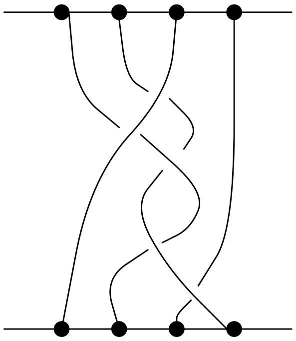
4 Braids

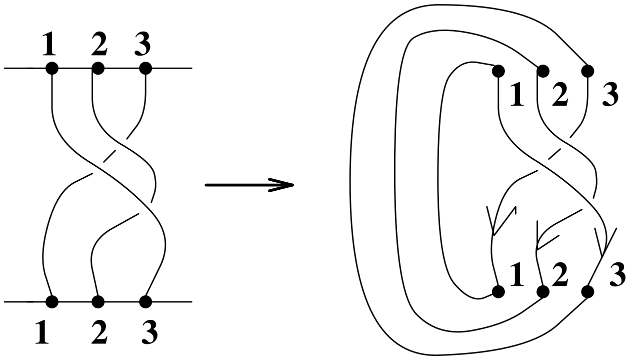
A *braid* is a set of n strings, all of which are attached to a bar at the top and the bottom. The diagram below shows a braid on four strings.



Each string always runs downward from the top bar to the bottom bar. Thus the individual strings cannot be knotted. However, every braid gives rise to a knot or link, by forming its *closure*.

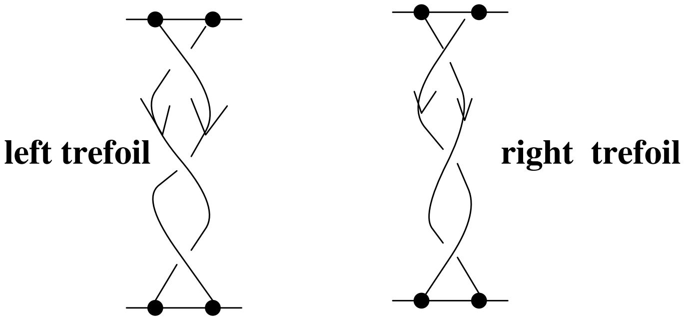
4.1 Closed Braids and Knots

To form the closure of a braid, number the ends of the strings from left to right along both bars. Thus each string has a number at the top and a number at the bottom, and these two numbers can be different. Now join the point numbered k on the top bar to the point numbered k on the bottom bar, for $1 \leq k \leq n$, by a straight string running round the back of the braid.



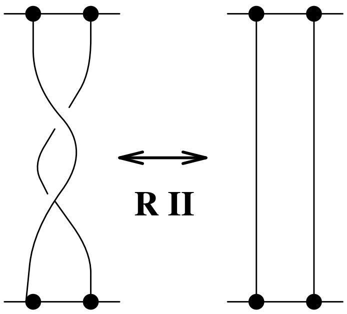
This gives a link called the closure of the braid. (You may have met the term “closure” in the context of sets in the plane or in a metric space. This is a different usage of the word, which has nothing whatever to do with that one.) The link has a natural orientation, given by taken the positive sense as running downwards on all the braid strings.

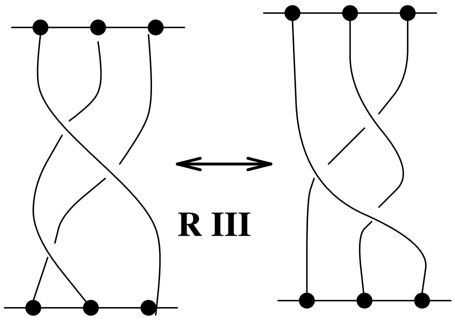
For example, the right and left trefoil knots are the closures of the braids shown below.



Before exploring the connection with knot theory, we shall spend a little time looking at braids themselves. We'll think of the top and bottom bars as parallel lines in $\mathbf{R}^3$ of the form $y = 0, z = a$ and $y = 0, z = b$ where $a > b$. The points $x = k$ for $k = 1, 2, \ldots, n$ can be taken as the points of attachment of the strings. The strings themselves are curves in $\mathbf{R}^3$ which run from one of the attachment points on the top bar to one of the attachment points on the bottom bar. The strings are not allowed to intersect each other, and each string meets every plane $z = c$ with $a > c > b$ exactly once. (This condition ensures that the strings run downwards.) As in knot theory, we have a choice as to the precise sort of curves we use, and it will be convenient to regard the strings as polygonal arcs made up of a finite number of straight line segments, even though we usually draw them as curves in practice.

Two braids are *equivalent* if one of them can be changed into the other by movements in space which always keep them in braid form. For example, we have the following "braid" analogues of the Reidemeister II and III moves.

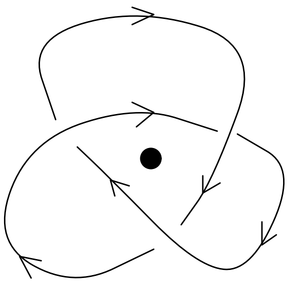




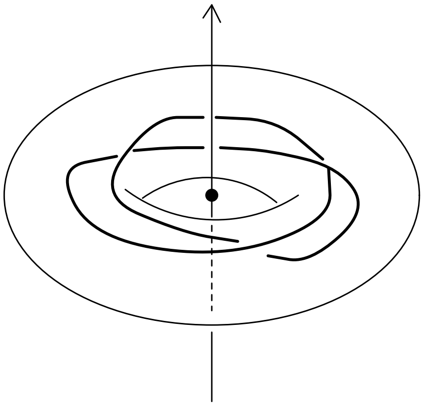
We are not allowed to pull the strings over or round the bars, or make loops in the strings, and of course the strings cannot pass through each other. Just as we informally think of an equivalence class of knots as different forms of the same knot, we think of an equivalence class of braids as different forms of the same braid. Clearly, if B and B' are equivalent braids, then their closures are equivalent links. It turns out that there is an analogue of Reidemeister's theorem for braids. This says that every braid equivalence can be obtained by a finite sequence of moves of the R II and R III types shown above, together with deformations of "braid diagrams" analogous to our planar isotopies for knots.

Suppose now that we are given a knot or link. Can it be represented as the closure of a braid? If so, then we may be able to apply the theory of braids to derive results about knots and links.

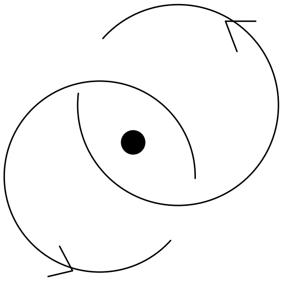
There is one important case where we can easily show that a knot is a closed braid. Suppose that there is a point O in the plane of the knot diagram, not lying on any arc of the diagram, with the property that an observer placed at the point O will see an insect travelling round the knot as always moving clockwise around O, or as always moving anticlockwise around O. We call such a point a *centre* for the knot diagram. The diagram below shows that the usual diagrams for the two trefoil knots have a centre.

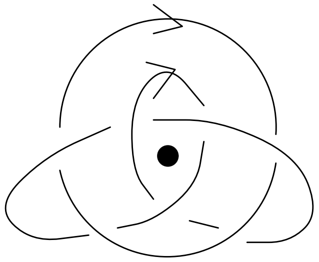


In this situation, we can take an axis perpendicular to the plane of the diagram through the centre, and we can imagine the knot winding around this axis. Thus the knot will be contained in a tube or solid torus T centred at O. Now choose a half plane in $\mathbf{R}^3$ containing the axis and cutting the plane of the diagram in a ray from O. By cutting the solid torus along the disc where it meets this half plane, and opening it out, we get a cylinder which contains a braid. The closure of this braid is clearly our original knot.



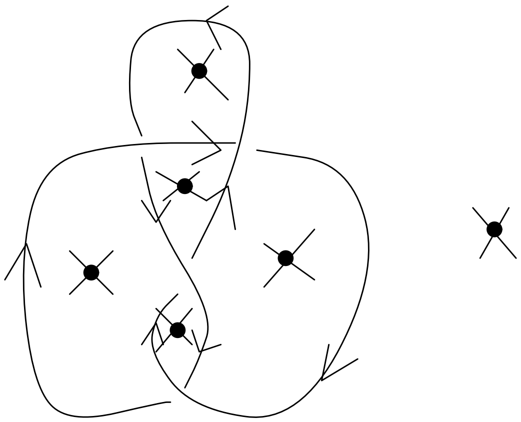
We can do the same for a link, provided that we can find a point O which is a common centre for each component of the link. Naturally, we call such a point O a *centre* for the link.





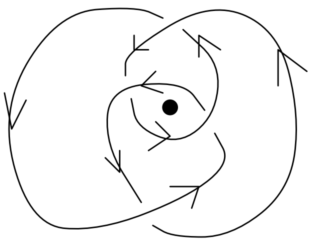
Thus we have seen that any link which has a centre may be represented as the closure of a braid, by carrying out the construction above with respect to that centre.

Of course, a centre may not exist, as you can easily check for the case of the standard diagram of the figure eight knot.



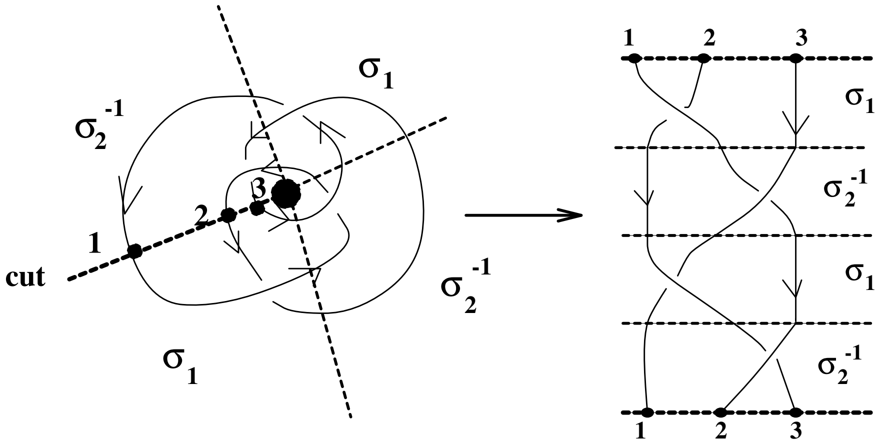
At this stage, it looks as though only certain special types of knots and links will be closed braids. However, this is not the case. It was proved in 1923 by Alexander (of polynomial fame) that *every* knot or link is the closure of a braid.

Alexander's method was to show that it is always possible to create a centre by changing a given link diagram to an equivalent diagram in a suitable way. For example, in the case of the figure eight knot we can use the alternative diagram

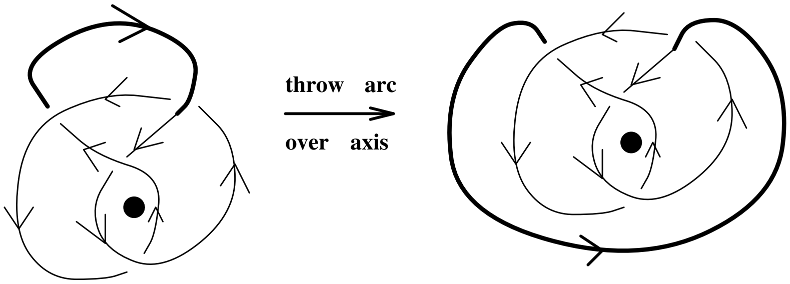


Using this diagram, we see that the figure eight knot is the closure of the braid $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1} = (\sigma_1\sigma_2^{-1})^2$.

Notice how we can read off the elementary braids by dividing the new diagram into sectors by radial lines (dotted) from the centre. These radial lines are chosen so that there is exactly one crossing point in each sector. It is not hard to see how to reorganise a diagram like this in the usual braid form, by cutting the diagram open along one of the radial lines, which then forms the top and bottom lines of the braid. Note that we can start reading the braid at any radial line, so all the cyclic permutations of a given word in the elementary braids will represent the same knot or link.



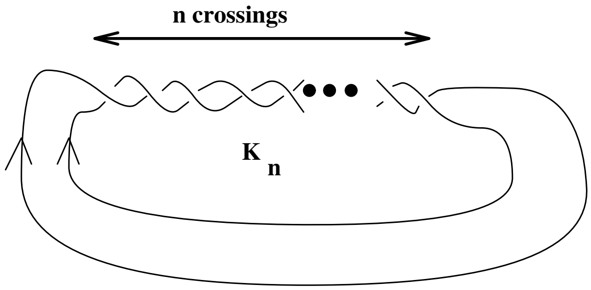
The procedure for obtaining a suitable diagram is to choose a centre in the plane of the diagram, together with a preferred direction of travel around this centre, *i.e.* clockwise or anticlockwise. We then modify the diagram by imagining an axis through the centre and perpendicular to the plane of the diagram, and by passing arcs of the diagram over this axis so as to obtain new diagrams. This can be seen in the case of the figure eight knot as shown below.



Of course it is best to choose the axis and the preferred direction around it so as to require the smallest possible disturbance of the original diagram. Notice also that this procedure does not rise to a unique closed braid representation for a given knot.

Alexander's theorem gives rise to an interesting link invariant, the *braid index* of a link K. This is the smallest number $b = b(K)$ for which K can be represented as the closure of a braid on b strings. If K is a link with n components, then clearly $b \geq n$.

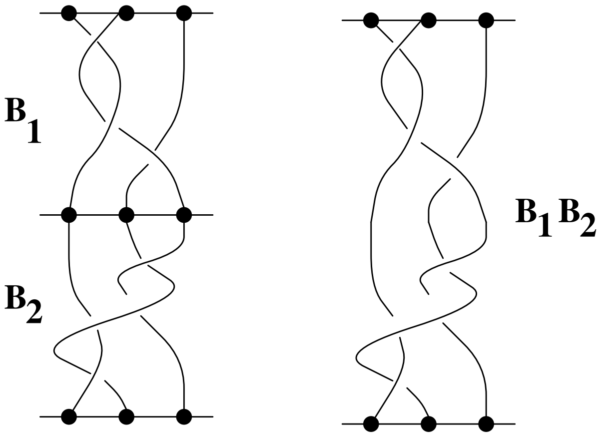
The only link of braid index 1 is obviously the unknot. The two trefoil knots and the two Hopf links have braid index 2, and in fact all the links of braid index 2 are easily described. (See Examples 3, Question 6.)



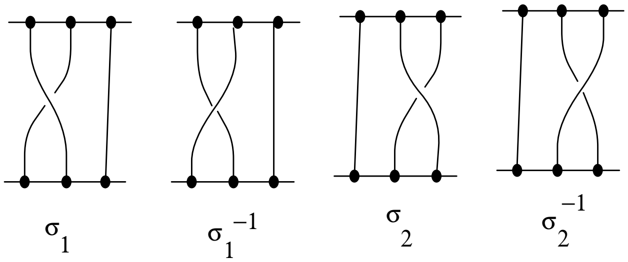
We can prove using the Conway polynomial that the figure eight knot is not on this list, so from our work above it follows that this knot has braid index 3.

4.2 Braid Groups

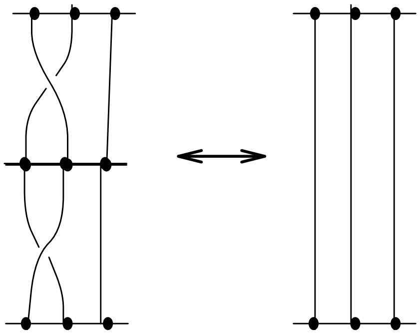
The advantage of representing a link as the closure of a braid is that there is a standard notation in which to describe braids. To describe this, we first introduce the *product* of two braids B_1 and B_2 with the same number of strings n. To form this product, we identify the points numberered $1, 2, \ldots, n$ on the bottom bar of B_1 with the correspondingly numbered points on the top bar of B_2. We then remove the middle bar, so that the product braid B_1B_2 runs from the top bar of B_1 to the bottom bar of B_2.



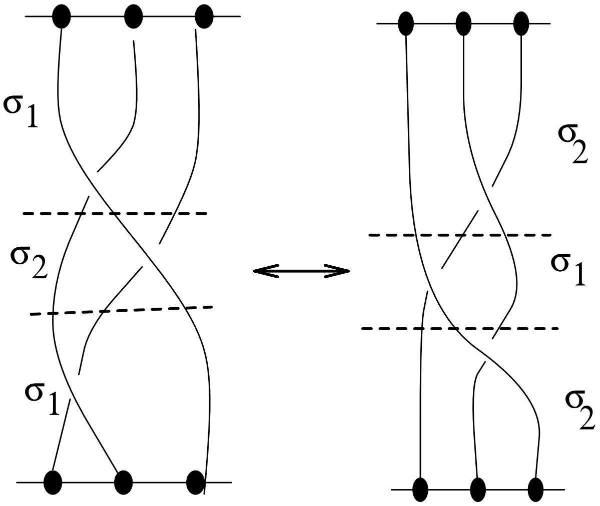
By means of this product, we can write any braid as a product of *elementary braids* and their inverses. The kth elementary braid σ_k is the braid formed by crossing the kth string over the $(k+1)$th string, as below. The inverse braid σ_k^{-1} is obtained by crossing the strings the other way.



The trivial braid (for consistency one might call it the "unbraid"!) is denoted by 1. With this notation, we can interpret the equation $\sigma_k \sigma_k^{-1} = 1$ to mean that the product of an elementary braid and its inverse is equivalent to the trivial braid. This is the algebraic counterpart of our Reidemeister II move for braids.



To see that an arbitrary braid is a product of elementary braids, simply deform the braid so that the crossings all occur on different planes $z = c_i$. We can then read off the product required directly form the braid diagram.



Thus we can use Alexander's theorem to obtain a combinatorial description of any knot or link. We first represent the knot or link as a closed braid, and then read off the braid as a "word" in the alphabet consisting of the σ_k's and their inverses. Adams suggests in his book that this is handy if you want to describe a knot to a friend over the phone!

From this point of view, our analogue of the Reidemeister III move takes the form

$$\sigma_k \sigma_{k+1} \sigma_k = \sigma_{k+1} \sigma_k \sigma_{k+1}. \qquad (1)$$

We also note that we have obvious relations

$$\sigma_k \sigma_l = \sigma_l \sigma_k, \text{ if } l \neq k+1, k-1. \qquad (2)$$

You should draw your own diagram to convince yourself that relations (2) hold.

We complete this brief discussion with a few remarks about the *braid groups* $\mathcal{B}_n$. If we fix the number of strings n, then the (equivalence classes of) n-string braids form a *group* (MT 2262) with respect to the product defined above. This is quite easy to see, though we do not go into details, since this course is supposed not to contain any group theory! What has to be checked is the associative law $(B_1B_2)B_3 = B_1(B_2B_3)$, the fact that the trivial braid 1 acts as a (two-sided) identity element, and the fact that to every braid B there is a corresponding inverse braid B^{-1} such that $BB^{-1} = B^{-1}B = 1$. Remember that all braids really mean equivalence classes of braids, so that these equations mean that when we take actual braids representing the equivalence classes, the two braids on either side of the equation are equivalent. The details are left to you to check.

From the group theory point of view, the nth braid group $\mathcal{B}_n$ is completely defined by the set of generators σ_k for $1 \leq k \leq n-1$ together with the set of defining relations (1) and (2). In the case $n = 2$, there is only one generator σ_1 and no defining relations, which means that $\mathcal{B}_2$ is an infinite cyclic group, *i.e.* it is isomorphic to the additive group $\mathbf{Z}$ of integers. However, in the case $n > 2$ the group $\mathcal{B}_n$ is a non-commutative infinite group. This is a very important family of groups in algebraic topology and in mathematical physics. The Jones polynomial for knots and links, which we shall come to in Chapter 6, has interesting connections with the representation theory of the braid groups. In fact, it was through the study of this topic that Vaughan Jones discovered his polynomial in 1983.