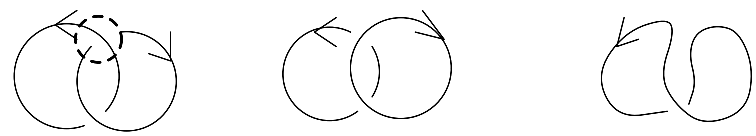
1. **(i)** Here K^- is the negative Hopf link H^-, K^+ is a split link and K^0 is the unknot. Hence $\nabla_{K^+}(z) = 0$ and $\nabla_{K^0}(z) = 1$, and so $\nabla_{H^-}(z) = -z$.



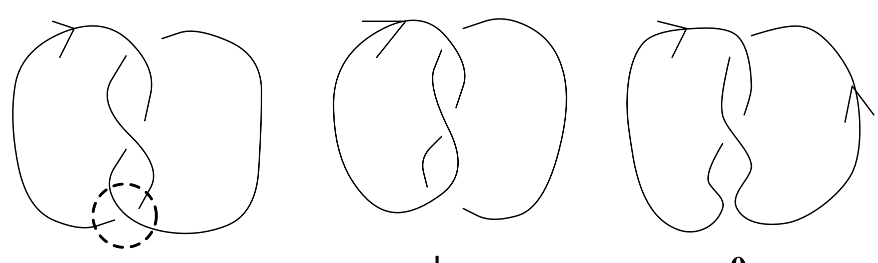
$K^- \sim H^-$ $\quad$ $K^+ \sim 0 + 0$ $\quad$ $K^0 \sim 0$

(ii) Here K^- is the negative trefoil knot T^-, K^+ is an unknot and $K^0 \sim H^-$. So we have $1 - \nabla_{T^-}(z) = z\nabla_{H^-}(z) = -z^2$, and so $\nabla_{T^-}(z) = 1 + z^2$.

$K^- \sim T^-$ $\quad$ $K^+ \sim 0$ $\quad$ $K^0 \sim H^-$

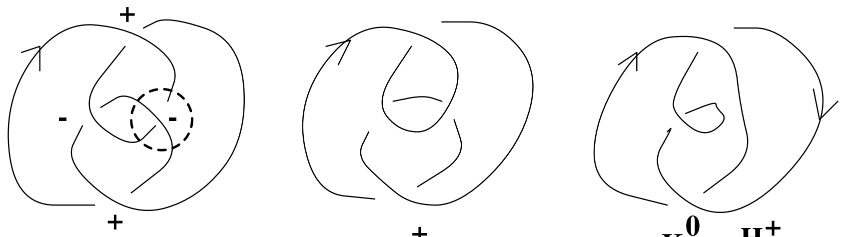
2. Let L be a link with more than two components. The result is true if L has no crossings, by Prop. 3.4. If we splice L at a crossing, then we get a link L^0 with more than one component, by Prop. 3.7. Hence $a_0(L^0) = 0$. From the exchange relation, it follows that $a_1(L)$ is unchanged by switching this crossing. But we can continue to switch crossings until we reach a split link, for which $a_1 = 0$ by Prop. 3.4. Hence $a_1(L) = 0$.

3. **(i)** In fact $K = K^-$ is a left trefoil knot, as you should recognise from Examples 1, Question 2! We get $\nabla_K(z) = 1 + z^2$.



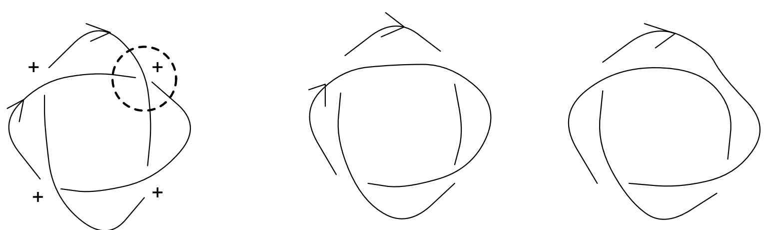
$K^- \sim T^-$ $\quad$ $K^+ \sim 0$ $\quad$ $K^0 \sim H^-$

(ii) This is another form of the figure eight knot. (Exercise: prove this!) We have $1 - \nabla_K(z) = z\nabla_{H^+}(z) = z^2$, so $\nabla_K(z) = 1 - z^2$.



$K = K^-$ $\quad K^+ \sim 0$ $\quad K^0 \sim H^+$

(iii) From the diagram below, with $L = K^+$ we have $\nabla_L(z) - \nabla_{H^+}(z) = z\nabla_{T^+}(z)$, so $\nabla_L(z) - z = z(1 + z^2)$ and hence $\nabla_L(z) = 2z + z^3$. (This L appears in an equivalent form as $K(4)$ in Question 6.)

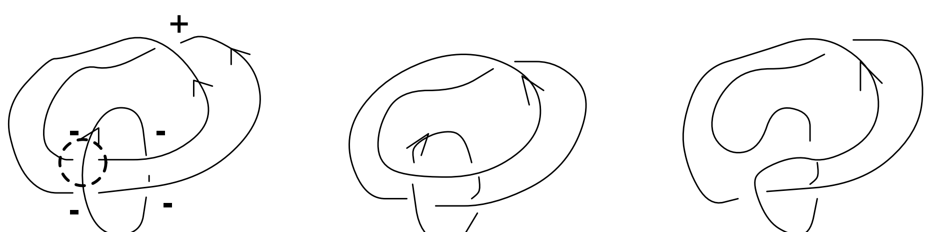


$K^+ = L$ $\quad K^- \sim H^+$ $\quad K^0 \sim T^+$

(iv) We have $\nabla_L(z) - \nabla_{K^-}(z) = z\nabla_{K^0}(z)$, but we must simplify K^- and K^0 further to evaluate these terms.

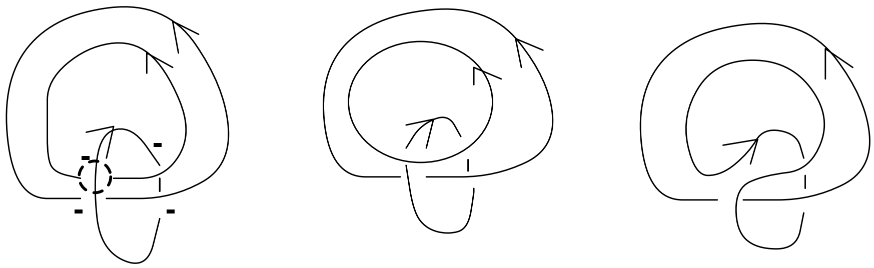
$L = K^+$ $\quad K^-$ $\quad K^0$

We take $K^0 = M^-$, getting $M^+ \sim H^-$ and M^0 an unknot.



$K^0 = M^-$ $\quad M^+ \sim H^-$ $\quad M^0 \sim 0$

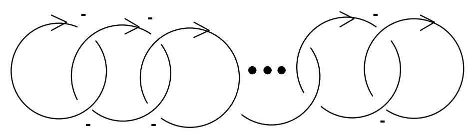
Thus $(-z) - \nabla_{K^0}(z) = z.1 = z$, so $\nabla_{K^0}(z) = -2z$.
We take $K^- = N^-$, getting N^+ a split link and $N^0 \sim H^-$.



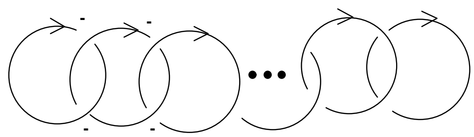
$K^- = N^-$ $\qquad$ N^+ = split link $\qquad$ $N^0 \sim H^-$

Thus $0 - \nabla_{K^-}(z) = z\nabla_{H^-}(z) = -z^2$, so $\nabla_{K^-}(z) = z^2$. Putting all this together, we get $\nabla_L(z) - z^2 = z(-2z)$, so $\nabla_L(z) = -z^2$.

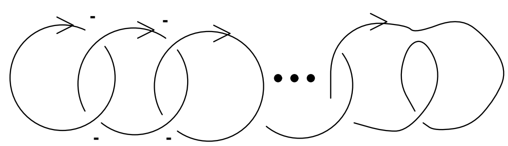
4. Taking $L(n) = K^-$, K^+ is a split link, so $\nabla_{K^+} = 0$, while K^0 is equivalent to $L(n-1)$.



$K^- = L(n)$



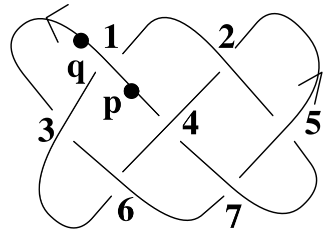
K^+ = split link



$K^0 \sim L(n-1)$

Hence $0 - \nabla_{L(n)}(z) = z\nabla_{L(n-1)}(z)$, giving a simple recursive relation. Since $L(1)$ is an unknot, we get $\nabla_{L(1)}(z) = 1$, $\nabla_{L(2)}(z) = -z$, (negative Hopf link H^-), $\nabla_{L(3)}(z) = z^2$, etc. It's now clear that $\nabla_{L(n)}(z) = (-z)^{n-1}$, by induction on n.

5. **(i)** Number the crossings as in the diagram below. Note that all 7 crossings are negative in K.



Then the standard unknotting sequence for the diagram K with the base point p is 2, 7, 3. For the base point q, the sequence is 1, 2, 7, 3. With notation S_iK for switched crossings and E_iK for spliced crossings, as in the notes, we get

$$\begin{aligned}\alpha(K;p) &= -\mathrm{Lk}(E_2K) - \mathrm{Lk}(E_7S_2K) - \mathrm{Lk}(E_3S_7S_2K)\\ \alpha(K;q) &= -\mathrm{Lk}(E_1K) - \mathrm{Lk}(E_2S_1K) - \mathrm{Lk}(E_7S_2S_1K) - \mathrm{Lk}(E_3S_7S_2S_1K)\end{aligned}$$

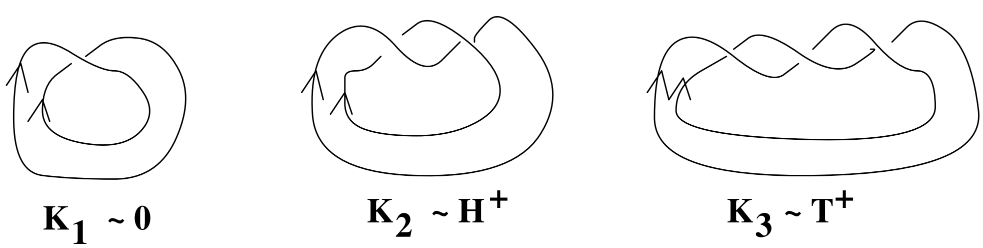
It is left to you to make diagrams to evaluate these linking numbers, as $\alpha(K;p) = -(-2) - (-2) - 0 = 4$ and $\alpha(K;q) = -(-2) - (-1) - (-1) - 0 = 4$.

(ii) The crossings c_1, c_2 in the standard unknotting sequence for p as base point are the crossings numbered 2 and 7. For q as base point, c_1 and c_2 are crossings 1 and 2. We now get (interchanging c_1 and c_2 in the sequences)

$$\begin{aligned}\alpha(K;p) &= -\mathrm{Lk}(E_7K) - \mathrm{Lk}(E_2S_7K) - \mathrm{Lk}(E_3S_2S_7K)\\ \alpha(K;q) &= -\mathrm{Lk}(E_2K) - \mathrm{Lk}(E_1S_2K) - \mathrm{Lk}(E_7S_1S_2K) - \mathrm{Lk}(E_3S_7S_1S_2K)\end{aligned}$$

Again, it is left to you to make diagrams to evaluate the linking numbers. The terms evaluate to $\alpha(K;p) = -(-2) - (-2) - 0 = 4$ and $\alpha(K;q) = -(-2) - (-1) - (-1) - 0 = 4$. (In general, the individual terms in the sums do not have to match up, but this example does not reveal this point.)

6. Note that all n crossings in K_n are positive, and $K_1 \sim$ unknot, $K_2 \sim H^+$, $K_3 \sim T^+$. By previous work $\nabla_1(z) = 1$, $\nabla_2(z) = z$, and $\nabla_3(z) = 1 + z^2$.



The statement about the number of components is easily proved by observing that each crossing exchanges the two strands entering at top left of the diagram, so that they emerge at top right from the sequence of crossings in the same order if n is even and the opposite way round if n is odd.

To get the recursion relation, take $K = K^+$ (using *any* crossing) and note that $K^- \sim K_{n-2}$, $K^0 \sim K_{n-1}$. Putting $z = 1$ gives $\nabla_n(1) = \nabla_{n-1}(1) + \nabla_{n-2}(1)$, the recursion relation for the Fibonacci sequence. Since we also have the correct initial values $\nabla_1(1) = \nabla_2(1) = 1$, the sequence $\nabla_n(1)$ *is* the Fibonacci sequence. Since this sequence is strictly increasing, no two of the polynomials $\nabla_n(z)$ are equal, and so no two of the links K_n are equivalent.