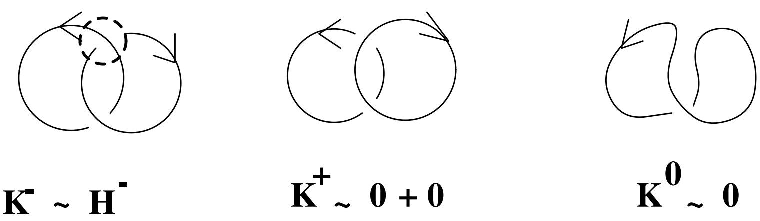
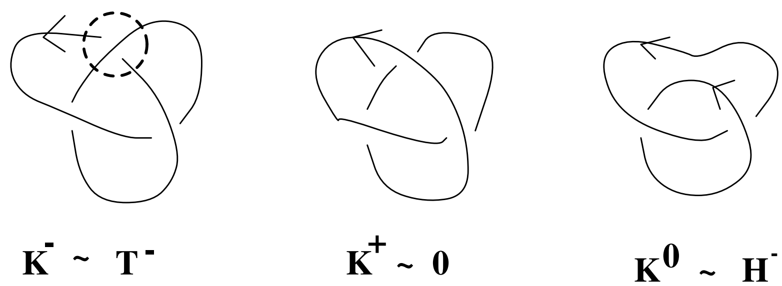
1. **(i)** Here K^- is the negative Hopf link H^-, K^+ is a split two-component link and K^0 is the unknot. Hence $f_{K^+}(A) = -A^{-2} - A^2$ and $f_{K^0}(A) = 1$, and so $f_{H^-}(A) = -A^{10} - A^2$.

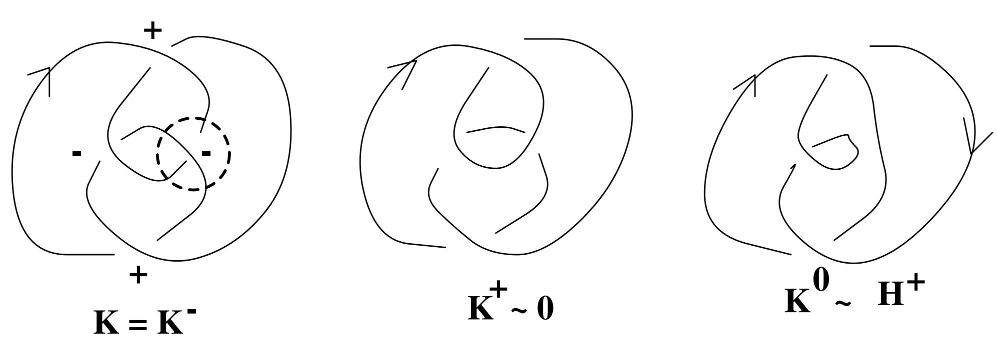


(ii) Here K^- is the negative trefoil knot T^-, K^+ is an unknot and $K^0 \sim H^-$. So we have $A^4 - A^{-4} f_{T^-}(A) = (A^{-2} - A^2) f_{H^-}(A) = (A^{-2} - A^2)(-A^{10} - A^2) = A^{12} - A^8 - 1 + A^4$. Hence $f_{T^-}(A) = -A^{16} + A^{12} + A^4$.



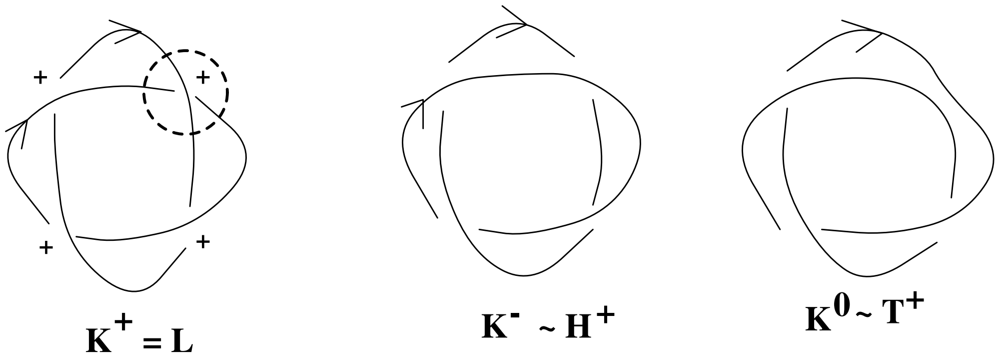
2. **(i)** $K = K^-$ is again a left trefoil knot, and the diagrams for Solutions 3, Question 3(i) show immediately that the calculation will be the same as in Question 1(ii) above.

(ii) (Figure eight knot) We have $A^4 - A^{-4} f_K(A) = (A^{-2} - A^2) f_{H^+}(A) = (A^{-2} - A^2)(-A^{-10} - A^{-2}) = -A^{-12} + A^{-8} - A^{-4} + 1$, so $f_K(A) = A^{-8} - A^{-4} + 1 - A^4 + A^8$.

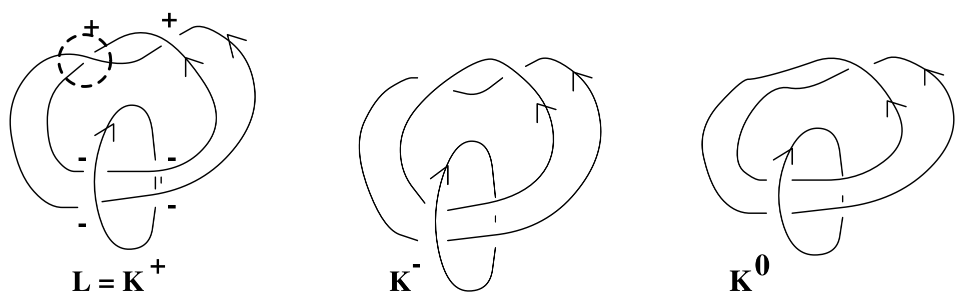


Note that $f_K(A) = f_K(A^{-1})$. This relation is forced by the achirality of the figure eight knot.

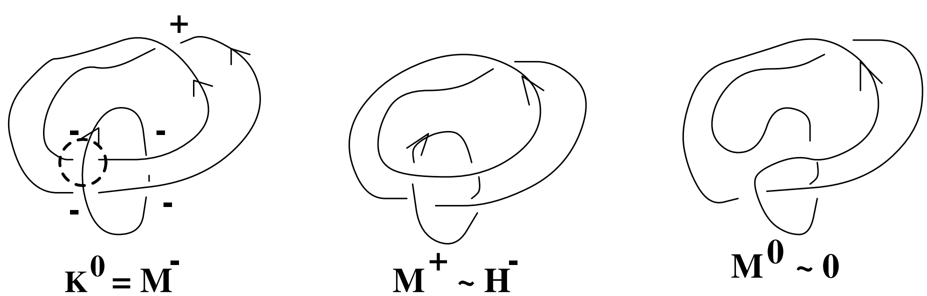
(iii) From the diagram below, with $L = K^+$ we have $A^4 f_L(A) - A^{-4} f_{H^+}(A) = (A^{-2} - A^2) f_{T^+}(A)$, so $A^4 f_L(A) - A^{-4}(-A^{-10} - A^{-2}) = (A^{-2} - A^2)(-A^{-16} + A^{-12} + A^{-4})$ and hence $f_L(A) = -A^{-22} + A^{-18} - A^{-14} - A^{-6}$. (This L appears in an equivalent form as $K(4)$ in Question 4.)



(iv) We have $A^4 f_L(A) - A^{-4} f_{K^-}(A) = (A^{-2} - A^2) f_{K^0}(A)$, but we must simplify K^- and K^0 further to evaluate these terms.



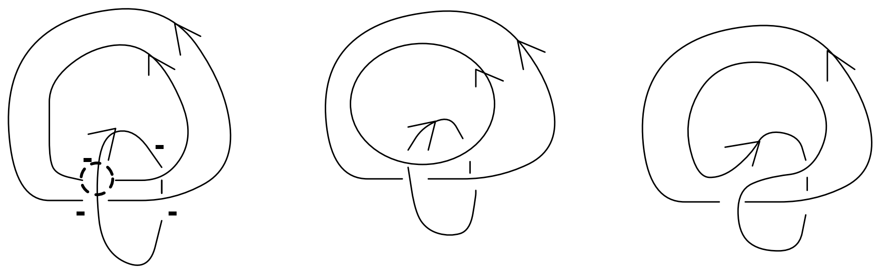
We take $K^0 = M^-$, getting $M^+ \sim H^-$ and M^0 an unknot.



Thus $A^4 f_{H^-}(A) - A^{-4} f_{K^0}(A) = A^{-2} - A^2$, so by using the result $f_{H^-}(A) = -A^{10} - A^2$ from Question 1 and simplifying, we get

$$f_{K^0}(A) = -A^{18} - A^{10} - A^2 + A^6.$$

We take $K^- = N^-$, getting $N^+ \sim H^- \cup 0$ and $N^0 \sim H^-$.



$K^- = N^-$ $\qquad$ $N^+ = H^- \cup 0$ $\qquad$ $N^0 \sim H^-$

Again we use the result $f_{H^-}(A) = -A^{10} - A^2$ from Question 1, together with Proposition 6.6 to evaluate $f_{H^- \cup 0}(A)$. This gives

$$A^4(-A^{-2} - A^2)(-A^{10} - A^2) - A^{-4} f_{K^-}(A) = (A^{-2} - A^2)(-A^{10} - A^2),$$

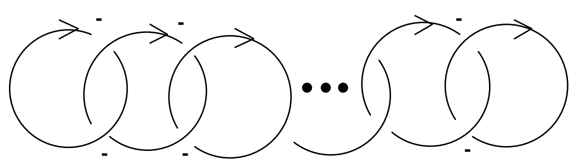
and simplifying this gives

$$f_{K^-}(A) = A^{20} + 2A^{12} + A^4.$$

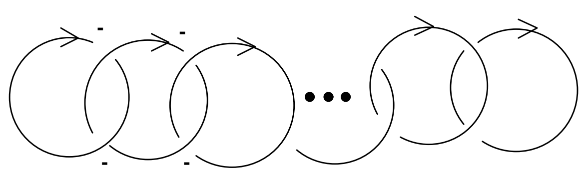
Putting all this together, we get $A^4 f_L(A) - A^{-4}(A^{20} + 2A^{12} + A^4) = (A^{-2} - A^2)(-A^{18} - A^{10} - A^2 + A^6)$, and on simplifying this we finally obtain

$$f_L(A) = A^{16} + A^8 + 2.$$

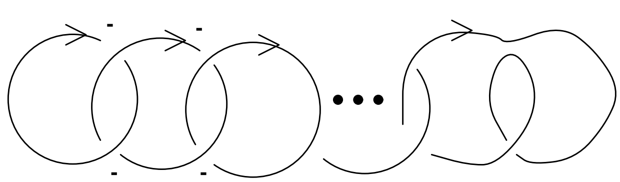
3. Taking $L(n) = K^-$, K^+ is a split link, so $f_{K^+} = 0$, while K^0 is equivalent to $L(n-1)$.



$K^- = L(n)$



$K^+ = L(n-1) \cup 0$



$K^0 \sim L(n-1)$

Hence $A^4 f_{L(n-1)\cup 0}(A) - A^{-4} f_{L(n)}(A) = (A^{-2} - A^2) f_{L(n-1)}(A)$, so using Proposition 6.6 we obtain the recursive relation

$$A^4(-A^{-2} - A^2) f_{L(n-1)}(A) - A^{-4} f_{L(n)}(A) = (A^{-2} - A^2) f_{L(n-1)}(A),$$

which simplifies to

$$f_{L(n)}(A) = (-A^{10} - A^2) f_{L(n-1)}(A)$$

.

Since $L(1)$ is an unknot, we get $f_{L(1)}(A) = 1$, $f_{L(2)}(A) = -A^{10} - A^2$, (negative Hopf link H^-), $f_{L(3)}(A) = (-A^{10} - A^2)^2$, etc. It's now clear that $f_{L(n)}(A) = (-A^{10} - A^2)^{n-1}$, by induction on n.

4. To get the recursion relation, take $K = K^+$ (using *any* crossing) and note that $K^- \sim K_{n-2}$, $K^0 \sim K_{n-1}$.

   Putting $a_n = f_{K_n}(e^{i\pi/12})$ in the recursion relation, we have

$$e^{i\pi/3} a_n - e^{-i\pi/3} a_{n-2} = (e^{-i\pi/6} - e^{i\pi/6}) a_{n-1}.$$

   This simplifies to the given relation

$$a_n = (-\frac{\sqrt{3}}{2} - \frac{1}{2} i) a_{n-1} + (-\frac{1}{2} - \frac{\sqrt{3}}{2} i) a_{n-2}.$$

   Note that all n crossings in K_n are positive, and $K_1 \sim$ unknot, $K_2 \sim H^+$, $K_3 \sim T^+$. Thus $f_1(A) = 1$, $f_2(A) = -A^{-10} - A^{-2}$, and $f_3(A) = -A^{-16} + A^{-12} + A^{-4}$. Hence $a_1 = 1$, $a_2 = -e^{-5i\pi/6} - e^{-i\pi/6} = i$ and $a_3 = -e^{-4i\pi/3} - 1 + e^{-i\pi/3} = -\sqrt{3}i$.

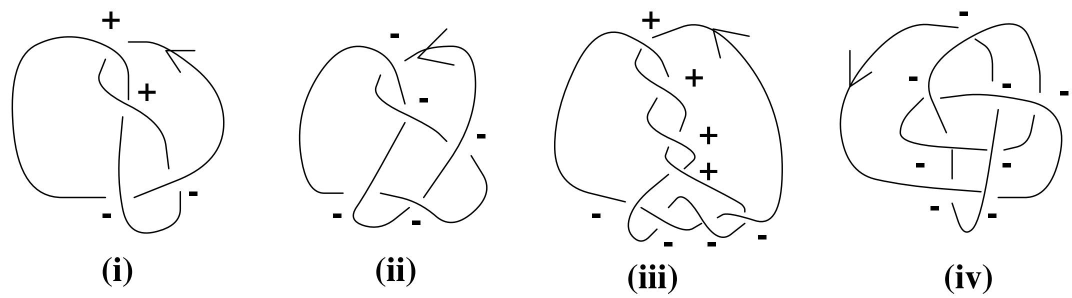
   Calculation with the recursion relation using the initial values $a_1 = 1$ and $a_2 = i$ gives

$$a_3 = -\sqrt{3}i,\ a_4 = i,\ a_5 = -1,\ a_6 = \sqrt{3},\ a_7 = -1 = -a_1,\ \text{and } a_8 = -i = -a_2.$$

   Continuing in this way we see that the value of a_n is determined by the residue class of n mod 12, as shown in the table below.

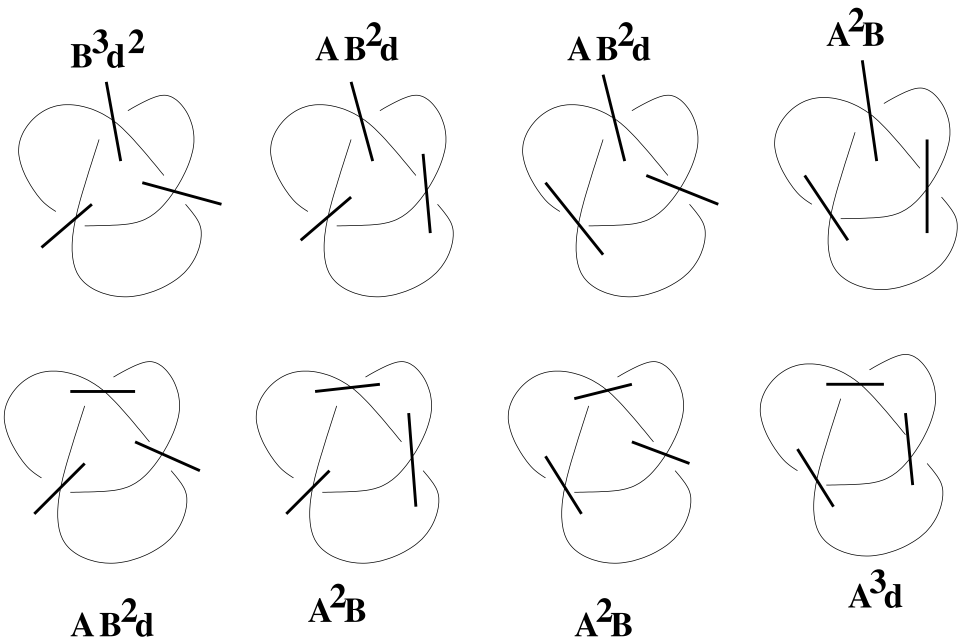
n mod 12	1	2	3	4	5	6	7	8	9	10	11	12
a_n	1	i	$-\sqrt{3}i$	i	-1	$\sqrt{3}$	-1	$-i$	$\sqrt{3}i$	$-i$	1	$-\sqrt{3}$

5. **(i)** The signs of the crossings are shown below. Summing these, the writhe is (i) 0, (ii) -5, (iii) 0, (iv) -8.



(ii) As noted on page 81, the linking number of the two component link obtained by adding a 'parallel strand' to a knot diagram is equal to the writhe of the diagram. So the answers are the same as in part (i) of the question.

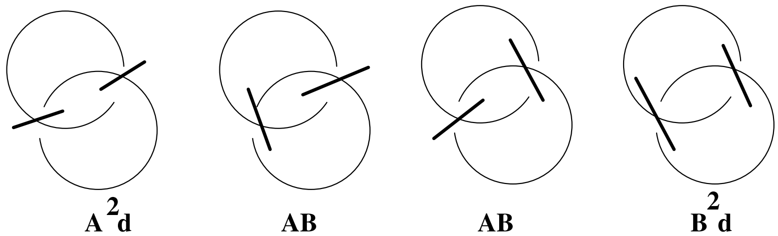
6. **(i)** (Positive trefoil knot)



$$\begin{aligned}\langle K\rangle &= d^2B^3 + 3A^2B + 3dAB^2 + dA^3\\ &= A^{-7} - A^{-3} - A^5.\end{aligned}$$

Since the writhe of the diagram (with either orientation) is 3, the Jones polynomial is $(-A^{-3})^3(A^{-7} - A^{-3} - A^5) = -A^{-16} + A^{-12} + A^{-4}$.

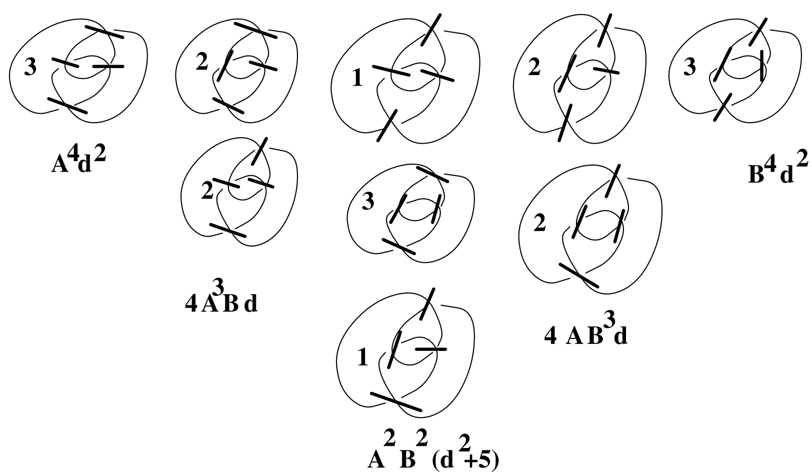
(ii) (Hopf link)



$$\begin{aligned}\langle K\rangle &= dA^2 + 2AB + dB^2 \\ &= -A^4 - A^{-4}.\end{aligned}$$

The writhe of the diagram is $2\mathrm{Lk(K)} = \pm 2$, since the two components cross at both crossings. Thus the Jones polynomial of the positive Hopf link is $(-A^{-3})^2(-A^4 - A^{-4}) = -A^{-2} - A^{-10}$, and the Jones polynomial of the negative Hopf link is $(-A^{-3})^{-2}(-A^4 - A^{-4}) = -A^{10} - A^2$.

(iii) (Figure eight knot)



(In the diagram above, the number on the diagram is the number of components of the corresponding state, and some diagrams have to be counted with appropriate multiplicity, since states which differ from those shown by obvious symmetries are omitted for clarity.)

$$\begin{aligned}\langle K\rangle &= d^2A^4 + 4dA^3B + (5 + d^2)A^2B^2 + 4dAB^3 + d^2B^4 \\ &= A^{-8} - A^{-4} + 1 - A^4 + A^8.\end{aligned}$$

Since the writhe of the diagram (with either orientation) is 0, the Jones polynomial is also $A^{-8} - A^{-4} + 1 - A^4 + A^8$. Note that $f_K(A) = f_K(A^{-1})$, which follows from the achirality of the figure eight knot.