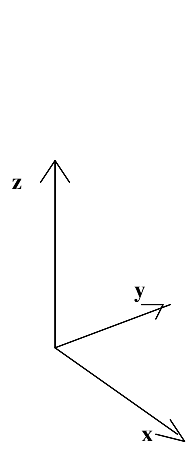
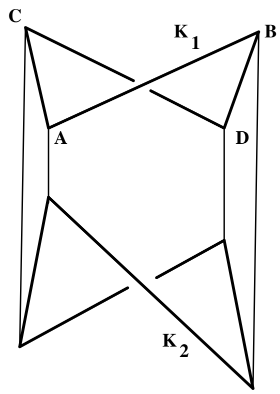
1. **(i)** Let's assume first that K_1 lies entirely above K_2, *i.e.* the z coordinate of each vertex of K_1 is greater than the z coordinate of the corresponding vertex of K_2. Consider the triangular prism with top K_1 and base K_2, so that the remaining 3 edges are line segments parallel to the z axis. Break each of the three quadrilateral faces of this prism into two triangles by a diagonal. These six triangles can be used in an appropriate order to give a sequence of Δ and Δ^{-1} moves connecting K_1 and K_2.



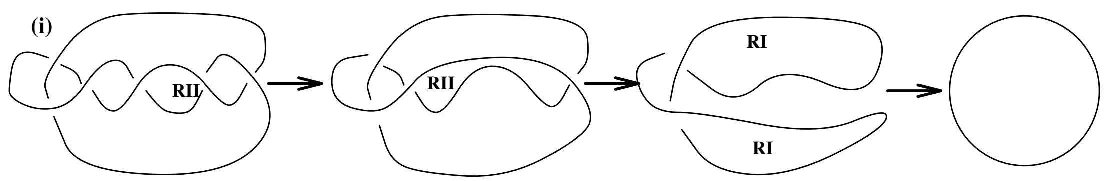
If K_1 is not entirely above K_2 to start with, it can be moved vertically until it is. For this, we can use a similar prism with parallel triangular faces, one of these faces being K_1.

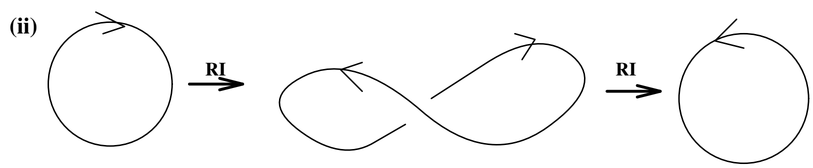
(ii) First note that we can move the knot K_1 vertically by using parallelograms in a vertical plane, with one edge of K_1 being taken as a side of the parallelogram. Each parallelogram is broken into two triangles by a diagonal, as in (i). For example, we can first raise the overpass AB, then raise the underpass CD by equal amounts, introducing extra vertical edges into the knot which project down to the points A, B, C and D. Now obvious further Δ^{-1} moves complete the parallel translation of K_1.

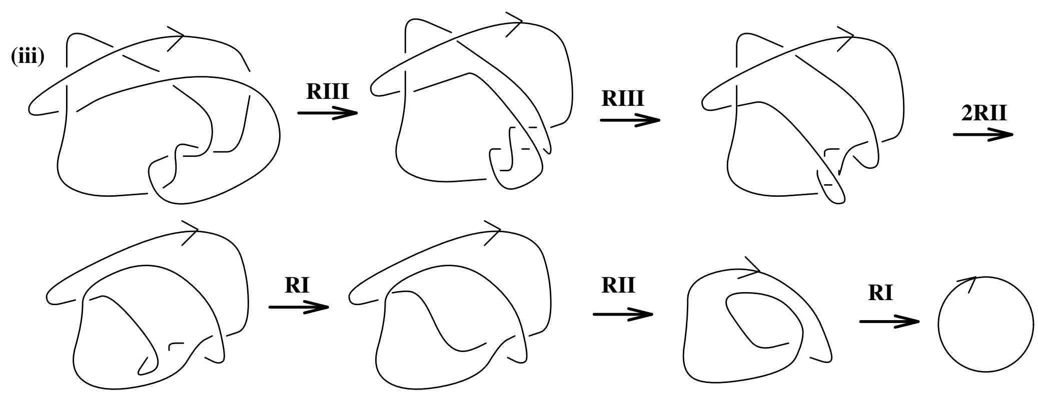
In this way, we may assume that K_1 lies entirely above K_2 to start with. Now use a similar technique to lower the edges of K_1 one at a time to coincide with the edges of K_2, again introducing extra vertical edges in the process. (Lower the underpass first, then the overpass.) Again some final Δ^{-1} moves will remove the extra vertical edges.



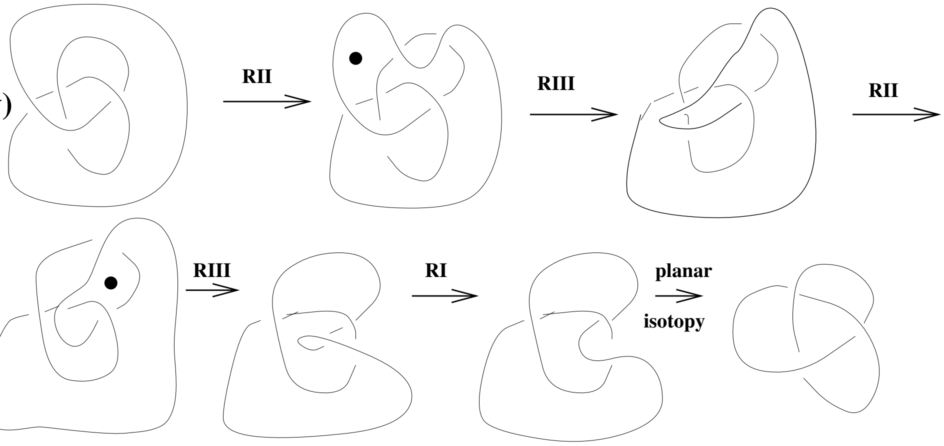
2. There are of course many possible sequences, of which examples are given below.



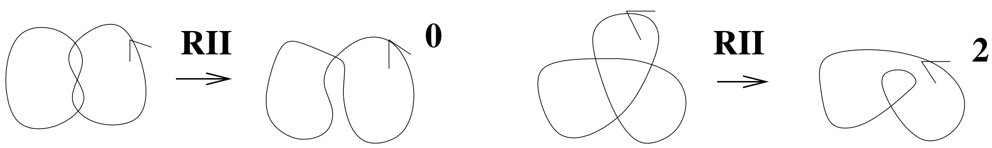




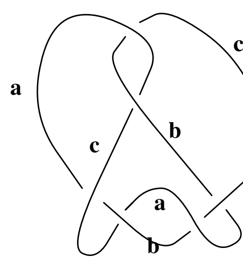
(iv)



3. The following diagrams show that the two trefoils are not regularly isotopic. The “degree” of the underlying knot universe cannot be changed by R moves II and III. Since one universe has degree 0 and the other has degree ± 2, depending on the choice of orientation, and an RI move changes this degree by 1, at least two RI moves are required for the transition between the knot diagrams. The sequence of moves shown in lectures in fact used two RI moves, one RII move and two RIII moves.



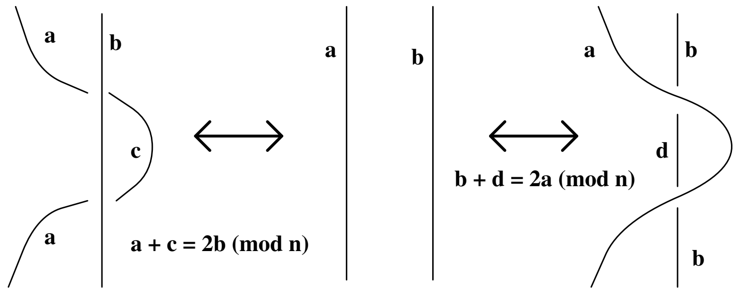
4. The first knot (6_1) is 3-colourable, the others (6_2 and 6_3) are not.



5. **(i)** We check invariance under the three Reidemeister moves, extending the proof of Proposition 2.4 of the notes.

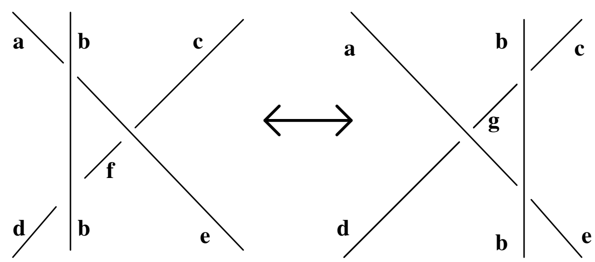
   The RI move is easy since only two arcs are involved and they have the same colour.

   For the RII move, Notice that the colours at one crossing determine the colours at the other crossing, in such a way that the incoming and outgoing arcs that are to be joined after the move have the same colour.



We may lose one colour from the diagram during this move , but we will always be left with a diagram containing at least two colours.

(**RIII move**)



Notice that only one arc changes colour. None of the others can change, because they lead to parts of the diagram that we have not drawn. The following equations (all to be taken mod n) correspond to the first diagram.

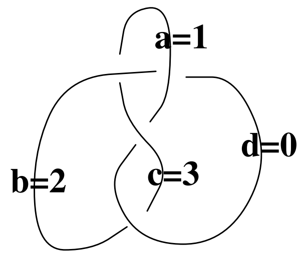
$$a + e = 2b, \quad d + f = 2b, \quad c + f = 2e.$$

The following equations correspond in the same way to the second diagram.

$$d + g = 2a, \quad g + c = 2b, \quad a + e = 2b.$$

We have to show that one set of equations has a solution if and only if the other one has. For this, note that $f = 2b - d = 2e - c$ from the first set of equations, and $g = 2a - d = 2b - c$ from the second set. Now if we assume that the first set of equations is consistent, we have $2b - d = 2e - c$. Hence $2a - d = 2(2b - e) - d = 4b - 2e - d = 2b - c$, and so the second set of equations is also consistent.

(ii) The diagram below shows a 5-colouring of the figure eight knot.



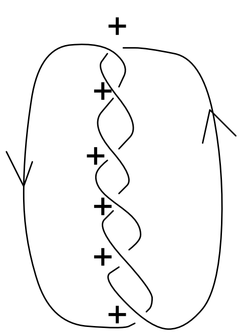
This can be found be trial and error in this case, but it is worth noting that standard techniques of linear algebra can be applied to this type of problem. Essentially, we have to solve the system of linear equations $a+c=2b$, $a+d=2c$, $b+c=2d$, $b+d=2a$ over the integers mod 5. This system obviously has the solution $a=b=c=d$ over the integers mod n for any n, corresponding to the trivial case where all arcs have the same colour. For $n=5$ however the system has rank 2 and there is a second independent solution, which is the one we want. Thus the general solution to the problem allows 20 possible legitimate 5-colourings, given by

$$(a,b,c,d)=\lambda(1,1,1,1)+\mu(1,2,3,0),$$

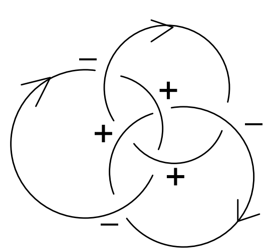
where λ and μ are in $\mathbf{Z}/5$ and $\mu \neq 0$.

To see that the figure eight knot is not equivalent to the unknot, we need to show that the unknot is *not* 5-colourable. In fact the unknot is not colourable mod n for any n, for the same reason as for the case $n=3$.

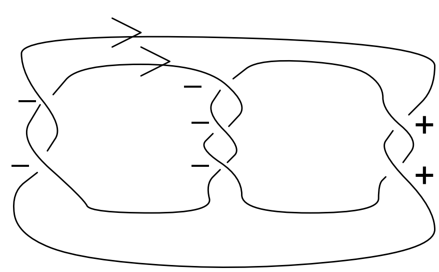
6. In case (i) the linking number is 3, in case (ii) the linking number is 0 for each pair of rings (naturally, since this example is the Borromean rings, see Chapter 1.1). In case (iii) the linking number is again 0 although this link is in fact not split. Note that in case (iii) the three self intersections of the component β do not contribute to the linking number $\mathrm{Lk}(\alpha,\beta)$.



(i)



(ii)



(iii)