\
 & & (1+p,p) & & \\
 & \swarrow & & \searrow & \\
(1,1+p) & & & & (1+p,0,1) \\
 & \searrow & & \swarrow & \\
 & & (1,1,1) & &
\end{array}
$$

Proposition 7.3 *Let λ be a $\mathbf{T}$-regular partition. Then the ω-vector of the spike monomial $s(\lambda)$ is the partition γ $\mathbf{T}$-conjugate to λ, and the polynomial $v(\lambda)$ is the sum of $(-1)^{\epsilon(\lambda)}s(\lambda)$ and monomials f such that $\omega(f) \prec \gamma$.*

Proof The proof is the same as that given in [16, proposition 4.5], with 2 replaced by p and λ' replaced by γ. For $\epsilon(\lambda)$, see Proposition 5.4**(iii)**. □

Corollary 5.8 and Proposition 7.3 together provide a 'topological' proof that the $\mathbb{F}_p[M_n]$-submodule of $\mathbf{P}^{d_c(\lambda)}$ generated by $s(\lambda)$ has a quotient module isomorphic to $L(\lambda)$. The next result provides a further comparison between the spike monomial $s(\lambda)$ and the polynomial $v(\lambda)$ in a special case. We conjecture that the corresponding statement holds for all $\mathbf{T}$-regular partitions λ.

Proposition 7.4 *Assume that $\lambda_i = (p-1)\kappa_i$ for $1 \leq i \leq n$, where $\kappa = (\kappa_1, \ldots, \kappa_n)$ is a column 2-regular partition. Then the submodule of $\mathbf{P}^{d_c(\lambda)}$ generated by the polynomial $v(\lambda)$ is contained in the submodule generated by the spike monomial $s(\lambda)$.*

The proof requires a preliminary lemma.

Lemma 7.5 *If $f \in \mathbb{F}_p[x_2, \ldots, x_n]$ and $1 \leq s \leq n$, then the $\mathbb{F}_p[M_n]$-submodule of $\mathbf{P}$ generated by $x_1^{p^s-1} \cdot f$ contains $[x_1, x_2^p, \ldots, x_s^{p^{s-1}}]^{p-1} \cdot f$.*

Proof For each linear form $v = a_1x_1+\ldots+a_sx_s$, where $a_i \in \mathbb{F}_p$ for $1 \le i \le s$, let $t_v : \mathbf{P} \to \mathbf{P}$ be the transvection mapping x_1 to v and fixing $x_2, \ldots, x_n$. We claim that the following equation holds in $\mathbb{F}_p[x_1, \ldots, x_s]$.

$$(-1)^s[x_1, x_2^p, \ldots, x_s^{p^{s-1}}]^{p-1} = \sum_v v^{p^s-1}. \tag{13}$$

Since t_v does not change the variables $x_2, \ldots, x_n$ which can occur in f, it follows from (13) that $\sum_v t_v$ is an element of the semigroup algebra $\mathbb{F}_p[M_n]$ which maps $x_1^{p^s-1} \cdot f$ to $(-1)^s[x_1, x_2^p, \ldots, x_s^{p^{s-1}}]^{p-1} \cdot f$.

To prove (13), first note that the right hand side is $GL_s(\mathbb{F}_p)$-invariant. Further, it is mapped to 0 by every singular matrix $g \in M_s$, since vectors $(a_1, \ldots, a_s)$ and $(a'_1, \ldots, a'_s)$ in $\mathbb{F}_p^s$ in the same coset of the kernel of g yield terms in (13) with the same image under g, and p divides the order of this coset. Arguing as in the first or second proof of Theorem 1.1, with s in place of n, it follows that (13) holds up to a (possibly zero) scalar.

Finally we verify that the monomial $m = x_1^{p-1}x_2^{p(p-1)} \cdots x_s^{p^{s-1}(p-1)}$ has coefficient $(-1)^s$ in the right hand side of (13). For each linear form v, we have $v^{p^s-1} = v^{p^{s-1}(p-1)} \cdots v^{p(p-1)} \cdot v^{p-1}$, where $v^{p^j(p-1)} = (a_1x_1^{p^j} + \ldots + a_sx_s^{p^j})^{p-1}$ for $0 \le j \le s-1$. The exponent $p-1$ in m must come from the last factor in this product, so we must choose the term $(a_1x_1)^{p-1} = x_1^{p-1}$ from the last factor, and $a_1 \ne 0$. In the same way, we must choose the term $(a_2x_2^p)^{p-1} = x_2^{p(p-1)}$ from the last but one factor, and $a_2 \ne 0$. Continuing in this way, we see that each of the $(p-1)^s$ linear forms v with all coefficients $a_i \ne 0$ gives a term containing m (with coefficient 1), while other choices of v give terms not containing m. Thus the scalar coefficient in (13) is $(-1)^s$. □

The following example shows how to apply Lemma 7.5 to a partition λ of the form $(p-1)\kappa$, so as to generate $v(\lambda)$ from $s(\lambda)$.

Example 7.6 Let $p = 3$ and let $\lambda = (6, 6, 4, 4, 2)$, so that $s(\lambda) = x^{26}y^{26}z^8t^8u^2$ and $v(\lambda) = x^2[x, y^3]^2[x, y^3, z^9]^2[y, z^3, t^9]^2[t, u^3]^2$.

Begin by permuting the variables, so as to work with the spike $u^8t^{26}z^{26}y^8x^2$. Apply Lemma 7.5 with $x_1 = y$ and $s = 2$ to generate $[y, x^3]^2 \cdot u^8t^{26}z^{26}x^2$. Repeat with $x_1 = z$ and $s = 3$ to generate $[z, y^3, x^9]^2 \cdot u^8t^{26}[y, x^3]^2x^2$, then with $x_1 = t$ and $s = 3$ to generate $[t, z^3, y^9]^2 \cdot u^8[z, y^3, x^9]^2[y, x^3]^2x^2$, and finally with $x_1 = u$ and $s = 2$ to generate $v(\lambda)$.

Proof of Proposition 7.4 We first observe (see [16, Proposition 4.9]) that the (multi)set of lengths of the antidiagonals of the column 2-regular partition κ is equal to the (multi)set of lengths of the rows. Hence the spike monomial $\tilde{s}(\lambda) = x_n^{p^{s_n}-1}x_{n-1}^{p^{s_{n-1}-1}} \cdots x_1^{p^{s_1}-1}$, where s_k is the length of the kth antidiagonal of the diagram of κ, can be obtained from $s(\lambda)$ by a suitable permutation of the variables. We can now obtain $v(\lambda)$ from $\tilde{s}(\lambda)$ by $n-1$ successive applications of Lemma 7.5, following the method illustrated by Example 7.6.

8 T-regular partitions and the Milnor basis

In this section we link the first occurrence polynomial $v(\lambda)$ and its leading monomial $s(\lambda)$ to the polynomial $p(\lambda') = \prod_{j=1}^{m} w(\lambda'_{(j)})^{p^{j-1}}$, which generates a submodule occurrence of $L(\lambda)$ in a higher degree. Here, as in Proposition 5.4, $\lambda_{(j)}$ is the partition given by the jth block of $p-1$ columns in the diagram of the $\mathbf{T}$-regular partition λ, and m is the length of γ, the $\mathbf{T}$-conjugate of λ. In the case $\lambda = (p-1)\kappa$, we also link the first submodule occurrence polynomial $w(\lambda')$ to $p(\lambda')$. The linking is achieved by Milnor basis elements in $\mathcal{A}_p$ which are combinatorially related to λ. We also obtain a relation between monomials in $\mathbf{P}$ and Milnor basis elements in terms of ω-vectors. These results extend some of the results of [16, Section 5].

As in Proposition 5.4, let $\lambda_i = a_i(p-1) + b_i$, where $a_i \geq 0$, $1 \leq b_i \leq p-1$. Following [16], for $R = ((b_1+1)p^{a_1}-1, \ldots, (b_n+1)p^{a_n}-1)$ we call the Milnor basis element $P(R)$ the *Milnor spike* associated to λ. We note that $\omega(R) = \gamma$. A Milnor spike is an admissible monomial [4]. For example, if $p=3$ and $\lambda = (4,3,1)$ then the corresponding Milnor spike is $P(8,5,1) = P^{32}P^8P^1$, and for the $\mathbf{T}$-conjugate partition $\gamma = (5,3)$ it is $P(17,5) = P^{32}P^5$. In this example, $\lambda'_{(1)} = (3,2)$ and $\lambda'_{(2)} = (2,1)$, so that $p(\lambda') = w(3)w(2)\cdot(w(2)w(1))^3 = [x_1, x_2^3, x_3^9]\cdot[x_1, x_2^3]^4\cdot x_1^3$.

Theorem 8.1 *Let λ be $\mathbf{T}$-regular with $\mathbf{T}$-conjugate γ.*

(i) *$P(R)s(\lambda) = (-1)^{\epsilon(\lambda)}P(R)v(\lambda) = p(\lambda')$, where $P(R)$ is the Milnor spike associated to $(\lambda_2, \ldots, \lambda_n)$.*

(ii) *If $\lambda = (p-1)\kappa$, where κ is column 2-regular, $P(S)w(\lambda') = p(\lambda')$, where $P(S)$ is the Milnor spike associated to $(\gamma_2, \ldots, \gamma_m)$.*

(iii) *There are formulae corresponding to* **(i)** *and* **(ii)** *for the Milnor spikes associated to λ and γ, with $p(\lambda')$ replaced by $p(\lambda')^p$.*

Remark 8.2 **(iii)** follows immediately from **(i)** and **(ii)** for degree reasons. The omission of the first terms in R and S corresponds to omitting the highest Steenrod power P^d in the admissible monomial forms of $P(R)$ and $P(S)$. In fact $d = \deg p(\lambda')$, so that $P^d p(\lambda') = p(\lambda')^p$. In the example $p=3$, $\lambda = (4,3,1)$ above, **(i)** states that $P^8P^1(x_1^8x_2^5x_3) = -P^8P^1(x_1^2\cdot[x_1,x_2^3]^2\cdot[x_2,x_3^3]) = [x_1,x_2^3,x_3^9]\cdot[x_1,x_2^3]^4\cdot x_1^3$. The case $\lambda = (4,3,1)$ is excluded from **(ii)**, but in fact $P^5w(\lambda') = -p(\lambda')$. We believe that **(ii)** holds, up to sign, for all $\mathbf{T}$-regular λ.

We begin by proving the equivalence of the two statements in **(i)**. For this we use the following generalization of [16, Theorem 5.9(i)]. The proof is based on Lemma 2.8, and follows that given in [16].

Theorem 8.3 *Let $x_1^{s_1}\cdots x_n^{s_n}$ be a monomial with ω-vector σ. Let $R = (r_1, \ldots, r_t)$ and let $\omega(R) = \rho$. If σ does not dominate ρ, then $P(R)(x_1^{s_1}\cdots x_n^{s_n}) = 0$.* □

Proof of Theorem 8.1(i) By Proposition 7.3, if the monomial f occurs in $v(\lambda)$ and $f \neq s(\lambda)$, then $\omega(f) \prec \gamma$. If $R = (r_1, \dots, r_n)$ where $r_i = (b_i+1)p^{a_i} - 1$, so that $P(R)$ is the Milnor spike associated to λ, then, as noted above, $\omega(R) = \gamma$. Hence, by Theorem 8.3, $P(R)$ takes the same value on $v(\lambda)$ and on its leading term $(-1)^{\epsilon(\lambda)} s(\lambda)$.

We evaluate $P(R)s(\lambda)$ by induction on the length m of γ. The base case $m = 1$ holds by our previous results, as follows. In this case, $\lambda = (p-1, \dots, p-1, b)$, with $1 \leq b \leq p-1$, and has length n, while **(i)** states that $P(R)s(\lambda) = w(\lambda')$, where $R = (p-1, \dots, p-1, b)$ has length $n-1$. By Proposition 3.2**(ii)**, $P(R)g = \widehat{P}^{(b+1)p_{n-1}-(n-1)}g$ when $\deg g \leq (n-1)(p-1) + b$, and we may choose $g = s(\lambda)$. Hence the result follows from Theorem 5.3.

For the inductive step, we use Proposition 5.4**(ii)** to write $s(\lambda) = f^p \cdot g$, where $g = v(\lambda_{(1)})$ and $f = s(\lambda^-)$. Hence $P(R)s(\lambda) = \sum (P(S)f)^p \cdot P(T)g$ by Lemma 2.8, where the sum is over sequences $S = (s_2, \dots, s_n)$, $T = (t_2, \dots, t_n)$ such that $r_i = ps_i + t_i$ for $2 \leq i \leq n$. Thus $t_n = b_1$, $s_n = 0$ and $t_i \geq p-1$ for $2 \leq i \leq n-1$. If $t_i \neq p-1$ for some $i < n$, then $P(T)$ has excess $\sum_i t_i > \deg v(\lambda_{(1)}) = \gamma_1$, so that $P(T)(v(\lambda_{(1)})) = 0$. Hence we may assume that $T = (p-1, \dots, p-1, b_1)$, so that $s_i = (b_i+1)p^{a_i-1} - 1$ for $2 \leq i \leq n-1$. By the argument for the case $m = 1$, $P(T)(v(\lambda_{(1)})) = w(\lambda'_{(1)})$, and by the induction hypothesis applied to λ^-, $P(S)s(\lambda^-) = p(\lambda^-)$. Since $p(\lambda) = w(\lambda'_{(1)}) \cdot p(\lambda^-)^p$, the induction is complete. □

Proof of Theorem 8.1(ii) Let $\lambda = (p-1)\kappa$, where κ is column 2-regular. Then $\gamma = (p-1)\kappa'$ has length $m = \kappa_1$, and $\lambda_{(i)} = ((p-1)^{\kappa'_i})$, so that $w(\lambda_{(i)}) = w(\kappa'_i)^{p-1}$. Also $S = (p^{\kappa'_2} - 1, \dots, p^{\kappa'_m} - 1)$, so that $P(S) = P^{t_2} \cdots P^{t_m}$, where $t_m = p^{\kappa'_m} - 1$ and $t_i = pt_{i+1} + p^{\kappa'_i} - 1$ for $1 \leq i < m$. We shall argue by induction on m, the case $m = 1$, where $P(S) = 1$, being trivial. For $2 \leq i \leq m$, let

$$W_i(\lambda') = w(\lambda'_{(1)}) \cdots w(\lambda'_{(i)}) \cdot w(\lambda'_{(i+1)})^p \cdots w(\lambda'_{(m)})^{p^{m-i}},$$

so that $W_1(\lambda') = p(\lambda')$ and $W_m(\lambda') = w(\lambda')$. We assume as inductive hypothesis on j that $P^{t_j} W_j(\lambda') = W_{j-1}(\lambda')$ for $j > i$, and prove this for $j = i$.

It follows from Lemma 2.1 that $P^r(w(n)^{p^i}) = 0$ unless $r = p^i(p_n - p_j)$, where $0 \leq j \leq n$. The largest of these values, equal to the degree of $w(n)^{p^i}$, is $p^i \cdot p_n$.

Since $w(\lambda'_{(i)})$ has degree $p^{\kappa'_i} - 1$, it follows by (downward) induction on i that t_i is the degree of $w(\lambda'_{(i)}) \cdot w(\lambda'_{(i+1)})^p \cdots w(\lambda'_{(m)})^{p^{m-i}}$. We may express t_i explicitly as the sum

$$t_i = \sum_{k=i}^{m} p^{k-i}(p^{\kappa'_k} - 1). \tag{14}$$

Hence one term in the expansion of $P^{t_i}(W_i(\lambda'))$ using the Cartan formula is $W_{i-1}(\lambda')$. We shall complete the proof by using Lemma 2.1 to show that all other terms in the expansion vanish. Thus we have to consider the possible ways to write t_i so that

$$(p-1)t_i = \sum_{v=1}^{p-1} \left(\sum_{k=1}^{i-1} (p^{\kappa'_k} - p^{j_{k,v}}) + \sum_{k=i}^{m} p^{k-i}(p^{\kappa'_k} - p^{j_{k,v}}) \right) \tag{15}$$

where $0 \leq j_{k,v} \leq \kappa'_k$ for $1 \leq k \leq m$. Equating (14) and (15) and simplifying, we obtain

$$(p-1)\left(\sum_{k=1}^{i-1} p^{\kappa'_k} + \sum_{k=i}^{m} p^{k-i} \right) = \sum_{v=1}^{p-1} \left(\sum_{k=1}^{i-1} p^{j_{k,v}} + \sum_{k=i}^{m} p^{k-i} \cdot p^{j_{k,v}} \right). \tag{16}$$

Since κ is column 2-regular, κ' is strictly decreasing and so $\kappa'_{i-1} > \kappa'_i \geq \kappa'_m + m - i > m - i$. Hence the m powers of p occurring in the left side of (16) are distinct. By uniqueness of base p expansions, there are also m distinct powers on the right of (16) and these are a permutation of the powers on the left. The argument is now completed as in the case $p = 2$ [16, Section 5]. □

We end with evaluations of certain Milnor basis elements on monomials. While [16, Lemma 5.6] generalizes easily to odd primes, this does not seem to be so useful here as the following (weak) generalization of [16, Proposition 5.8].

Proposition 8.4 *Let $R = (r_1, r_2, \ldots)$ where $r_i = p-1$ if $i = b_1, \ldots, b_m$ and $r_i = 0$ otherwise. Then*

$$P(R)(x_1 \cdots x_n)^{p-1} = \begin{cases} [x_1^{p^{b_1}}, \ldots, x_n^{p^{b_n}}]^{p-1}, & \text{if } m = n, \\ [x_1, x_2^{p^{b_1}}, \ldots, x_n^{p^{b_{n-1}}}]^{p-1}, & \text{if } m = n-1. \end{cases}$$

Proof This is proved by induction on $|R|$. The base of the induction is Theorem 1.1, which is the case $m = n-1$, $b_i = i$ for $1 \leq i \leq n-1$. Given a sequence $R = (r_1, \ldots, r_{j-1}, 0, p-1, p-1, \ldots, p-1)$, let $R' = (r_1, \ldots, r_{j-1}, p-1, 0, p-1, \ldots, p-1)$, so that $|R| - |R'| = (p-1)(p^{j+1}-1) - (p-1)(p^j - 1) = (p-1)^2 p^j$. We claim that $P^{p^j(p-1)} \cdot P(R')$ and $P(R)$ have the same value on any polynomial of degree $n(p-1)$. To prove this, we use Milnor's product formula to expand $P^{p^j(p-1)} \cdot P(R')$ in the Milnor basis. The Milnor matrix

	r_1	$\ldots$	r_{j-1}	0	0	$p-1$	$\ldots$	$p-1$
0	0	$\ldots$	0	$p-1$	0	0	$\ldots$	0

shows that $P(R)$ occurs with coefficient 1 in the product. Since $P(R)$ is the unique Milnor basis element of minimal excess $(n-1)(p-1)$ in degree $|R|$, this proves our claim.

Applying the induction hypothesis to $P(R')$, we have $P(R)(x_1 \ldots x_n)^{p-1} = P^{p^j(p-1)}[x_1, x_2^{p^{b_1}}, \ldots, x_i^{p^j}, \ldots, x_n^{p^{b_{n-1}}}]^{p-1}$ where R and R' differ in the ith term, i.e. $b_i = j$ for R' and $b_i = j+1$ for R. By the Cartan formula, this is

$[x_1, x_2^{p^{b_1}}, \ldots, x_i^{p^{j+1}}, \ldots, x_n^{p^{b_{n-1}}}]^{p-1}$, and this completes the induction for the case $m = n-1$. The case $m = n$ is proved similarly. □

Proposition 8.4 serves as the base of induction for the following generalization of [16, Theorem 5.9(ii)] to odd primes. The proof, by induction on the length of the ω-vector σ, is essentially the same as in [16].

Theorem 8.5 *Let $R_0 = (r_0, r_1, \ldots, r_t)$, $R = (r_1, \ldots, r_t)$ and $f = x_1^{s_1} \cdots x_n^{s_n}$, where the base p expansion of each term r_i and exponent s_j contains only the digits 0 and $p-1$. Assume that f and R_0 have the same ω-vector σ. Then $P(R)f = \prod_{k=1}^{m} \Delta_k^{p^{k-1}(p-1)}$, where m is the length of σ and $\Delta_k = [x_{i_1}^{p^{j_1}}, \ldots, x_{i_\kappa}^{p^{j_\kappa}}]$ is the Vandermonde determinant of order $\kappa = \sigma_k/(p-1)$ defined by the subsequences $(s_{i_1}, \ldots, s_{i_\kappa})$ of $(s_1, \ldots, s_n)$ and $(r_{j_1}, \ldots, r_{j_\kappa})$ of R_0 consisting of the terms whose kth base p place is $p-1$.*

Example 8.6 Using the tables

r_0	$p-1$	0	$p-1$
r_1	$p-1$		
r_2	$p-1$		
σ	$3(p-1)$	0	$p-1$

x_1	$p-1$	0	$p-1$
x_2	$p-1$		
x_3	$p-1$		
σ	$3(p-1)$	0	$p-1$

r_0	$p-1$		
r_1	$p-1$	0	$p-1$
r_2	$p-1$		
σ	$3(p-1)$	0	$p-1$

we obtain $P(p-1,p-1)x_1^{(p^2+1)(p-1)}x_2^{p-1}x_3^{p-1} = [x_1, x_2^p, x_3^{p^2}]^{p-1} \cdot x_1^{p^2(p-1)}$ and $P((p^2+1)(p-1), p-1)x_1^{(p^2+1)(p-1)}x_2^{p-1}x_3^{p-1} = [x_1, x_2^p, x_3^{p^2}]^{p-1} \cdot (x_1^p)^{p^2(p-1)}$.

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