

3 Knot Polynomials I

3.1 The Conway Polynomial

We have seen in Chapter 2 that linking numbers are a useful means of distinguishing oriented links. However the information they contain is necessarily very limited, and of course we get no information about knots themselves this way. In this chapter we shall see how invariants of knots, and more powerful invariants of links, can be obtained by using linking numbers of links derived from the original one by switching crossings.

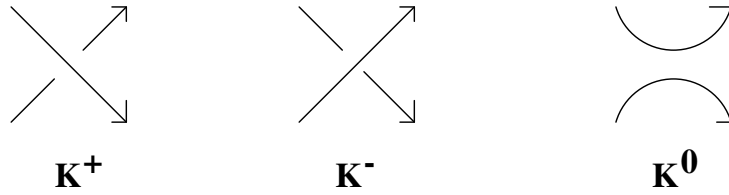
The invariant we shall discuss takes the form of a *polynomial* associated with the knot K . The variable in this polynomial is usually denoted by z , but this is not supposed to imply that we will be substituting complex numbers for z (although there would be nothing wrong with doing so). The polynomial will have the form

$$\nabla_K(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n,$$

where a_0, a_1, \dots, a_n are integers. In the usual notation from algebra, $\nabla_K(z) \in \mathbf{Z}[z]$, where this just means the ring of all such polynomials. The degree n of ∇_K and the coefficients a_0, a_1, \dots, a_n depend of course on K , and we will often write them as $a_0(K), a_1(K), \dots$ when we need to be reminded of this. Each coefficient $a_i(K)$ is in fact a separate and perfectly good invariant of K all on its own. In other words, what we are going to have is a sequence of integer valued knot or link invariants $a_0(K), a_1(K), \dots$. For a given knot or link K , there will be only a finite number of nonzero terms $a_i(K)$ in this sequence. The polynomial notation ties all of these invariants together, in a way that allows us to work with them all at once by using the standard algebra of polynomials.

Here are the three axioms for the Conway polynomial.

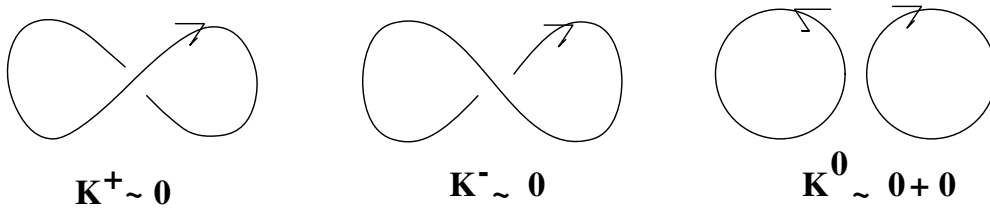
- (Axiom 1: Invariance)** To each oriented knot or link K there is associated a polynomial $\nabla_K(z) \in \mathbf{Z}[z]$, such that $\nabla_K(z) = \nabla_{K'}(z)$ if K and K' are equivalent.
- (Axiom 2: Normalisation)** If 0 denotes the standard unknot, *i.e.* a positively oriented circle, then $\nabla_0(z) = 1$.
- (Axiom 3: The exchange relation)** Let K^+, K^- and K^0 be three knots or links that have identical diagrams except in the neighbourhood of one crossing, where they differ as follows:



Then

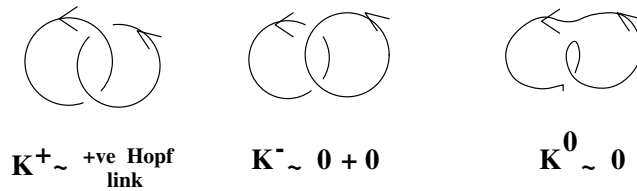
$$\nabla_{K^+}(z) - \nabla_{K^-}(z) = z\nabla_{K^0}(z).$$

We can start right away to do some calculations.



Example 3.1 Here K^+ and K^- are both unknots, so $\nabla_{K^+} = \nabla_{K^-} = \nabla_0$ by Axiom 1, which = 1 by Axiom 2. Hence $\nabla_{K^0} = 0$ by Axiom 3.

This simple example shows that the Conway polynomial distinguishes the unknot from the trivial link with two components. Of course, this is not much of an achievement! The next example calculates ∇ for a positive Hopf link.

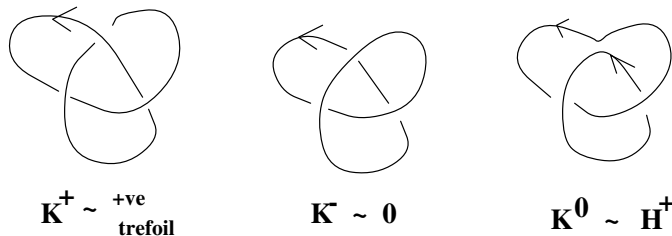


Example 3.2 Here $K^+ = H^+$, the positive Hopf link, K^0 is an unknot, and K^- is a trivial two-component link. So using the previous result we have

$$\nabla_{H^+} - 0 = z.1,$$

so $\nabla_{H^+} = z$.

With this preparation, we are ready to tackle the positive trefoil knot T^+ .



Example 3.3 Here $K^+ = T^+$, the positive trefoil knot, K^- is an unknot, and K^0 is a positive Hopf link. So using the previous result we have

$$\nabla_{T^+} - \nabla_0 = z\nabla_{H^+}$$

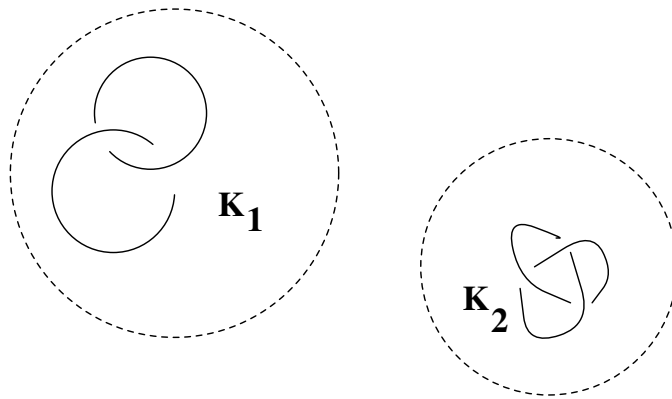
so $\nabla_{T^+} - 1 = z.z = z^2$, and $\nabla_{T^+} = 1 + z^2$.

The strategy in these calculations can be summarised as follows:

- (i) Choose a crossing c in your given knot or link K . If the crossing is positive, let $K = K^+$, and if the crossing is negative, let $K = K^-$. To be definite, let's assume we started with a positive crossing.
- (ii) Form the other diagrams, K^- and K^0 , that are needed for the exchange relation.
- (iii) Notice that K^0 has one crossing fewer than K^+ and K^- . Hence we should be able to calculate ∇_{K^0} first. In other words, we will argue by induction on the number of crossings.
- (iv) Since K^- has the same number of crossings as $K = K^+$, the induction hypothesis won't apply to K^- . So we have to rely on K^- being a "simpler" knot than K in some sense. Ideally, we would like to choose the crossing c so that K^- is a trivial link, but this is not always possible.
- (v) Assuming that such a good choice of crossing c can be made, ∇_{K^-} can be taken as known, and ∇_K calculated from the exchange relation.

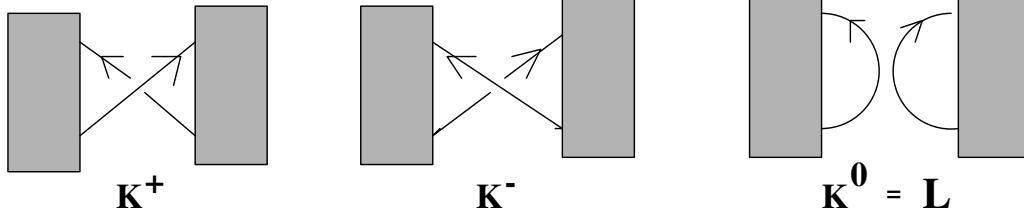
Later we'll see how to fix the apparent difficulty in (iv), so that the method will always work. But this should be enough to get you started with some practical examples.

At this point, it's convenient to introduce the notation $K_1 \cup K_2$ for a **split link** which is made up of the two separate links K_1 and K_2 . This means that K_1 and K_2 can be enclosed in separate, non-intersecting spheres in \mathbf{R}^3 .



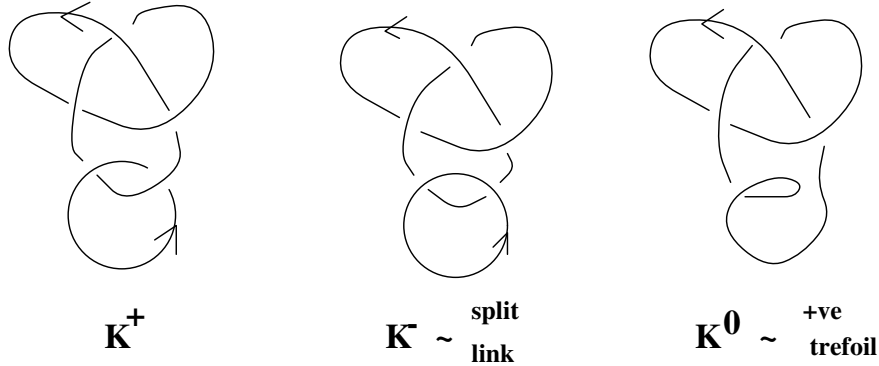
Proposition 3.4 $\nabla_L(z) = 0$ for any split link L .

Proof This is an easy generalisation of Example 3.1.



We take $K^0 = L$, the given split link, with its strands arranged as shown above. We can then form the associated links K^+ and K^- as shown. Now K^+ and K^- are equivalent by twisting the right hand part of the diagram through 2π . Hence $\nabla_{K^+} = \nabla_{K^-}$ by Axiom 1. By Axiom 3, $z\nabla_L = \nabla_{K^+} - \nabla_{K^-} = 0$. \square

Here is an example which utilises this proposition.



Example 3.5 Here we chose the crossing c so that K^- is a split link. Since K^0 is a positive trefoil knot, we have $\nabla_K - 0 = z\nabla_{T^+}$. Since $\nabla_{T^+} = 1 + z^2$, we conclude that $\nabla_K = z + z^3$.

A link diagram with no crossings must of course be a trivial link. Hence, as a particular case, Proposition 3.4 takes care of the base case of the induction referred to in step (iii) of the method outlined above. We are now ready to formalise this method, so as to prove that ∇_K is uniquely determined by the three axioms, for any knot or link K .

Proposition 3.6 Let ∇ and ∇' be two functions from the set of all knot or link diagrams to the polynomial ring $\mathbf{Z}[z]$ which satisfy Axioms 1,2 and 3. Then $\nabla = \nabla'$.

Proof Recall from Proposition 1.1 that every diagram for a knot can be transformed into a diagram for the unknot by switching a suitably chosen set of crossings. This argument can easily be extended to show that every link diagram

can be changed into the diagram of a trivial link (with the same number of components) in the same way. The unknotting number $u(K)$ is the minimum number of switches required to achieve this, if *all* diagrams for links equivalent to K are considered, but here we just consider one given diagram.

Let's call the minimum number of switches required to untie this particular diagram the *untying number* of the diagram — of course, this is not a knot invariant. Now assume as induction hypothesis that $\nabla_{K'} = \nabla'_{K'}$ for every diagram K' such that either

1. K' has fewer crossings than K , or
2. K' has the same number of crossings as K , but K' has a smaller untying number than K .

Given K , there is some set of $u(K)$ crossings that can be switched so as to unknot K . Let c be one of these crossings. If c is positive, take $K^+ = K$, and if c is negative, take $K^- = K$. In either case, take K' to be the result of switching crossing c in K . Then the untying number of K' is 1 less than the untying number of K , while K^0 has fewer crossings than K . Thus $\nabla_{K'} = \nabla'_{K'}$ and $\nabla_{K^0} = \nabla'_{K^0}$, by the induction hypothesis. Now it follows from Axiom 3 that $\nabla_K = \nabla'_K$. \square

If you think about this proof, you will see that the exact form of the exchange relation is irrelevant. In other words, the uniqueness argument would work if Axiom 3 were replaced by some other relation between ∇_{K^+} , ∇_{K^-} and ∇_{K^0} . So what is special about the particular relation $\nabla_{K^+} - \nabla_{K^-} = z\nabla_{K^0}$?

The point is that we have yet to prove that the three axioms are *consistent*. If we replace the exchange relation by a different relation chosen at random, then it is highly unlikely that we will have a knot invariant. That is to say, if we use the method illustrated by the examples above with a diagram for an equivalent knot, or another diagram for the same knot, or even use the same knot and the same diagram but make a different choice of the crossings to be switched, then in all probability we will get a different polynomial at the end of our calculation. The amazing thing about the particular relation given in Axiom 3 is that it does give results which are independent of all these possible choices. In other words, Axioms 2 and 3 are consistent with Axiom 1. Were it not for this fact, there would be no point whatever in doing calculations such as we have done in this section. Most of our work in Chapter 3 will be directed towards proving consistency of the three axioms, and hence establishing the theory of the Conway polynomial on a sound footing. Meanwhile, you should convince yourself of the truth of the consistency result by picking a knot or link K and verifying that you get the same value for $\nabla_K(z)$ using

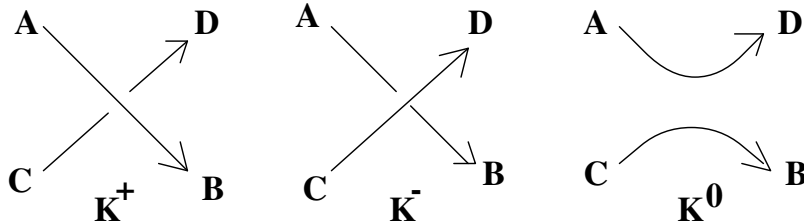
- two different choices of crossing switches in the same diagram,
- two different diagrams for the same knot or link K .

It's a good idea to make string models of a few of your favourite knots and links. With these, it is easy to generate different looking diagrams to practise on. This exercise will also provide you with practice in drawing the knot diagrams themselves from the models. In this way you will rapidly gain in your understanding of the relationship between the knot or link K and its diagrams, as well as skill in calculating the Conway polynomial.

The following observation will be useful in the next section.

Proposition 3.7 *Let K^+ , K^- and K^0 be three knot or link diagrams related as in Axiom 3 by changing the crossing c of K^+ . If K^+ has n components, then K^- also has n components, and K^0 has either $n + 1$ or $n - 1$ components, according to whether the two arcs of K^+ which cross at c belong to the same component or not.*

Proof If the two strands of K^+ which cross at c belong to the same component of K^+ , then clearly the same is true for K^- . However, in this case, the two strands of K^0 which lie close to c belong to different components of K^0 . To see this, label the arrows in the K^+ diagram as AB and CD , as below. Then the points A, B, C, D occur in cyclic order $ABCD$ as we traverse this component of K^+ or K^- . However, in K^0 we pass from A to D and back to A again without reaching B or C .



Conversely, if the two strands of K^+ which cross at c belong to different components of K^+ , then the same is true for K^- , but the two strands of K^0 which lie close to c belong to the same component of K^0 . This can be seen easily by colouring one of the components of K^+ red and the other green: we have to join red to green in forming K^0 .

3.2 The coefficients a_0 , a_1 and a_2

In this section we shall study the geometric meaning of the coefficients a_0 , a_1 and a_2 in the Conway polynomial $\nabla_K(z)$.

Proposition 3.8 $a_0(K) = 1$ if K is a knot, and $a_0(L) = 0$ if L is a link with more than one component.

Proof The right hand side $z\nabla_{K^0}$ of the exchange relation is a polynomial with z as a factor, and so it has zero constant term. Hence $a_0(K^+) = a_0(K^-)$,

i.e. the invariant a_0 is not changed when a crossing is switched. Since we can always obtain a trivial link by switching crossings, the result now follows from Axiom 2 and Proposition 3.4. \square

Of course, this only shows that a_0 is not a very exciting invariant, but at least it is easy to understand. Let's move on to a_1 .

Proposition 3.9 $a_1(L) = \text{Lk}(L)$ if L is a link with two components, and $a_1(L) = 0$ if L is a knot or a link with more than two components.

Proof This time we equate the coefficients of z in the exchange relation for a crossing point c . This gives

$$a_1(K^+) - a_1(K^-) = a_0(K^0).$$

By Proposition 3.8, we deduce

$$a_1(K^+) - a_1(K^-) = \begin{cases} 1, & \text{if } K^0 \text{ is a knot,} \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

Now let's do the case of a knot. This goes rather like the proof of Proposition 3.8. By Proposition 3.7, if K^+ is a knot, then so is K^- , but K^0 is a 2-component link, and hence $a_1(K^+) = a_1(K^-)$ by (1). So for a knot, a_1 is unchanged by switching, and hence has the same value as it has for an unknot, namely 0.

Now suppose that K^+ is a 2-component link, and that the two components cross at c . By Proposition 3.7, K^0 is a knot. Hence $a_1(K^+) - a_1(K^-) = 1$, by (1). Now since the crossing c contributes $+1/2$ to $\text{Lk}(K^+)$ and c contributes $-1/2$ to $\text{Lk}(K^-)$, we also know that $\text{Lk}(K^+) - \text{Lk}(K^-) = 1$. Thus the result we want is true for K^+ if and only if it is true for K^- . The truth of the result for 2-component links is therefore not affected by switching crossings where the two components cross. However, we can choose these switches so as to change any given 2-component link L to a split 2-component link S . Since $a_1(S) = 0$ by Proposition 3.4, and obviously $\text{Lk}(S) = 0$, the result is true for S , and hence also for L .

The case of a link with more than two components is left for you as an exercise (Examples 3). \square

The next coefficient a_2 in the Conway polynomial $\nabla_K(z)$ is a more subtle invariant, and we shall devote quite a lot of attention to it. We shall see that it can also be interpreted in terms of linking numbers. However, these linking numbers are not directly obtained from K itself, but rather from links derived from K by appropriate changes in a given diagram for K .

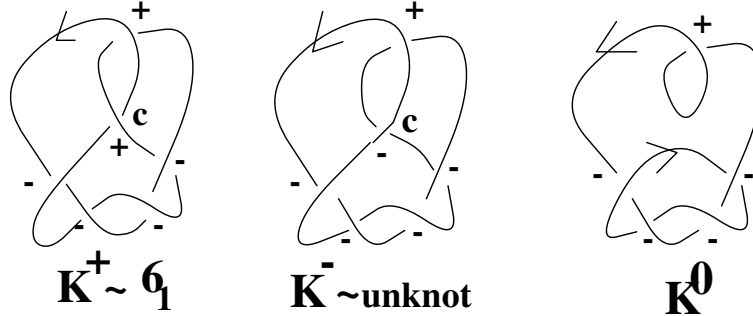
Let's begin by equating the coefficients of z^2 in the exchange relation for a crossing point c . This gives

$$a_2(K^+) - a_2(K^-) = a_1(K^0).$$

By Proposition 3.9, we deduce

$$a_2(K^+) - a_2(K^-) = \begin{cases} \text{Lk}(K^0), & \text{if } K^0 \text{ is a 2-component link,} \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

Example 3.10 Let $K = K^+$ be the knot 6_1 . Then switching the crossing marked c in the diagram below will make K^- an unknot, while K^0 is a 2-component link with linking number -2 .

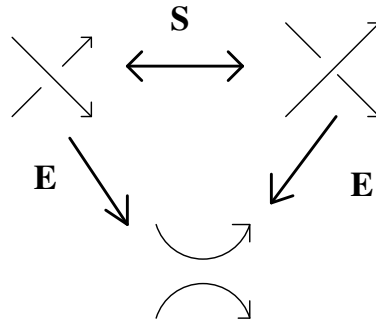


Hence by (2) $a_2(6_1) = -2$.

In more complicated examples, we may need to apply the exchange relation in the form (2) several times in order to evaluate a_2 for the given knot or link K , by “switching” and “splicing” a sequence of crossings c_1, c_2, \dots, c_n in the given diagram for K .

We shall use the following notation. If K is a knot or link diagram where certain crossings have been numbered as c_1, c_2, \dots, c_n , then $S_i K$ will denote the diagram obtained by switching c_i in K , and $E_i K$ will denote the diagram obtained by eliminating c_i in K by a splice.

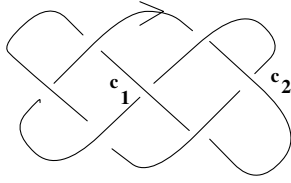
We also denote by $\epsilon_i(K)$ (or simply by ϵ_i if it is clear what K is) the sign of crossing c_i in K .



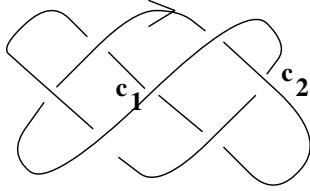
Thus for any knot or link K with indexed crossings, Axiom 3 becomes

$$a_{n+1}(K) - a_{n+1}(S_i K) = \epsilon_i a_n(E_i K).$$

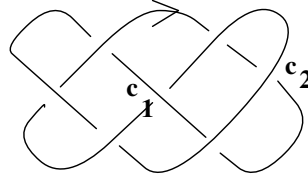
Here are some examples to show how these definitions work, taking K as the 7_4 knot in your table. Note that all crossings in the diagram below are positive.



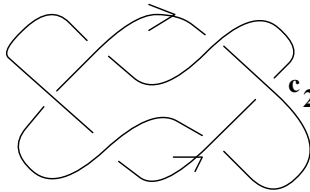
$K=7_4$



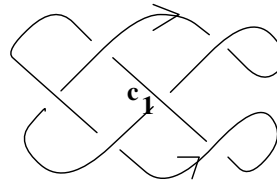
$S_1 K$



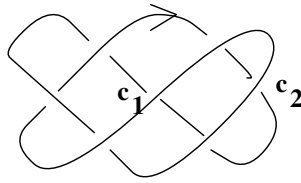
$S_2 K$



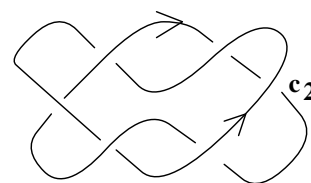
$E_1 K$



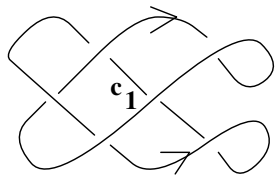
$E_2 K$



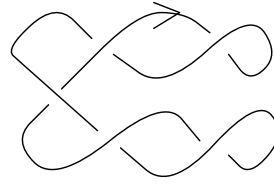
$S_2 S_1 K = S_1 S_2 K$



$E_1 S_2 K = S_2 E_1 K$



$E_2 S_1 K = S_1 E_2 K$



$E_1 E_2 K = E_2 E_1 K$

We can work out $a_2(K)$ for $K =$ the 7_4 knot as follows:

$$\begin{aligned} a_2(K) - a_2(S_1 K) &= \text{Lk}(E_1 K) \\ a_2(S_1 K) - a_2(S_2 S_1 K) &= \text{Lk}(E_2 S_1 K) \end{aligned}$$

Since $S_2 S_1 K$ is an unknot, adding these equations gives

$$a_2(K) = \text{Lk}(E_1 K) + \text{Lk}(E_2 S_1 K) = 3 + 1 = 4.$$

In general, when K is a knot, $a_2(K)$ can be expressed as a sum of signed linking numbers as in the example above. In this example, both c_1 and c_2 are positive crossings. In the general case, the sign of the crossing c_i will multiply $a_1(E_i S_{i-1} \dots S_1 K)$.

If K is a knot, then $S_{i-1} \dots S_1 K$ is also a knot, and as above, $E_i S_{i-1} \dots S_1 K$ is a 2-component link. Thus a_1 of this link will just be its linking number.

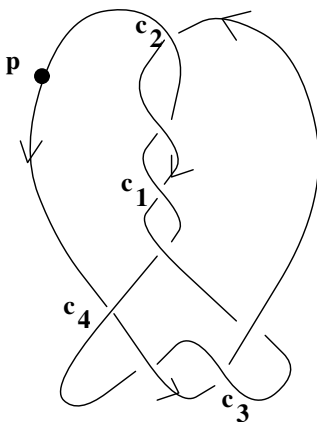
3.3 Standard unknotting sequences

In this section we use the method of Section 3.2 to *define* an invariant $\alpha(K)$ for a knot K . Eventually, we shall be able to identify the invariant $\alpha(K)$ as the coefficient $a_2(K)$ in the Conway polynomial of K . In this rather indirect way, we shall then have proved that $a_2(K)$ is indeed a knot invariant. To keep things simple, K will *always* denote a knot in Sections 3.3–3.6, but everything we do can be generalised to links.

We know from Proposition 1 of Chapter 1 that every knot can be transformed into the unknot by switching a suitable selection of crossings. In order to define $\alpha(K)$, we shall restrict this selection of crossings to be what we shall call a *standard unknotting sequence*.

To define a standard unknotting sequence, we use the method given in the proof of Proposition 1. That is, we first choose a base point p on K which is *not* a crossing point. Starting from p and following the given orientation, we traverse the knot K . Every crossing is met twice in this journey, once as an overpass and once as an underpass. We shall switch precisely those crossings which are met as an *underpass* the first time they are met on the journey. We denote by K^p the unknot obtained by switching this set of crossings.

Now label the crossings which were switched as c_1, c_2, \dots, c_n , starting from the *last* one to be switched and working back to the first one. This gives a switching sequence S_1, S_2, \dots, S_n such that $S_n S_{n-1} \dots S_1 K = K^p$ is unknotted. Note that in traversing K from p , the switched crossings are met in the order c_n, c_{n-1}, \dots, c_1 . The example below illustrates this for the knot 8_3 .



This sequence c_1, c_2, \dots, c_n will be called the *standard unknotting sequence* for the knot K with base point p . It is important to notice that the sequence depends on the choice of p .

Definition 3.11 *Let K be an oriented knot, and let c_1, \dots, c_n be a standard unknotting sequence for K . Let $\epsilon_i = \epsilon_i(K)$ and let $X_i = E_i S_{i-1} \dots S_1 K$ for $1 \leq i \leq n$. Define*

$$\alpha(K) = \sum_{i=1}^n \epsilon_i \text{Lk}(X_i).$$

We collect the properties of α that we aim to prove in a theorem. The proof of this will occupy most of Sections 3.4–3.7.

Theorem 3.12 *Given an oriented knot K with base point p , $\alpha(K)$ has the following properties.*

1. $\alpha(K)$ is independent of the order of the crossings c_1, c_2, \dots, c_n .
2. $\alpha(K)$ is independent of the choice of the base point p .
3. $\alpha(K)$ is unchanged by all three Reidemeister moves, and hence it is an invariant of the knot K .
4. $\alpha(K) = a_2(K)$.

For the first step, we can reduce our task by applying a little algebra. Recall that any permutation of n objects c_1, \dots, c_n can be obtained by a suitable sequence of transpositions. (A transposition consists of exchanging one of the objects with either of its neighbours, *i.e.* c_1 with c_2 , c_2 with c_3 , and so on.) So let's look at the effect on the formula for $\alpha(K)$ of exchanging c_k with c_{k+1} . First of all, this clearly does not alter X_i for $i < k$. Since $S_{k+1} S_k = S_k S_{k+1}$, it does not alter X_i for $i > k + 1$ either. So we have to prove that the two remaining terms agree, *i.e.*

$$\epsilon_k \text{Lk}(E_k K') + \epsilon_{k+1} \text{Lk}(E_{k+1} S_k K') = \epsilon_{k+1} \text{Lk}(E_{k+1} K') + \epsilon_k \text{Lk}(E_k S_{k+1} K'),$$

where $K' = S_{k-1} \dots S_1 K$ is the result of switching the first (unaltered) part of the sequence.

If we write $j = k + 1$, this equation appears more symmetrically as

$$\epsilon_k \text{Lk}(E_k K') + \epsilon_j \text{Lk}(E_j S_k K') = \epsilon_j \text{Lk}(E_j K') + \epsilon_k \text{Lk}(E_k S_j K').$$

Rearranging the terms, and noting that $E_k S_j = S_j E_k$, $E_j S_k = S_k E_j$ we have

$$\epsilon_k \{\text{Lk}(E_k K') - \text{Lk}(S_j E_k K')\} = \epsilon_j \{\text{Lk}(E_j K') - \text{Lk}(S_k E_j K')\}.$$

Now all four links here are 2-component links, so the linking numbers can be identified with the a_1 coefficients in their Conway polynomials, by Proposition 3.9. Using the exchange relation (Axiom 3) on the left hand side, we have

$$\epsilon_k \{a_1(E_k K') - a_1(S_j E_k K')\} = \epsilon_k \epsilon_j a_0(E_j E_k K').$$

Similarly, the right hand side evaluates to $\epsilon_j \epsilon_k a_0(E_k E_j K')$. Since $E_j E_k = E_k E_j$, the equation is true. \square

We have now achieved our first aim, in showing that we may arrange the crossing switches in any order when we calculate $\alpha(K)$ for a knot K with a given base point p .

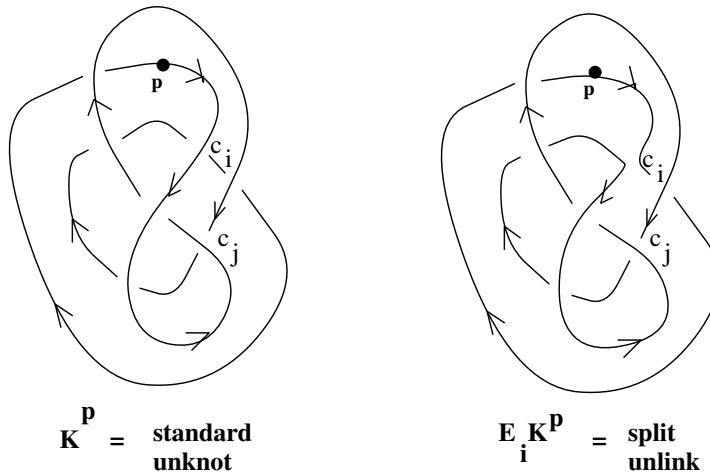
3.4 Showing that $\alpha(K)$ is independent of the base point

In this section we shall prove that $\alpha(K)$, as defined for a knot K by Definition 3.11, is independent of the choice of the base point p on K . We shall write $\alpha(K; p)$ for the sum in Definition 3.11, to indicate the dependence of this expression on p .

To do this, we need to extend the argument of Proposition 1, Chapter 1 to see what happens to a “standard” unknot when one of the crossings nearest the base point is spliced.

Proposition 3.13 *Let K^p be a standard unknot obtained by switching crossings in a knot diagram K with base point p . Let c be the first crossing after p for the given orientation, and let c' be the last crossing before p . Then the links L and L' obtained from the unknot K^p by splicing at c and c' respectively are trivial links.*

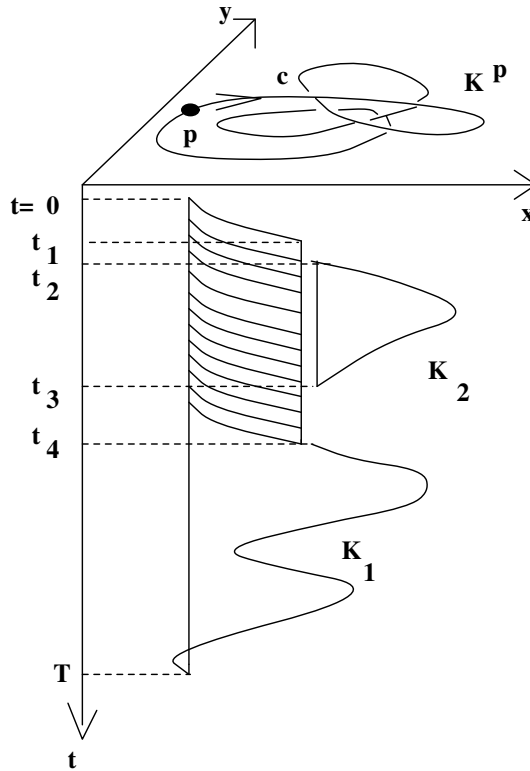
The diagram below gives an example of this situation.



Proof As in the proof of Proposition 1, we can define a function on the diagram of K^p which increases steadily as we traverse K^p from p . The value of

this function at a point x can be thought of as the time at which an imaginary insect walking along the knot reaches x . If $x = c$, a crossing point, then the insect passes this point twice, first on the overpass and later on the underpass. Thus we obtain an unknot which projects to the diagram K^p by using this function to lift the curve of the diagram into the third dimension, and closing the resulting path by a line segment perpendicular to the plane of the diagram.

Let us consider the effect of eliminating the very first crossing c after p in K^p by a splice. It will be convenient to have the insect pause for thought each time it passes c , say from $t = t_1$ to $t = t_2$ when it is on the overpass and from $t = t_3$ to $t = t_4$ when it is on the underpass. Now set our insect on its travels round the link which results from the splice at c . On the component containing the base point p , it goes from its position just before t_1 directly to its position just after t_4 . Its journey on the other component corresponds to its path from just after t_2 to just before t_3 .



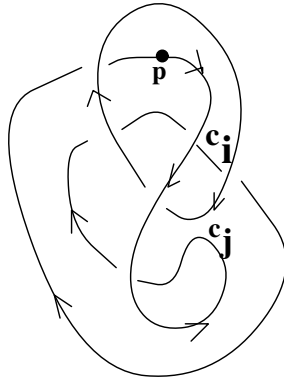
Now think of the corresponding curves in \mathbf{R}^3 , closed up by straight line segments parallel to the t -axis. These form the components K_1 and K_2 of a link. Let's name them so that the projection of K_1 contains p . Then we can move K_1 by replacing the part of K_1 consisting of the line segment from $t = T$ to $t = 0$ over p , the path from $t = 0$ to $t = t_1$, and the line segment from $t = t_1$ to $t = t_4$ by a shorter route, made up of a line segment from $t = T$ to $t = t_4 - t_1$ and a parallel path from there to $t = t_4$. We can do this because c is the first crossing after p .

This condition ensures that the “rectangle” over which we have moved K_1 does not contain any points of K_2 . (See the diagram on the previous page.)

If we arrange things so that $t_3 < t_4 - t_1$, then the two components of the link then have non-overlapping t -coordinates after moving K_1 . Hence we have separated the two components of the link, which must therefore be a trivial link.

Finally, the case where the spliced crossing comes immediately before the basepoint is done by reversing the orientation and using the same argument. \square

It is important to notice that this argument only applies to these two particular crossings in the unknot diagram. Splicing K^p at a different crossing will give a 2-component link with unknotted components, but the two components may be non-trivially linked, as the example below shows.



$E_j K^p = \text{-ve Hopf link}$

We are now ready to begin looking at the effect of changing the base point. Clearly, moving p along a section of K containing no crossings does not alter any of the terms in the sum, so we really have to consider the effect of sliding p past a crossing c . There are two cases, depending on whether p is on the overpass or on the underpass at c . The second case turns out to be a little easier, so we do that one first.

Case 1: p on the underpass at c



Here K and K^p differ at c , *i.e.* c occurs in the switching sequence for K when p is chosen as base point. If we slide the base point past c to q , then we have



and we see that c does not occur in the switching sequence for K when q is chosen as base point. Thus, if c_1, \dots, c_{n-1}, c_n is the standard unknotting sequence for K with base point p , $c = c_n$, and c_1, \dots, c_{n-1} is the standard unknotting sequence for K with base point q . Hence from Definition 3.11 we have

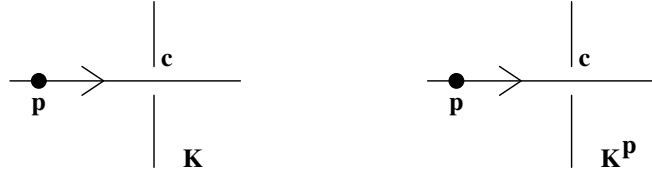
$$\alpha(K; p) - \alpha(K; q) = \epsilon \text{Lk}(E_n S_{n-1} \dots S_1 K),$$

where $\epsilon = \epsilon_n K$ is the sign of the crossing c .

Now $S_n S_{n-1} \dots S_1 K = K^p$, a standard unknot, so $E_n S_{n-1} \dots S_1 K = E_n K^p$. By Prop. 3.13, $E_n S_{n-1} \dots S_1 K$ is a trivial link. Hence $\text{Lk}(E_n S_{n-1} \dots S_1 K) = 0$. Thus $\alpha(K; p) = \alpha(K; q)$, completing the proof of Case 1.

Case 2: p on the overpass at c

Here K and K^p are the same at c , i.e. c does not occur in the switching sequence for K when p is chosen as base point. If we slide the base point past c to q , then we have



and we see that c does occur in the switching sequence for K when q is chosen as base point.



If the switching sequence from K to K^q is $c_1, \dots, c_i, \dots, c_n$, where $c_i = c$, then the switching sequence from K to K^p is $c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n$.

We can simplify this situation by changing the order of the crossings in the switching sequences, so that c is the *last* crossing to be switched. Our work in Section 3.4 tells us that (for a given base point) $\alpha(K)$ does not depend on the order of the switching sequence. Thus, for example, we could take the switching sequence as $S_{i+1}, \dots, S_n, S_1, \dots, S_i$ for the base point q , and as $S_{i+1}, \dots, S_n, S_1, \dots, S_{i-1}$ for the base point p .

We are now in a similar situation to Case 1, *i.e.*

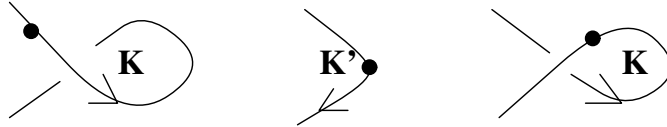
$$\alpha(K; q) - \alpha(K; p) = \epsilon \text{Lk}(E_i S_{i-1} \dots S_1 S_n \dots S_{i+1} K).$$

Note that $S_i S_{i-1} \dots S_1 S_n \dots S_{i+1} K = K^q$, a standard unknot. Hence by Proposition 3.13, $E_i S_{i-1} \dots S_1 S_n \dots S_{i+1} K$ is a trivial link. (This time, c is the last crossing *before* the base point q in K^q .) Thus $\text{Lk}(E_i S_{i-1} \dots S_1 S_n \dots S_{i+1} K) = 0$, so that $\alpha(K; q) = \alpha(K; p)$. This completes the proof in Case 2. \square

3.5 Invariance of $\alpha(K)$ under Reidemeister moves

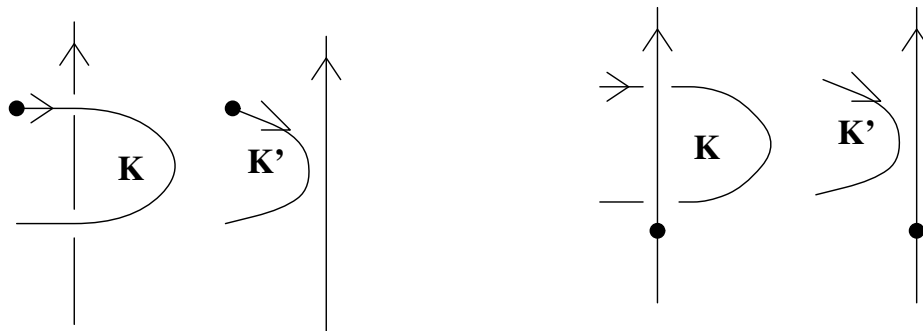
We now know that $\alpha(K)$ is independent of the base point. We shall put this information to good use, by choosing the base point to our best advantage in checking the behaviour of α under each of the three Reidemeister moves in turn. Ideally we should like to arrange matters so that none of the crossings involved in the Reidemeister move are part of the standard unknotting sequence for this choice of p .

RI move Position the base point p just before the overpass. The new crossing is not part of the switching sequence for K .



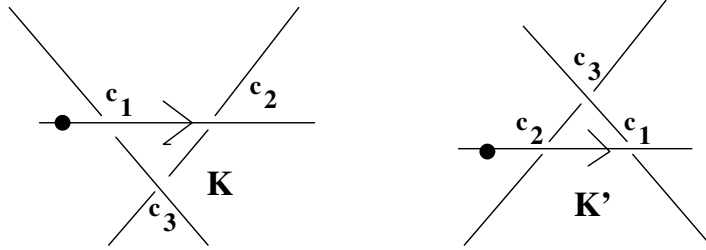
Let K' be the knot diagram obtained by untwisting the loop. Then every term in the definition of $\alpha(K)$ matches up with a corresponding term in the definition of $\alpha(K')$, since the same crossings are involved in the standard unknotting sequences for K and K' . Moreover, corresponding terms $\epsilon_i \text{Lk}(X_i)$ in these definitions are equal, because the diagrams to which they refer are related in every case by a corresponding RI move, and we already know that linking numbers are invariant under RI moves.

RII move The same argument works for RII moves. Again, we position the base point p just before the overpass. The two new crossings then do not form part of the switching sequence for K .



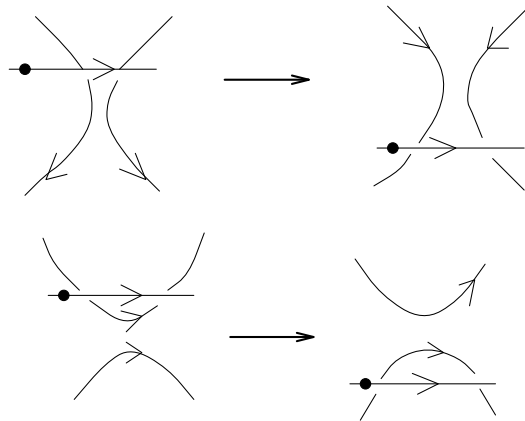
The rest of the argument follows the same pattern as for the RI move.

RIII move The same idea almost works, but not quite. By positioning the base point p as shown, we can ensure that neither c_1 nor c_2 occurs in the standard unknotting sequence for K .



If c_3 does not occur in this sequence either, then all is well and the argument is completed as in the other two cases. However, we must consider the case where c_3 is involved in the standard unknotting sequence. (Clearly, this will be so for K if and only if it is also so for K' .) The terms in the definition of $\alpha(K)$ still match up with the terms in the definition of $\alpha(K')$. However, in a given term $\epsilon_i \text{Lk}(X_i)$, the crossing c_3 may be switched or it may be spliced. Again, whatever happens to this crossing in a term of $\alpha(K)$ must also happen to it in the corresponding term $\epsilon_i \text{Lk}(X'_i)$ of $\alpha(K')$.

Now if c_3 is switched, we can still do the RIII move to change X_i into X'_i . Hence the terms are equal, by the usual argument that linking numbers are invariant under RIII moves. And if c_3 is spliced, we are looking at a simple equivalence, or else two RII moves, as shown below.



In each case, we see that $\text{Lk}(X_i) = \text{Lk}(X'_i)$. This completes the proof of invariance of $\alpha(K)$.

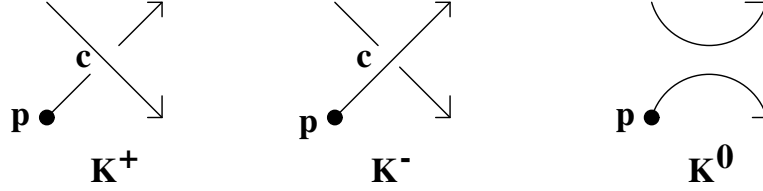
3.6 Consistency of the axioms for $\nabla_L(z)$

We are finally ready to prove the last property of $\alpha(K)$, the fact that it can be identified with the coefficient $a_2(K)$ in the Conway polynomial.

Proposition 3.14 *Given K^+ , K^- and K^0 as in Axiom 3, where K^+ and K^- are knots, then*

$$\alpha(K^+) - \alpha(K^-) = \text{Lk}(K^0).$$

Proof Position the base point p on the underpass of K^+ just before the crossing c .



Then, as in Case 1 of Section 3.5, c is the last crossing, c_n say, to be switched in the standard unknotting sequence for K^+ , and the standard unknotting sequence for K^- is obtained from this by omitting the switch at c . Now reverse the unknotting sequences, so that c is the *first* crossing to be switched for K^+ , and the rest of the sequence is the same as that for K^- . This is justified by (1) of Theorem 3.12. We now have

$$\begin{aligned} \alpha(K^+) &= \text{Lk}(E_n K^+) + \sum_{i=1}^{n-1} \epsilon_i \text{Lk}(E_i S_{i+1} \dots S_{n-1} S_n K^+), \\ \alpha(K^-) &= \sum_{i=1}^{n-1} \epsilon_i \text{Lk}(E_i S_{i+1} \dots S_{n-1} K^-). \end{aligned}$$

Since $K^0 = E_n K^+$, the first term on the right of the first equation is $\text{Lk}(K^0)$, and since $K^- = S_n K^+$, the two summations on the right hand sides are identical. \square

We are at last ready to complete the proof of Theorem 3.12 for knots.

Proposition 3.15 $\alpha(K) = a_2(K)$ for all knots K , and hence $a_2(K)$ is a knot invariant.

Proof By Proposition 3.9 we can write $a_1(K^0)$ for $\text{Lk}(K^0)$ when $K (= K^+$ or $K^-)$ is a knot. Now the inductive argument of Proposition 3.6 completes the proof. \square

To set up the Conway polynomial in full generality, we need to work with links and not only with knots. The following remarks give a sketch of how this can be

done. The details are left to you (a possible project!). The essential geometric ideas are already contained in the cases we have discussed in detail above.

The next step is to generalise the invariant α and the results of Sections 3.4–3.6 to arbitrary links. To do this, you need to define standard unknotting sequences for links. We now need a separate base point for each component, and we have to show that α does not depend on the choice of these base points, or on the order in which the components are taken. Also, in place of Lk for two component links, we have to use a_1 in the definition of α . This is consistent, by Proposition 3.9.

At this stage, we have established a_2 as a link invariant for all links. Now we have to repeat all this for the coefficients a_3, a_4, \dots . This may seem to be a daunting task! After all, a_2 has clearly been a lot harder than a_1 .

However, this is not the case. By following through the method of Sections 3.4–3.6, all the coefficients a_n can be established as invariants by a single recursive argument. That is, we define standard unknotting sequences just as before, but now assume that a_0, a_1, \dots, a_{n-1} have been proved invariant. Then we define $\alpha(K)$ by the formula of Definition 3.11, but with a_{n-1} in place of Lk. (Remember that a_1 generalises Lk.) All the arguments go through as before, making use of the recursive property

$$a_n(K) - a_n(S_i K) = \epsilon_i K a_{n-1}(E_i K).$$

From this point on, we shall assume that this programme goes through, so that we are ready to use the Conway polynomial as an important and powerful tool to distinguish knots and links.

3.7 Application to chirality of links

We conclude this chapter by giving one application of the Conway polynomial. It turns out that this invariant can not distinguish between a *knot* and its mirror image (see Section 1.1), but can distinguish certain two component *links*, for example, the Whitehead link (see Section 2.4) from their mirror images.

We begin with a general observation about Conway polynomials.

Proposition 3.16 *For any link L with n components,*

$$\nabla_L(-z) = (-1)^{n+1} \nabla_L(z).$$

Put differently, this says that the Conway polynomial of a knot, or of a link with an *odd* number of components, is an *even* polynomial, while the Conway polynomial of a link with an *even* number of components is an *odd* polynomial. For example, either trefoil knot has Conway polynomial $1 + z^2$, while the right and left Hopf links have Conway polynomials z and $-z$ respectively. (Recall that a polynomial is odd if it involves only odd powers of z , and is even if it involves

only even powers of z . The zero polynomial is both odd and even.) Thus the Conway polynomial is successful in distinguishing the two mirror-image Hopf links, but does not succeed in distinguishing the two trefoil knots.

Proof Carry out induction using the exchange relation, following the method of Proposition 3.6. To start the induction, we check the statement for links with no crossings. We may assume that $L = L^+$, the case $L = L^-$ being similar. By the induction hypothesis, we may assume that the result holds for L^- and for L^0 . Now if L has an even number of components, so does L^- , while L^0 has an odd number of components, by Proposition 3.7. So $\nabla_{L^-}(z)$ is odd and $\nabla_{L^0}(z)$ is even, and hence

$$\nabla_L(z) = \nabla_{L^-}(z) + z\nabla_{L^0}(z)$$

is odd. The case where L has an odd number of components is similar, and is left to you to check. \square

Next we consider mirror image knots and links. As in Chapter 1, we shall denote by L^* the mirror image of the link L . Recall that L^* is obtained from L by switching *all* the crossings in L .

Proposition 3.17 *For any link L ,*

$$\nabla_{L^*}(z) = \nabla_L(-z).$$

Proof Again we argue by induction using the exchange relation. Again the result is trivial for links with no crossings. As in the preceding proof, we take $L = L^+$, the case $L = L^-$ being similar. Then

$$\nabla_L(z) - \nabla_{L^-}(z) = z\nabla_{L^0}(z),$$

and similarly

$$\nabla_{L^*}(z) - \nabla_{L^{-*}}(z) = -z\nabla_{L^{0*}}(z),$$

noting that each crossing changes sign when we take mirror images. Now, by the induction hypothesis, we may assume that

$$\nabla_{L^{0*}}(z) = \nabla_{L^0}(-z), \quad \nabla_{L^{-*}}(z) = \nabla_{L^-}(-z).$$

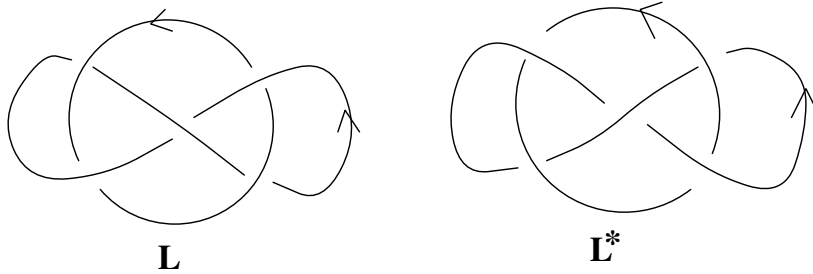
Putting these statements together, we get $\nabla_{L^*}(z) = \nabla_L(-z)$, which does the induction step. \square

As a result of Propositions 3.16 and 3.17, we can say

Proposition 3.18 *If L is an oriented link with an even number of components, and if $\nabla_L(z) \neq 0$, then L and its mirror image L^* are not equivalent.*

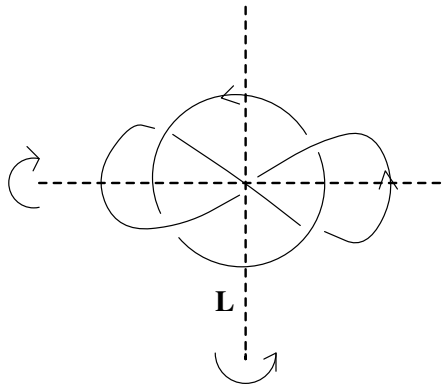
Proof Since $\nabla_{L^*}(z) = \nabla_L(-z) = -\nabla_L(z)$, if L^* and L are equivalent then $\nabla_L(z) = -\nabla_L(z)$, and hence $\nabla_L(z) = 0$. \square

Example 3.19 The mirror image Whitehead links



are not equivalent. It is easy to check that $\nabla_L(z) = z^3$, while $\nabla_{L^*}(z) = -z^3$.

It is interesting to note that the two Whitehead links are not equivalent, even if we regard both of them as *unoriented* links. This is in contrast to the situation for the two Hopf links, where of course the links become equivalent if we disregard the orientations.



This can be seen using the symmetries of the unoriented link L under rotation through 180° about the horizontal and vertical axes shown below, and about an axis through the centre of the diagram perpendicular to the plane of the paper. These symmetries reverse the orientations of one or both components of the link L .

Historical note: The Alexander polynomial

As mentioned in Section 2.1, in 1928 J. W. Alexander discovered a polynomial invariant of knots and links using methods of algebraic topology. The Alexander polynomial $\Delta_K(t)$ was a highly successful invariant. For example, it succeeded in distinguishing between all the 35 knots of up to eight crossings given in your knot table.

The Alexander polynomial was calculated from a diagram of the knot or link as a determinant with polynomial entries, the number of rows and columns in the

determinant being equal to the number of crossings in the diagram. This made it difficult to calculate except for knots with a small crossing number. Another inconvenient property of $\Delta_K(t)$ was that its definition depended on a number of choices, which could alter its value by changing its sign or multiplying it by a power of the variable t .

In 1970 the British mathematician J. H. Conway discovered a remarkably simple recursive scheme for calculating the Alexander polynomial. This was the scheme we have described by the three axioms in Section 3.1. It led to the Conway polynomial $\nabla_K(z)$. Conway used his polynomial to help tabulate all knots of up to 11 crossings, and all links of up to 10 crossings. To recover the Alexander polynomial $\Delta_K(t)$ from the Conway polynomial, all we have to do is to make the simple change of variables $z = \sqrt{t} - 1/\sqrt{t}$.

For example, from the Conway polynomial $1 + z^2$ for the (left or right) trefoil knot, we obtain the Alexander polynomial $1 + (t - 2 + t^{-1}) = t - 1 + t^{-1}$. This is a Laurent polynomial in t , so to put it in Alexander's normalised form, which a polynomial in t with positive constant term, we multiply by t to get $1 - t + t^2$. Similarly, from the Conway polynomial $1 - z^2$ for the figure eight knot, we obtain the Alexander polynomial $1 - 3t + t^2$.

We can think of the Conway polynomial as a natural normalisation of the Alexander polynomial, making it independent of the choices involved in Alexander's construction. As we have seen in Proposition 3.18, the Conway polynomial is capable of distinguishing certain links with an even number of components from their mirror images. However, this invariant can not distinguish any *knot* from its mirror image. Until the discovery of the Jones polynomial, which we shall meet in Chapter 6, the problem of finding knot invariants capable of distinguishing between mirror image knots was considered very difficult. Although the German topologist Max Dehn had given in 1914 a very clever geometric proof that the left and right trefoil knots were not equivalent, his argument did not lead to a general method that would work for other knots.