6 Knot Polynomials II

6.1 The Jones polynomial

In 1985, the New Zealand mathematician Vaughan Jones described a new polynomial invariant of oriented knots and links which was able to distinguish knots and links with the same Conway polynomial. The Jones polynomial successfully distinguishes all knots of up to ten crossings, and it also distinguishes many knots from their mirror images. As we shall see, the Jones polynomial can be computed by a recursive procedure similar to that for the Conway polynomial.

The Jones polynomial can have negative as well as positive powers of its variable. Thus it is not really a polynomial in the ordinary sense of the word, but an object which is more accurately described in algebra as a *Laurent polynomial.* (You will probably have encountered Laurent series in connection with the expansion of a function $f(z)$ of a complex variable with a pole at $z = a$ in powers of $z - a$.) As in the case of the Conway polynomial, the coefficients in the Jones polynomial are integers, so for our purposes it is convenient to make the following definition.

Definition 6.1 A *Laurent polynomial* in a variable A is an expression of the form $f(A) = f_{-r}A^{-r} + f_{-r+1}A^{-r+1} + \ldots + f_sA^s$, for some integers $r, s \geq 0$, where the coefficients f_i are integers for $-r \leq i \leq s$.

We can think of a Laurent polynomial as the sum of a polynomial in A and a polynomial in A^{-1}. For example, the Laurent polynomial $A^{-7} - A^{-3} - A^5$ is the sum of the polynomial $-A^5$ in A and the polynomial $A^{-7} - A^{-3}$ in A^{-1}.

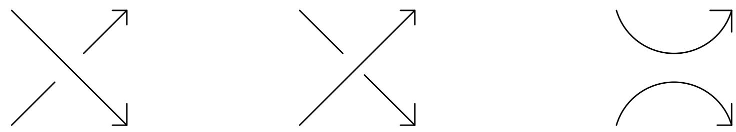
We can add, subtract and multiply Laurent polynomials in the same way as for ordinary polynomials, and it is easy to see that the set of all Laurent polynomials is a commutative ring with identity element 1. We denote this ring by $\mathbf{Z}[A, A^{-1}]$.

Next we shall state the axioms for the Jones polynomial.

(Axiom 1: Invariance) To each oriented knot or link K there is associated a polynomial $f_K(A) \in \mathbf{Z}[A, A^{-1}]$, such that $f_K(A) = f_{K'}(A)$ if K and K' are equivalent.

(Axiom 2: Normalisation) If 0 denotes the standard unknot, *i.e.* a positively oriented circle, then $f_0(A) = 1$.

(Axiom 3: The exchange relation) Let K^+, K^- and K^0 be three knots or links that have identical diagrams except in the neighbourhood of one crossing, where they differ as follows:

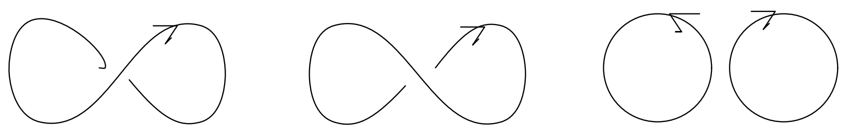


K^+ K^- K^0

Then

$$A^4 f_{K^+}(A) - A^{-4} f_{K^-}(A) = (A^{-2} - A^2) f_{K^0}(A).$$

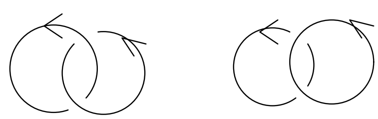
We can start right away to do some calculations.



$K^+ \sim 0$ $K^- \sim 0$ $K^0 \sim 0 + 0$

Example 6.2 Here K^+ and K^- are both unknots, so $f_{K^+}(A) = f_{K^-}(A) = f_0(A)$ by Axiom 1, which $= 1$ by Axiom 2. Hence $f_{K^0}(A) = (A^4 - A^{-4})/(A^{-2} - A^2) = -A^{-2} - A^2$ by Axiom 3.

This simple example shows that the Jones polynomial distinguishes the unknot from the trivial link with two components. Of course, this is not much of an achievement! The next example calculates f for a positive Hopf link.



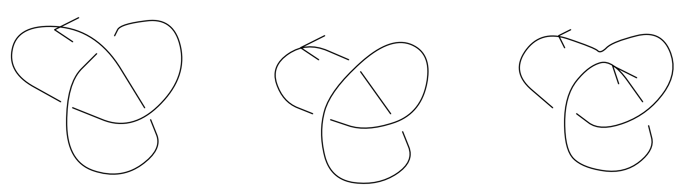
$K^+ \sim$ +ve Hopf link $K^- \sim 0 + 0$ $K^0 \sim 0$

Example 6.3 Here $K^+ = H^+$, the positive Hopf link, K^0 is an unknot, and K^- is a split two-component unlink. So using the previous result we have

$$A^4 f_{H^+}(A) - A^{-4}(-A^{-2} - A^2) = (A^{-2} - A^2).1,$$

so $f_{H^+}(A) = -A^{-10} - A^{-2}$.

With this preparation, we are ready to tackle the positive trefoil knot T^+.



$K^+ \sim$ +ve trefoil $K^- \sim 0$ $K^0 \sim H^+$

Example 6.4 Here $K^+ = T^+$, the positive trefoil knot, K^- is an unknot, and K^0 is a positive Hopf link. So using the previous result we have

$$A^4 f_{T^+}(A) - A^{-4} f_0(A) = (A^{-2} - A^2) f_{H^+}(A)$$

so $A^4.f_{T^+}(A) - A^{-4}.1 = (A^{-2} - A^2)(-A^{-10} - A^{-2})$, and hence $f_{T^+}(A) = -A^{-16} + A^{-12} + A^{-4}$.

The strategy in these calculations is exactly the same as for the Conway polynomial — only the answers are different! You may find it worth refreshing your memory by looking back at Section 3.2, where it is shown that the three axioms determine the Conway polynomial uniquely. Exactly the same recursive argument (a double induction on the number of crossings in a diagram, and the untying number of the diagram) can be used to show that the following is true.

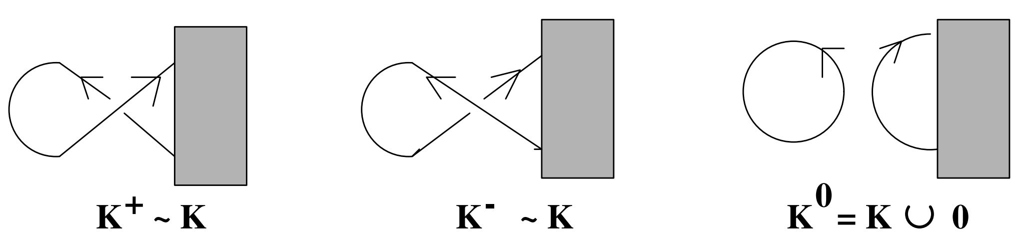
Proposition 6.5 *Let f and f' be two functions from the set of all knot or link diagrams to the Laurent polynomial ring $\mathbf{Z}[A, A^{-1}]$ which satisfy Axioms 1, 2 and 3. Then $f = f'$.*

To get this induction argument started, we have to deal with the case where the crossing number is zero. This means, of course, that we are dealing with a diagram which represents a trivial link. Recall that in the case of the Conway polynomial ∇, we showed that $\nabla_L(z) = 0$ for any split link L. In particular, this holds for a trivial link, and this, together with Axiom 2, gets the induction going.

Recall the notation $K_1 \cup K_2$ for a split link which is made up of the two separate links K_1 and K_2. In particular, $K \cup 0$ is a split link with the unknot 0 as one component which can be split off from all the rest by enclosing it in this way inside its own spherical home. Extending this notation, we can write $0 \cup 0 \cup \ldots \cup 0$ (n terms) for a trivial link with n components. Since this is a bit cumbersome, we'll shorten it to just $n \cdot 0$.

The next result generalises Example 6.2, and should be compared with Proposition 3.4.

Proposition 6.6 *For any oriented knot diagram K, $f_{K\cup 0}(A) = (-A^{-2} - A^2) f_K(A)$.*



Proof We take $K^0 = K \cup 0$, the given split link, with its strands arranged as shown above. We can then form the associated links K^+ and K^- as shown. Now K^+ and K^- are both equivalent to K, so $f_{K^+}(A) = f_{K^-}(A) = f_K(A)$, by

Axiom 1. By Axiom 3, $(A^4 - A^{-4})f_K(A) = (A^{-2} - A^2)f_{K^0}(A)$, and the result follows. □

The next result shows that the Jones polynomial of a trivial link is determined uniquely by the three axioms, thus starting the inductive argument that the same is true for an arbitrary link.

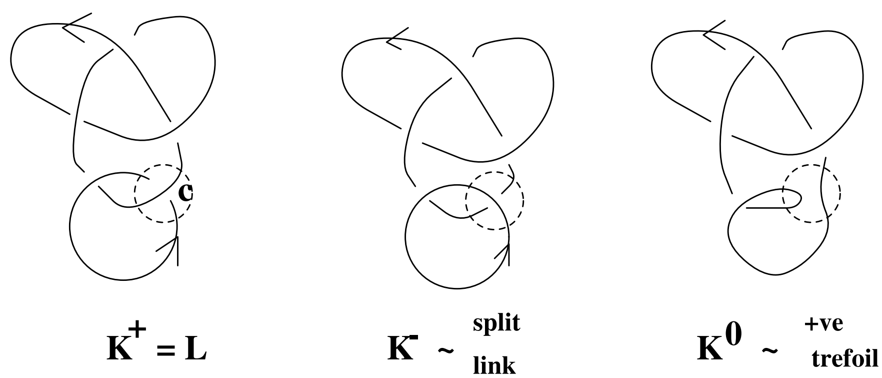
Corollary 6.7 *For any positive integer n,*

$$f_{n\cdot 0}(A) = (-A^{-2} - A^2)^{n-1}.$$

Proof This follows from the preceding result by induction on n. □

The following example should be compared with Example 3.5.

Example 6.8 Consider the following link L.



Here we choose the crossing c so that K^- is a split link $T^+ \cup 0$, and $K^0 \sim T^+$, where T^+ is the positive trefoil knot. Thus we have $A^4 f_L(A) - A^{-4} f_{T^+\cup 0}(A) = (A^{-2} - A^2)f_{T^+}(A)$. Thus

$$\begin{aligned} A^4 f_L(A) &= A^{-4}(-A^{-2} - A^2)f_{T^+}(A) + (A^{-2} - A^2)f_{T^+}(A) \\ &= (-A^{-6} - A^2)f_{T^+}(A), \end{aligned}$$

so that $f_L(A) = (-A^{-10} - A^{-2})f_{T^+}(A)$.

Evidently, the same calculation goes through for any other knot K in place of the positive trefoil knot T^+: the Jones polynomial of L is the product of the Jones polynomials of the positive Hopf link and of K. This is also true for the Conway polynomial (see Example 3.5).

Exercise 6.9 Show that if we reverse the arrows on *all* the components of a link diagram, then we don't change either the Conway polynomial or the Jones polynomial. (Use an inductive argument on the crossing number and untying number of diagrams, as in the proofs of Propositions 3.16 and 3.17 for the Conway polynomial.)

Thus neither of the two polynomials is capable of distinguishing between an oriented knot or link and its reverse. However, examples exist (*e.g.* 8_{17}, see Section 1.3) where a knot and its reverse are not equivalent as oriented knots.

Note that the sign of a crossing is not changed if the orientations of *both* strands at the crossing are reversed. Thus by Exercise 6.9 we may regard both the Conway and Jones polynomials as invariants of the *unoriented* knot type. The same is not true for links, however, since both polynomials can distinguish between the positive and negative Hopf links, and these links become identical if the arrows are disregarded.

The next example is very important in terms of the history of knot theory. It shows that the Jones polynomial distinguishes between the two mirror image trefoil knots. Up to 1985, the problem of finding an invariant that would distinguish these two knots was considered to be very difficult. For example, classical knot invariants (such as the Alexander polynomial, or anything to do with complements such as the fundamental group) are the same for any knot K and its mirror image K^*. It follows from Propositions 3.16 and 3.17 that the Conway polynomial cannot distinguish mirror images for knots, although Proposition 3.18 shows that it can often achieve this for links with an even number of components.

You are by now expert enough to check this important example for yourself, and it is strongly recommended that you do so. The method is exactly the same as for the Conway polynomial: see Examples 3, Question 1.

Exercise 6.10 Show that for the negative Hopf link H^-,

$$f_{H^-}(A) = -A^{10} - A^2,$$

and deduce that, for the negative trefoil knot T^-,

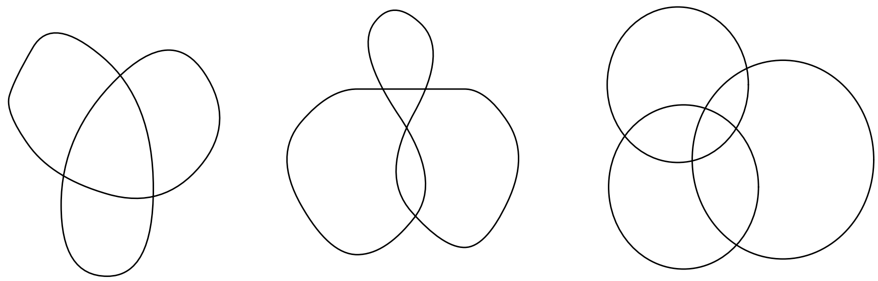
$$f_{T^-}(A) = -A^{16} + A^{12} + A^4.$$

6.2 Link universes and states

Our next task is to show the existence of the Jones polynomial. We do this by adopting a very efficient (but at first sight indirect) approach due to the Chicago mathematician Louis H. Kauffman. This approach is motivated by surprising relations between knot theory and a branch of physics called statistical mechanics. We exploit this by borrowing some physical terminology. Note that in Sections 6.2 and 6.3 we work with *unoriented* knots and links.

We start by giving a formal definition of a knot or link universe, an idea that we already met in Section 2.2. Recall that for formal purposes we may consider all knots and links to be polygonal.

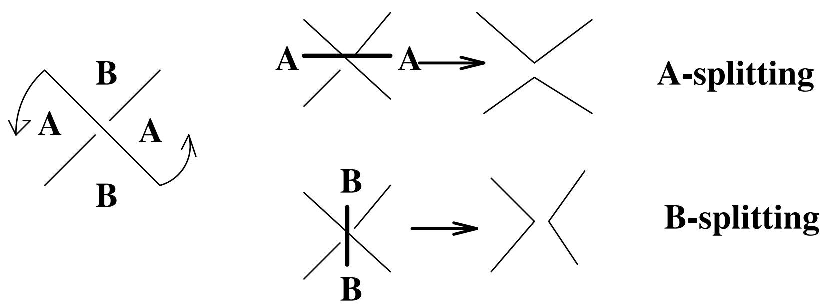
Definition 6.11 A *link universe* is a union of (one or more) polygons in $\mathbf{R}^2$ with a finite number of double points (called *crossings*), which are not vertices of any of the polygons, and no other multiple points.



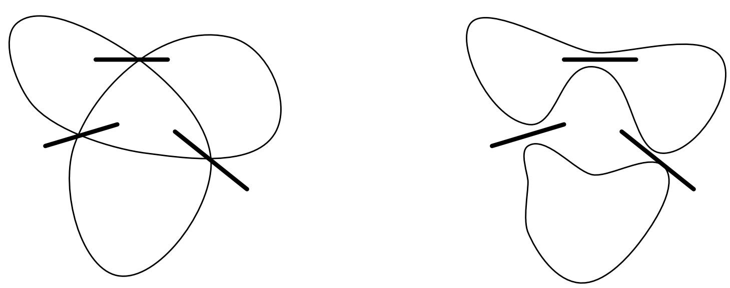
trefoil universe **figure eight knot universe** **Borromean rings universe**

Now the choice of overpass and underpass at a crossing is equivalent to a choice between the two opposite pairs of local regions of the complement of the universe which meet at that crossing. The convention is to mark the two regions swept out by anticlockwise rotation of the top string as A, and the other two as B.

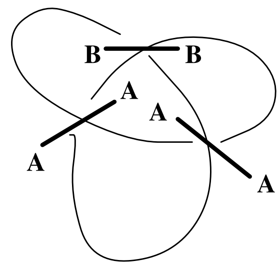
At each crossing we can draw a "splitting marker" joining either the A regions or the B regions. This marker can be interpreted as an instruction to splice the crossing in one of the two possible ways, so that one pair of opposite regions is joined. We refer to this as an *A-splitting* if the A regions are joined, and as a *B-splitting* if the B regions are joined.



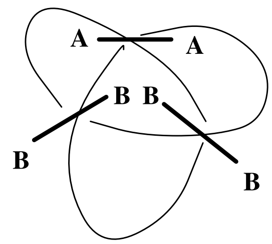
Definition 6.12 A *state* of a link universe is a choice of splitting marker for each crossing.



For a given link diagram D with n crossings, there will be 2^n possible states of its underlying universe U. For each state S and each crossing x of D, the splitting marker at x will give either an A-splitting or a B-splitting at x. For the state S of the trefoil universe shown above, the standard diagrams D^+ and D^- for the two trefoil knots T^+ and T^- are marked as shown below.



negative trefoil knot diagram D -



positive trefoil knot diagram D +

We can think of A and B as measures of the probabilities that the A or B splitting will be chosen at a crossing.

Definition 6.13 Let S be a state of the underlying universe of the link diagram D. The *expectation* of D on S is $\langle D|S\rangle = A^iB^j$, where i is the number of A splitting markers and j is the number of B splitting markers in the state S.

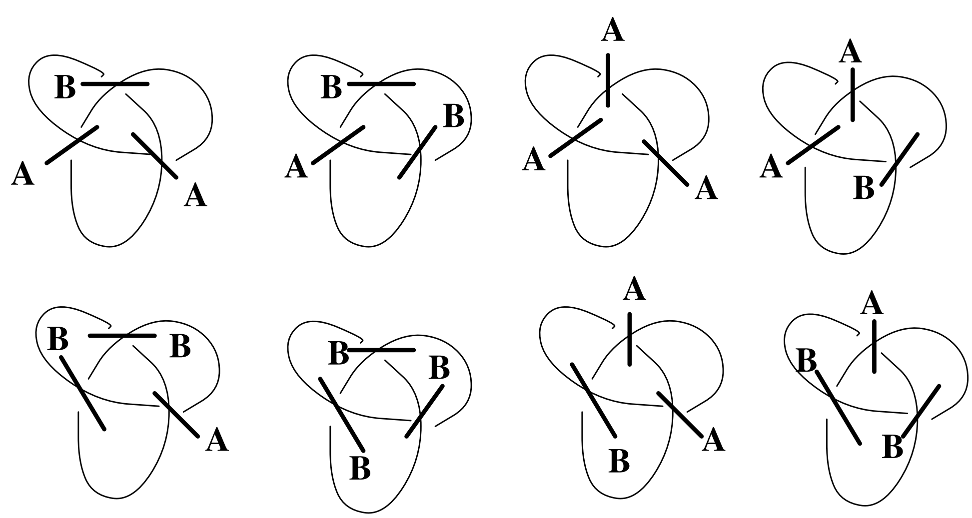
For each state S, we now split the diagram at each crossing as indicated by the markers. Since this procedure eliminates all the crossings, this gives the diagram of a trivial link. We denote by $|S|$ the number of components of this unlink.

Definition 6.14 Let D be an unoriented knot or link diagram. Then the *generalised bracket* of D is the polynomial in A, B and d defined by

$$\langle D\rangle = \sum_S \langle D|S\rangle d^{|S|-1},$$

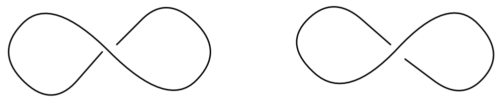
where the sum is over all states of the universe U of D.

Example 6.15 For the negative trefoil knot diagram D^-, $\langle D^-\rangle = 3AB^2 + 3A^2Bd + B^3d + A^3d^2$. The eight states are shown below.



Exercise 6.16 Write down the generalised bracket $\langle D^+\rangle$.

The generalised bracket $\langle D\rangle$ is certainly not an invariant of the knot or link represented by the diagram D. For example, the standard unknot diagram, a circle, has no crossings and only one state, so the bracket polynomial is 1, but the unknot diagrams



have one crossing and two states with bracket polynomials $A + Bd$ and $B + Ad$ respectively. This shows that $\langle D\rangle$ is not invariant under Reidemeister I moves. It is equally easy to show that it is not invariant under RII or RIII moves — this is left to you.

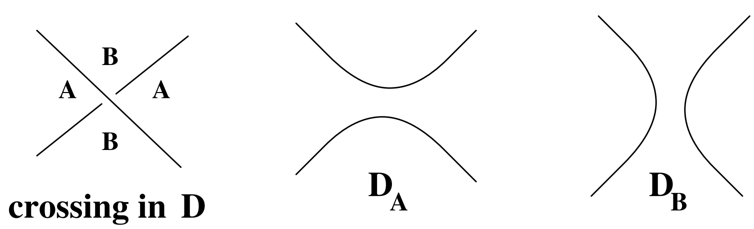
The next result allows us to calculate the generalised bracket polynomial by recursion.

Proposition 6.17 (Axioms for the generalised bracket $\langle D\rangle$)

(i) $\langle 0\rangle = 1$, *where* 0 *is an unoriented circle.*

(ii) *For any link diagram* D, $\langle D \cup 0\rangle = d\langle D\rangle$.

(iii) *If* D *is any link diagram and* D_A, D_B *are the diagrams obtained by splitting* D *along the* A *and* B *markers at the same crossing point* x, *then* $\langle D\rangle = A\langle D_A\rangle + B\langle D_B\rangle$.



Proof (i) Clear.

(ii) Adding 0 introduces no extra crossings, so each state S of D corresponds to a state S' of $D \cup 0$ so that the crossing markers are the same in S and S'. Thus $\langle D|S\rangle = \langle D \cup 0|S'\rangle$. Since S' has one more component than S, $|S'| = |S| + 1$. The result follows.

(iii) Divide all states of D into two sets $\mathcal{S}_A$, $\mathcal{S}_B$ according to whether the marker at the crossing x joins the A regions or the B regions. If S is a state of D and $S \in \mathcal{S}_A$, then S corresponds to a state S' of D_A in which the markers at all crossing points other than x agree. Clearly $|S| = |S'|$ and $\langle D|S\rangle = A\langle D_A|S'\rangle$. Hence

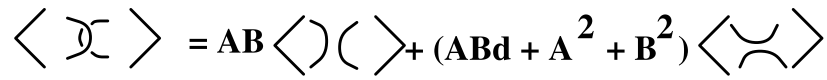
$$\sum_{S\in\mathcal{S}_A} \langle D|S\rangle d^{|S|-1} = A\langle D_A\rangle.$$

Similarly for $\mathcal{S}_B$. □

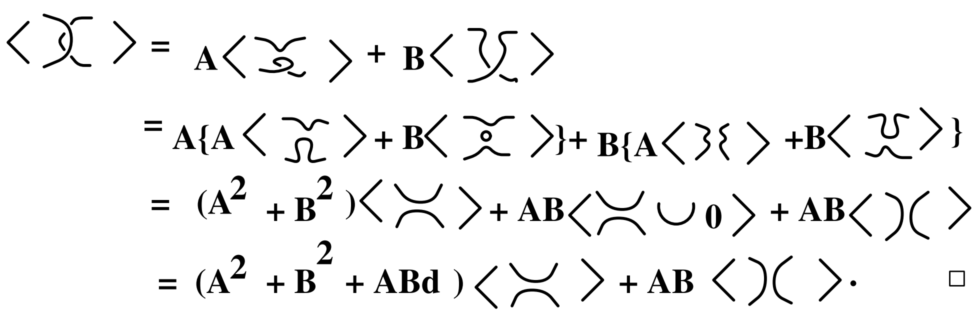
6.3 The Kauffman Bracket

The next step is to choose values of A, B and d so that the generalised bracket polynomial $\langle D \rangle$ becomes invariant under Reidemeister moves RII and RIII. Later, we'll add a correction factor to obtain an invariant under Reidemeister I moves as well.

Rather than naming all the knot diagrams which occur, we adopt a natural diagrammatic notation which shows only the parts of the knot diagrams involved in the manipulations we do. In the equations which follow, it is always assumed that the rest of the diagrams are identical in each case.

Proposition 6.18 

Proof By Proposition 6.17



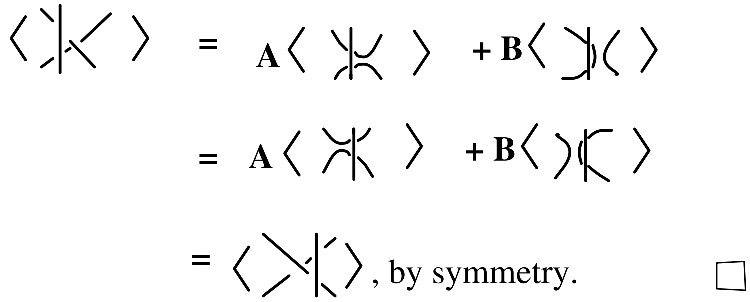
It follows that if $AB = 1$ and $ABd + A^2 + B^2 = 0$ then $\langle D \rangle$ is invariant under RII.

Definition 6.19 The *Kauffman bracket* $\langle D \rangle \in \mathbf{Z}[A, A^{-1}]$ is obtained from the generalised bracket $\langle D \rangle \in \mathbf{Z}[A, B, d]$ by substituting $B = A^{-1}$ and $d = -A^2 - A^{-2}$.

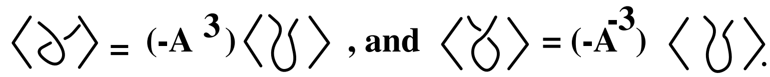
Notice that we shall not distinguish this specialisation of the bracket polynomial by a different notation. The Kauffman bracket is a Laurent polynomial in A.

Proposition 6.20 *The Kauffman bracket $\langle D \rangle$ is invariant under RIII.*

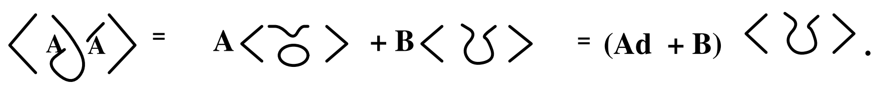
Proof We calculate using Propositions 6.17 and 6.18.



Recalling Section 2.2, the Kauffman bracket is an invariant of *regular isotopy*. (This just means that it is invariant under RII and RIII.) It is easy to see that it is not invariant under RI, as the examples mentioned at the end of Section 6.2 show: for the circle, $\langle O \rangle = 1$, while for the twists with one crossing we get $\langle D \rangle = B + Ad = A^{-1} - A(A^2 + A^{-2}) = -A^3$ or $A + Bd = A - A^{-1}(A^2 + A^{-2}) = -A^{-3}$, depending on the direction of the twist. The next result generalises this example to an arbitrary RI move.

Proposition 6.21 

Proof



The other case is left to you. □

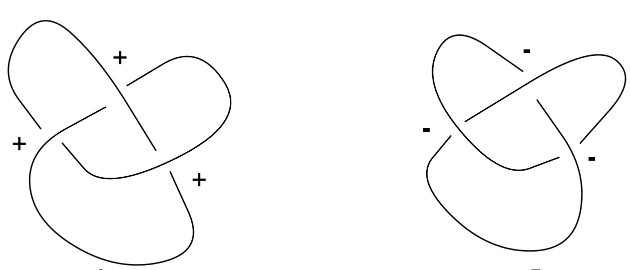
6.4 The writhe of an oriented link diagram

In Chapter 5, we discussed the writhe $\mathrm{Wr}(R)$ of a ribbon R in $\mathbf{R}^3$. Here we are concerned with a simpler concept, since a link diagram is already the projection of a set of polygons in $\mathbf{R}^3$ on to a plane in $\mathbf{R}^3$. To distinguish these concepts by notation, we shall use $\mathrm{wr}(D)$ rather than $\mathrm{Wr}(D)$ for the writhe of the link diagram D.

Definition 6.22 For an oriented link diagram D, the *writhe* $\mathrm{wr}(D) = \sum_x \epsilon(x)$, where the sum is over all crossings in D.

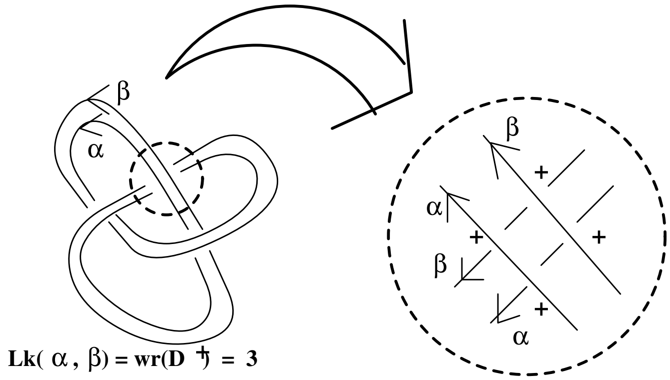
Note that the one crossing twist has writhe $+1$ if the twist is one way and -1 if it is the other way, regardless of the choice of orientation.

For example, $\mathrm{wr}(D^+) = +3$ for the positive trefoil knot diagram D^+, and $\mathrm{wr}(D^-) = -3$ for the negative trefoil knot diagram D^-. (Note that we can't use this calculation to distinguish between the two trefoil knots, since the writhe is not an invariant of knot type.)



wr(D+) = 3 wr(D-) = -3

The writhe of a knot diagram D has an invariant interpretation as the linking number of the two component link L whose diagram is obtained by adding a "parallel" curve to D. This is indicated by the diagram below. However, the writhe $\mathrm{wr}(D)$ is not an invariant of the knot K represented by D.



We shall show that the writhe, like the Kauffman bracket, is a regular isotopy invariant (see Section 2.2). Since it is obviously unchanged by planar isotopies, this means that we have to prove invariance under RII and RIII.

Proposition 6.23 $\mathrm{wr}(D)$ *is unchanged by RII and RIII.*

Proof Modify the corresponding proof for linking numbers (Proposition 2.8). □

The behaviour of the writhe $\mathrm{wr}(D)$ under Reidemeister I moves is implicit in earlier examples, but let's restate it.

Proposition 6.24 *If D' is obtained from D by an RI move which introduces a positive crossing, then* $\mathrm{wr}(D') = \mathrm{wr}(D) + 1$*, and if D' is obtained from D by an RI move which introduces a negative crossing, then* $\mathrm{wr}(D') = \mathrm{wr}(D) - 1$. □

6.5 Invariance proof for the Jones polynomial

We now have two regular isotopy invariants of a link diagram D, the Kauffman bracket $\langle D\rangle$ and the writhe $\mathrm{wr}(D)$. By combining them suitably, we can obtain an invariant under Reidemeister I moves as well. The following theorem shows that $f_D(A)$ is the Jones polynomial $f_K(A)$ of the oriented link K. By uniqueness (Proposition 6.5), we need only prove that $f_D(A)$ satisfies the three axioms of Section 6.1.

Theorem 6.25 *Let K be an oriented link, let D be a diagram for K, and let*

$$f_D(A) = (-A^{-3})^{\mathrm{wr}(D)}\langle D\rangle.$$

Then f is a function from the set of all oriented link diagrams to the ring of Laurent polynomials $\mathbf{Z}[A, A^{-1}]$ which satisfies Axioms 1, 2 and 3 of Section 6.1.

Proof Invariance under RII and RIII moves is clear from previous results, so suppose D' is obtained from D by an RI move which introduces a positive crossing.

Then $\mathrm{wr}(D') = \mathrm{wr}(D) + 1$, by Prop. 6.24, and $\langle D' \rangle = (-A^3)\langle D \rangle$, by Prop. 6.21. So $f_{D'}(A) = (-A^{-3})^{\mathrm{wr}(D')}\langle D' \rangle = (-A^{-3})^{\mathrm{wr}(D)+1}(-A^3)\langle D \rangle = (-A^{-3})^{\mathrm{wr}(D)}\langle D \rangle = f_D(A)$.

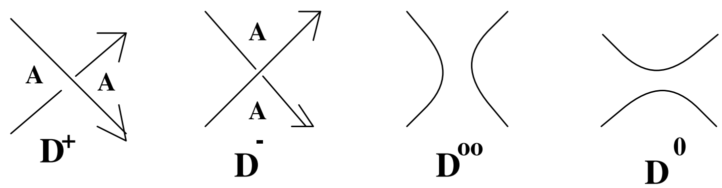
Of course, there is a similar argument (left to you) for a negative crossing.

This proves that $f_D(A)$ satisfies Axiom 1, *i.e.* $f_D(A)$ is an invariant of oriented link type. We may therefore write it as $f_K(A)$.

Axiom 2 presents no problem, since $\langle 0 \rangle = 1$ and $\mathrm{wr}(0) = 0$. For Axiom 3, the exchange relation, we start by using Proposition 6.17 to evaluate

$$\langle D^+ \rangle = A^{-1}\langle D^\infty \rangle + A\langle D^0 \rangle, \quad \langle D^- \rangle = A\langle D^\infty \rangle + A^{-1}\langle D^0 \rangle,$$

where D^∞ and D^0 are the unoriented diagrams shown below.



(Note that the orientations are not needed in this part of the calculation, and that in fact there is no "natural" way to orient D^∞.)

Thus

$$A\langle D^+ \rangle - A^{-1}\langle D^- \rangle = (A^2 - A^{-2})\langle D^0 \rangle.$$

Substituting $(-A^3)^{\mathrm{wr}(D)} f_D(A)$ for $\langle D \rangle$ in each term, we have

$$A(-A^3)^{\mathrm{wr}(D^+)} f_{D^+}(A) - A^{-1}(-A^3)^{\mathrm{wr}(D^-)} f_{D^-}(A) = (A^2 - A^{-2})(-A^3)^{\mathrm{wr}(D^0)} f_{D^0}(A).$$

Now $\mathrm{wr}(D^+) = \mathrm{wr}(D^0) + 1$ and $\mathrm{wr}(D^-) = \mathrm{wr}(D^0) - 1$, so the exchange relation is obtained by dividing this equation by $(-A^3)^{\mathrm{wr}(D^0)}$ and changing signs. □

To summarise, Kauffman's approach provides a direct definition of the Jones polynomial, which could be used as a method of calculation in place of the recursive method we used in examples earlier. This apparent advantage is an illusion, however, as the method for calculating the Kauffman bracket $\langle D \rangle$ for a diagram D of K using Proposition 6.17 is itself recursive. For a direct calculation on a link with n crossing points, we would have to evaluate a sum over all 2^n states of the corresponding link universe U — a formidable task.

On the other hand, Kauffman's formula $f_K(A) = (-A^{-3})^{\mathrm{wr}(D)}\langle D \rangle$ shows how f_K is built up from two invariants $\langle D \rangle$ and $\mathrm{wr}(D)$ of regular isotopy. An orientation on K is not needed for $\langle D \rangle$, but only for the writhe. In particular, the effect of reversing the orientation on *all* components of a link diagram is to leave the writhe (and hence the Jones polynomial) unchanged, so we see that, for knots, (but not for links in general) the Jones polynomial is an invariant of the *unoriented* knot type.

6.6 Application to chirality of links

Unlike the Conway polynomial, the Jones polynomial can distinguish between a *knot* and its mirror image. The following result is the analogue of Proposition 3.17.

Proposition 6.26 *For any link L,*

$$f_{L^*}(A) = f_L(A^{-1}).$$

Proof As for Proposition 3.17, we argue by induction using the exchange relation. By Corollary 6.7, the result is true for links with no crossings, and this is used to start the induction.

We shall take $L = L^+$, the case $L = L^-$ being similar. Then

$$A^4 f_L(A) - A^{-4} f_{L^-}(A) = (A^{-2} - A^2) f_{L^0}(A),$$

and similarly

$$A^{-4} f_{L^*}(A) - A^4 f_{L^{-*}}(A) = (A^2 - A^{-2}) f_{L^{0*}}(A),$$

noting that each crossing changes sign when we take mirror images. Now, by the induction hypothesis, we may assume that

$$f_{L^{0*}}(A) = f_{L^0}(A^{-1}), \quad f_{L^{-*}}(A) = f_{L^-}(A^{-1}).$$

Putting these statements together, we get $f_{L^*}(A) = f_L(A^{-1})$, which does the induction step. □

Exercise 6.27 Use Theorem 6.25 to give an alternative proof of this result.

This result gives us a very useful means of narrowing down the search for achiral links.

Corollary 6.28 *If L is an achiral link, then $f_L(A) = f_L(A^{-1})$.*

Although The Jones polynomial is a strong invariant, note that the above result cannot be used to prove that a given link L is achiral. However, if it is found that $f_L(A) = f_L(A^{-1})$, then in practice one should certainly suspect that the reason for this is indeed that L is achiral.

Proposition 3.16 also has an analogue for the Jones polynomial.

Proposition 6.29 *For any link L with an odd number of components, $f_L(A)$ is a polynomial in A^4, and for any link L with an even number of components, $f_L(A)$ is an odd polynomial in A^2,* i.e. *all the exponents are* $\equiv 2 \bmod 4$.

Note that Corollary 6.7 again provides the base case for an inductive proof. The details of the proof are left to you as an exercise.

6.7 Two variable polynomial invariants

We have seen that there are two polynomial invariants of knots and links, the Conway polynomial and the Jones polynomial, both satisfying the same kind of exchange relation. It is natural to try to generalise this recursive scheme, in an attempt to find further invariants of the same type. In the mid 1980's, this idea occurred to several groups of researchers at the same time. The new polynomial, christened the HOMFLY polynomial after the initials of its discoverers, is denoted by $P_K(l, m)$. It is a Laurent polynomial with integer coefficients in the two variables l and m, *i.e.* negative powers of both variables can occur.

As for the Conway and Jones polynomials, the HOMFLY polynomial is normalised by taking $P_0(l, m) = 1$ for the unknot 0. The exchange relation is

$$lP_{K^+}(l, m) + l^{-1}P_{K^-}(l, m) = -mP_{K^0}(l, m).$$

By substituting $l = i$ and $m = -iz$, where $i^2 = -1$, we can recover the Conway polynomial from the HOMFLY polynomial. Similarly, by substituting $l = iA^4$ and $m = i(A^2 - A^{-2})$, we can recover the Jones polynomial from the HOMFLY polynomial. It follows that whenever two knots or links can be distinguished by either the Conway polynomial or by the Jones polynomial, then they can also be distinguished by the HOMFLY polynomial. However, it turns out that the HOMFLY polynomial captures some information that is missed by both the Conway and the Jones polynomials.

You can now follow the methods of Sections 3.1 and 6.1 to calculate $P_K(l, m)$ for all your favourite knots and links! The table below gives some examples.

knot or link K	$P_K(l, m)$
Split two component unlink $2 \cdot 0$	$-m^{-1}(l + l^{-1})$
Positive Hopf link H^+	$-ml^{-1} + m^{-1}l^{-1} + m^{-1}l^{-3}$
Negative Hopf link H^-	$-ml + m^{-1}l + m^{-1}l^3$
Positive trefoil knot T^+	$-2l^{-2} - l^{-4} + m^2l^{-2}$
Negative trefoil knot T^-	$-2l^2 - l^4 + m^2l^2$
Figure eight knot	$-l^{-2} - 1 - l^2 + m^2$

As you might guess from this table, the HOMFLY polynomials of a link K and its mirror image K^* are very simply related, namely by replacing l by l^{-1}.

The HOMFLY polynomial is not the only interesting generalisation of the Jones polynomial. By introducing a second variable into the Kauffman bracket, Kauffman discovered another two variable polynomial invariant of knots and links which also generalises it.

Despite all these new invariants, we still cannot distinguish successfully between all pairs of non-equivalent knots and links. This subject is a very active area of research at present, with connections to quantum mechanics and statistical mechanics in physics as well as to other branches of mathematics such as graph theory, functional analysis and group theory.