Frank Peterson's problem

Find a minimal generating set for

$$P(n) = \mathbb{F}_2[t_1, \ldots, t_n]$$

as a module over the Steenrod algebra $\mathcal{A}_2$.

For topologists, this problem is part of the study of BV, where V is an elementary abelian 2-group V of rank n.

For representation theorists, it is part of the study of the structure of the symmetric power representations of $G(n) = GL(n, \mathbb{F}_2)$.

V is a vector space of dimension n over $\mathbb{F}_2$, and $P(n) = \sum_{d \geq 0} P^d(n)$ is the symmetric algebra on the dual V^*, with $G(n)$ acting by linear substitutions. For all d, the Steenrod operation Sq^k is a $\mathbb{F}_2 G(n)$-module map $P^d(n) \to P^{d+k}(n)$.

Masaki Kameko's conjecture

Kameko conjectured that for all n and d

$$\dim Q^d(n) \leq 1 \cdot 3 \cdot 7 \cdots (2^n - 1)$$

where $Q^d(n) = P^d(n)/\mathcal{A}_2^+ P(n)$.

Equality has been proved (Tran Ngoc Nam, Reg Wood) in degrees

$$d = (2^{\lambda_1} - 1) + (2^{\lambda_2} - 1) + \ldots + (2^{\lambda_n} - 1),$$

which are 'generic' in the sense that the differences $\lambda_i - \lambda_{i+1}$ are sufficiently large.

$[G(n) : L(n)] = 1 \cdot 3 \cdot 7 \cdots (2^n - 1)$, where $L(n)$ is the lower triangular subgroup of $G(n)$. The cosets can be identified with complete flags in V. Working in homology, Michael Crabb and John Hubbuck proved that for generic d, $G(n)$ acts on $Q^d(n)$ by its permutation representation on complete flags.

Odd primes

The Peterson problem generalises to all primes. The representation theory view suggests that we extend farther, to prime powers $q = p^c$.

For this we use Larry Smith's algebra of Steenrod qth powers $\mathcal{A}_q \subset \mathcal{A}_p$. Now $[G(n) : L(n)]$ is

$$(1+q)(1+q+q^2)\cdots(1+q+\ldots+q^{n-1})$$

and we can generalize Kameko's conjecture.

But Martin Crossley has shown that for $p > 2$, $d = p^2 - 1$, $\dim Q^d(2) = 2p - 1 > p + 1$, so this fails.

However, we can hope to extend some features of the case $p = 2$.

Two results for all n and q

Theorem 1 *Let* $\lambda = (n-1,\ldots,2,1,0)$*, so that*

$$d_1 = (q^{n-1}-1)+(q^{n-2}-1)+\ldots+(q-1).$$

Then $\dim Q^{d_1}(n) \geq q^{n(n-1)/2}$*, and* $Q^{d_1}(n)$ *has a quotient space on which* $G(n)$ *acts by its Steinberg representation* $St(n)$*.*

For $q=2$*,* $Q^{d_1}(n) \cong St(n)$*.*

It is possible that equality holds for all q.

Theorem 2 *Let* $\lambda = (2n-2,\ldots,4,2,0)$ *so that*

$$d_2 = (q^{2n-2}-1)+(q^{2n-4}-1)+\ldots+(q^2-1).$$

Then $\dim Q^{d_2}(n) \geq [G(n):L(n)]$*, and* $Q^{d_2}(n)$ *has a quotient space on which* $G(n)$ *acts by its flag representation* $Fl(n)$*.*

We have not been able to prove equality for $q=2$. As mentioned above, equality does not hold for odd primes.

The dual Peterson problem

For simplicity, assume that $q = 2$. Then

$$P(n) = H^*(\mathbb{R}P^\infty \times \ldots \times \mathbb{R}P^\infty; \mathbb{F}_2).$$

Its dual is the **divided power algebra**

$$D(n) = H_*(\mathbb{R}P^\infty \times \ldots \times \mathbb{R}P^\infty; \mathbb{F}_2),$$

with basis $u_1, \ldots, u_n$ dual to $t_1, \ldots, t_n$.

The action of $\mathcal{A}_2$ on $D(n)$ is given by

$$Sq_k(u^{(r)}) = \binom{r-k}{k} u^{(r-k)}, \quad u \in D^1(n),$$

where $u^{(r)} \sim u^r / r!$.

Problem: Give a basis for the Steenrod kernel

$$K(n) = \{f \in D(n) : Sq_k(f) = 0 \text{ **for all** } k > 0\}.$$

$K(n)$ is a subring of $D(n)$. For all n and d, the vector spaces $K^d(n)$ and $Q^d(n)$ have the same dimension.

Blocks and monomials (for $q = 2$)

A monomial in $P(n)$ corresponds to a **block** whose rows are the reverse binary expansions of its exponents. Thus

$$t_1^{10} t_2^5 t_3^3 \longleftrightarrow \begin{array}{cccc} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & \\ 1 & 1 & & \end{array}$$

Similarly, a (divided) monomial in $D(n)$ corresponds to a $(0,1)$-block

$$u_1^{(10)} u_2^{(5)} u_3^{(3)} \longleftrightarrow \begin{array}{cccc} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & \\ 1 & 1 & & \end{array}$$

Multiplication in $D(n)$ is by addition of blocks as matrices mod 2, so $1 + 1 = 0$ (no 'carry digits'), e.g.

$$u_1^{(10)} = u_1^{(8)} u_1^{(2)}, \quad u_1^{(6)} u_1^{(4)} = 0.$$

.

Theorem of Crabb and Hubbuck

$K(n)$ has a subring generated by monomials (Bill Singer's **spikes**) with blocks

$$\begin{matrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & \cdots & 1 & \\ 1 & \cdots & 1 & & \end{matrix}$$

in $K(1) \times \cdots \times K(1)$. This subring is called the **ring of lines**.

Theorem (Crabb-Hubbuck) For $q = 2$, there is an injection $Fl(n) \hookrightarrow K^d(n)$ in generic degrees d.

(Here genericity only requires the 'overlaps' $\lambda_i - \lambda_{i+1}$ to grow logarithmically.)

$K^d(n)$ and $Q^d(n)$ are transpose duals as modules over $\mathbb{F}_2 G(n)$. The Steinberg representation $St(n)$ and the flag representation $Fl(n)$ are their own transpose duals.

Schubert cells

The set of flags over $\mathbb{F}_q$ is the disjoint union of **Schubert cells** indexed by permutations ρ of $\{1, 2, \ldots, n\}$. The Schubert cell $Sch(\rho)$ is a vector space over $\mathbb{F}_q$.

Given a matrix $g \in G(n)$, let W_i be the span of the first i rows of g. Then

$$W : W_1 \subset W_2 \subset \ldots \subset W_{n-1}$$

is a flag. Addition of higher to lower rows does not change W, so we can identify W with the coset $L(n)g$. Each coset has a unique row-reduced representative, such as

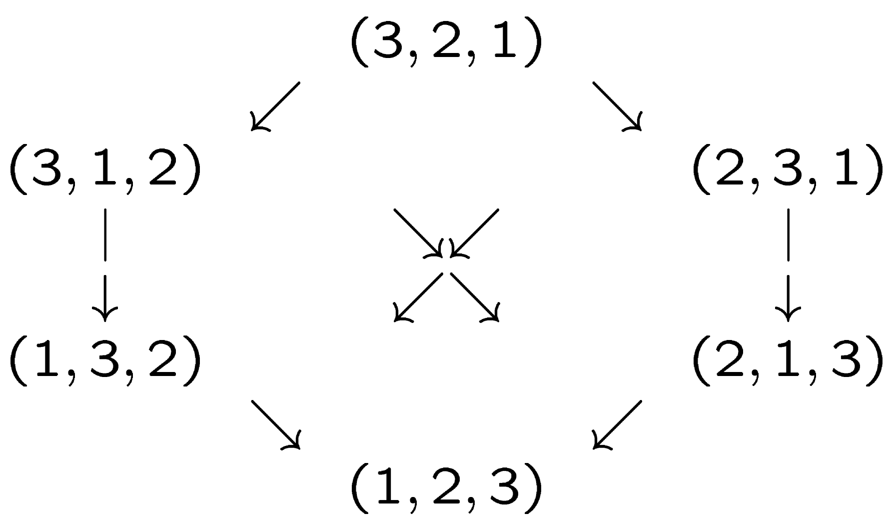
$$\begin{pmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

where the entries below the final 1 in each row are 0's and the $*$'s are arbitrary elements of $\mathbb{F}_q$. Reading the final 1's in row order gives ρ. The example shows $Sch(3, 4, 1, 2)$.

The Bruhat order

The dimension of $Sch(\rho)$ is $\ell(\rho)$, the length of ρ as a reduced word in the usual generating transpositions $\sigma_i = (i, i+1)$, $1 \leq i \leq n-1$, or (equally) the number of pairs (i, j) with $i < j$ and $\rho(i) > \rho(j)$.

The Schubert cells have a natural partial order, the **Bruhat order**, shown below for $n = 3$.



We denote the permutation $(n, n-1, \ldots, 1)$ of maximal length $n(n-1)/2$ by ρ_0.

The Crabb-Hubbuck map

Given $\lambda : \lambda_1 > \lambda_2 > \ldots > \lambda_n \geq 0$, we define $\phi^\lambda : Fl(n) \longrightarrow K^d(n)$ for $d = \sum_{i=1}^n (q^{\lambda_i} - 1)$ by

$$\phi^\lambda(W) = \prod_{i=1}^n w_i^{(q^{\lambda_i}-1)},$$

where the flag W is identified with the coset $L(n)g$, and w_i is the ith row of g. Standard relations in $D(n)$ ensure that ϕ^λ is well-defined.

Crabb and Hubbuck prove that for generic λ the **flag polynomials** $\phi^\lambda(W)$ are linearly independent. We aim to prove that this is true in degree $d = d_2$, when $\lambda_i - \lambda_{i+1} = 2$ for all i.

This is the minimal case in which ϕ^λ can be injective. If $\lambda_i - \lambda_{i+1} = 1$ and $\sigma = \sigma_i \circ \rho$, then each flag polynomial for $Sch(\rho)$ is the sum of q flag polynomials for $Sch(\sigma)$.

The Steinberg module

For $d = d_1$, $\lambda = (n-1, \ldots, 2, 1, 0)$. Hence $\mathrm{Im}(\phi^\lambda)$ is spanned by the polynomials for the top Schubert cell $Sch(\rho_0)$. To prove these linearly independent, we show that $\mathrm{Im}(\phi^\lambda)$ contains a copy of the Steinberg module $St(n)$.

$St(n)$ is irreducible, with dimension $q^{n(n-1)/2}$. It is generated in $\mathbb{F}_q G(n)$ by the idempotent

$$e(n) = (-1)^n \overline{U}(n) \overline{S}(n),$$

with $\overline{U}(n)$ the sum of upper triangular and $\overline{S}(n)$ the signed sum of permutation matrices.

If W is the reference flag, $\phi^\lambda(W)$ is the spike

$$u_1^{(q^{n-1}-1)} u_2^{(q^{n-2}-1)} \cdots u_{n-1}^{(q-1)}.$$

An inductive calculation shows that $\phi^\lambda(W)e(n)$ is the signed sum of all $n!$ 'spikes', and so it is nonzero. Hence $St(n)$ appears in $\mathrm{Im}(\phi^\lambda)$, and so ϕ^λ is injective on $Sch(\rho_0)$.

It follows that $\dim Q^{d_1}(n) \geq q^{n(n-1)/2}$.

Young tableaux and monomials

Upper bounds on $Q^d(n)$ or $K^d(n)$ seem to be much harder to obtain. However, for $q = 2$, we can show that $Q^{d_1}(n) \cong St(n)$. Our argument uses Young tableaux, which we associate with binary blocks representing monomials in $P(n)$.

In the first column of a tableau of shape λ we record the positions of the 1's which appear in the first column of the block, and similarly for the other columns.

Example Let $n = 3$, $q = 2$, $\lambda = (2, 1, 0)$. Then the Steenrod quotient $Q^4(3)$ has a basis of monomials corresponding to the 8 blocks

$$\begin{array}{l}1\,1\\1\\0\end{array},\ \begin{array}{l}1\,1\\0\\1\end{array},\ \begin{array}{l}1\\1\,1\\0\end{array},\ \begin{array}{l}1\\0\,1\\1\end{array},\ \begin{array}{l}0\\1\,1\\1\end{array},\ \begin{array}{l}0\\1\\1\,1\end{array},\ \begin{array}{l}1\\1\\0\,1\end{array},\ \begin{array}{l}1\\0\\1\,1\end{array}.$$

We associate to these the tableaux

$$\begin{array}{l}1\,1\\2\end{array},\ \begin{array}{l}1\,1\\3\end{array},\ \begin{array}{l}1\,2\\2\end{array},\ \begin{array}{l}1\,2\\3\end{array},\ \begin{array}{l}2\,2\\3\end{array},\ \begin{array}{l}2\,3\\3\end{array},\ \begin{array}{l}1\,3\\2\end{array},\ \begin{array}{l}1\,3\\3\end{array}.$$

Semi-standard tableaux and ω-vectors

These **semi-standard** Young tableaux increase **strictly** down columns and **weakly** along rows. We call the corresponding blocks or monomials semi-standard as well. Since

$$Sq^1(t_1 t_2 t_3) = t_1^2 t_2 t_3 + t_1 t_2^2 t_3 + t_1 t_2 t_3^2$$

we may omit the non-semi-standard monomial $t_1^2 t_2 t_3$ from our basis.

$$\begin{matrix} 0 & 1 \\ 1 & \\ 1 & \end{matrix} \quad \longleftrightarrow \quad \begin{matrix} 2 & 1 \\ 3 & \end{matrix}$$

We call the vector of column sums of a binary block its ω-**vector**. The action of $\mathcal{A}_2^+$ on $P(n)$ lowers the ω-vector in left lexicographic order, giving a filtration of $P^d(n)$ by $\mathcal{A}_2$-submodules. We denote by $P^\omega(n)$ the filtration quotient given by monomials with ω-vector ω. Similarly we have a filtration quotient $D^\omega(n)$ of $D^d(n)$. When ω is weakly decreasing, $P^\omega(n)$ and $D^\omega(n)$ contain spikes.

An upper bound on some $Q^d(n)$

Our column-position correspondence matches blocks with weakly decreasing ω to tableaux which increase strictly on columns. A block is semi-standard $\iff$ the ω-vector of the sub-block consisting of its first i rows is weakly decreasing for all i.

Theorem 3 *When $P^d(n)$ has a unique weakly decreasing ω-vector, $Q^d(n)$ is spanned by semi-standard blocks.*

When $d = d_1$, Theorem 3 applies with $\omega = (n-1, \ldots, 2, 1)$. The well-known **hook formula**, which counts semi-standard tableaux, gives

$$2^{n(n-1)/2} = \dim St(n).$$

The proof of Theorem 3 exploits a standard combinatorial move, the χ-**trick**, together with careful control of the error terms which arise using inductions on both the left and right lexicographic orderings on ω-vectors.

The truncated flag polynomial $\overline{\phi}^{\lambda}(W)$

We wish to show that all the flag polynomials are linearly independent when $d = d_2$. If we truncate by omitting all terms with the 'wrong' ω-vector, we find a simple relation between the flag polynomials for $d = d_1$ and $d = d_2$.

We can separate the odd and even columns of a block with $\omega = \mu = (n-1, n-1, \ldots, 2, 2, 1, 1)$ to get two blocks with $\omega = \nu = (n-1, \ldots, 2, 1)$.

$$\begin{matrix} 1 & 1 & 0 & 1 & & \\ 0 & 1 & 1 & 1 & 1 & \\ 1 & 0 & 1 & & & \\ 1 & 1 & 0 & 0 & 0 & 1 \end{matrix} \quad \mapsto \left(\begin{matrix} 1 & & \\ 0 & 1 & 1 \\ 1 & 1 & \\ 1 & & \end{matrix} , \quad \begin{matrix} 1 & 1 & \\ 1 & 1 & \\ 0 & & \\ 1 & 0 & 1 \end{matrix} \right).$$

Conversely, we can interlace the columns of two blocks to obtain a single larger block. Extending linearly, we get an isomorphism

$$\pi : \ D^{\nu}(n) \otimes D^{\nu}(n) \longrightarrow D^{\mu}(n)$$

of vector spaces, and

$$\overline{\phi}^{\lambda}(W) = \pi(\overline{\phi}^{\nu}(W) \otimes \overline{\phi}^{\nu}(W)),$$

where $\lambda = \mu' = (2n-2, \ldots, 4, 2, 0)$.

Sketch proof of Theorem 2

We have seen that for $d = d_1$ the flag polynomials for $Sch(\rho_0)$ form a basis for $\mathrm{Im}(\phi^\nu)$. This remains true after truncation. We can therefore write a truncated flag polynomial for $d = d_2$ as a linear combination of 'tensor products' of these $q^{n(n-1)/2}$ polynomials.

We obtain a $[G(n) : L(n)] \times q^{n(n-1)}$ matrix X over $\mathbb{F}_q$, with rows corresponding to the flags W, and with columns corresponding to ordered pairs of flags (W_1, W_2) in $Sch(\rho_0)$, giving the expansion of the truncated flag polynomials as a sum of the polynomials $\pi(\overline{\phi}^\nu(W_1) \otimes \overline{\phi}^\nu(W_2))$.

Working downwards in the Bruhat order, it is possible to find a non-singular triangular submatrix of X. Hence X has linearly independent rows.

The 0-Hecke algebra $H(n)$

Let $s(\rho) \in Fl(n)$ be the sum of the flags in $Sch(\rho)$, a formal sum of row-reduced matrices. The elements $s(\rho)$ form a semigroup under the product induced by matrix multiplication, with generators $\sigma_i = (i, i+1)$ and defining relations

$$\sigma_i^2 = \sigma_i, \quad \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1,$$

$$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ if } |i-j| = 1.$$

Thus the semigroup algebra is a deformation of the group algebra of $S(n)$ obtained by replacing each relation $\sigma_i^2 = e$ by $\sigma_i^2 = \sigma_i$. This is the **0-Hecke algebra** $H(n)$.

$H(n)$ acts on the left of $Fl(n)$ so as to commute with the action of $G(n)$ on the right. It is the algebra of $\mathbb{F}_q G(n)$-endomorphisms of $Fl(n)$.

To define this action, identify $\rho \in H(n)$ with $s(\rho) \in Fl(n)$ and flags with row-reduced matrices. Now multiply in $\mathbb{F}_q G(n)$ and row reduce the result to obtain an element of $Fl(n)$.

Decomposition of $H(n)$

P. N. Norton (1979) showed that the 0-Hecke algebra $H(n)$ is a direct sum of 2^{n-1} indecomposable left ideals. This gives corresponding information about $Fl(n)$, i.e.

$$Fl(n) \cong \bigoplus_J Fl_J(n), \qquad J \subseteq \{1, 2, \ldots, n-1\},$$

where the $Fl_J(n)$ are indecomposable $\mathbb{F}_q G(n)$-modules. For example, for $n = 3$ and $q = 2$ we have 4 summands

$$\begin{array}{ccc} & St & \\ & \oplus & \\ V & & V^* \\ | & \oplus & | \\ V^* & & V \\ & \oplus & \\ & T & \end{array}$$

where T is the trivial $\mathbb{F}_2 G(3)$-module, V and V^* are the natural module and its dual, and St is the Steinberg module. This case was treated in detail by Michael Boardman (1991).

Embedding the summands of $Fl(n)$

The decomposition of $Fl(n)$ and computations for $q = 2$ and $n \leq 4$ encourage us to try to extend our results to other degrees d.

Two summands $Fl_J(n)$ are irreducible, namely $St(n)$, for $J = \{1, 2, \ldots, n-1\}$, and the trivial module, for $J = \emptyset$.

Conjecture 4 *For $J \subseteq \{1, 2, \ldots, n-1\}$ let*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 1, & \text{if } i \in J, \\ 2, & \text{if } i \notin J. \end{cases}$$

Then ϕ^λ is injective on $Fl_J(n)$.

Conjecture 4 describes 'generic' cases for injectivity of ϕ^λ on $Fl_J(n)$. There are also 'non-generic' cases where this happens. For example, the trivial module appears for $\lambda = (0)$.

Future work

We hope to show that each $Fl_J(n)$ has irreducible socle, with a generator s_J given by an element of $H(n)$, regarded as a sum of flags in $Fl(n)$, and to prove that $\phi^\lambda(s_J) \neq 0$ by identifying its leading monomial. This appears to be determined simply by J and λ. For example in the Steinberg case $J = \{1, 2, \ldots, n-1\}$ it is a spike, and for the generic case $J = \emptyset$ its block

$$\begin{matrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & \\ 1 & 1 & 0 & 1 & & \\ 1 & 1 & 1 & & & \end{matrix},$$

shown for $n = 4$, has $1, 2, \ldots, n-1$ 0's in alternate antidiagonals.

For $q = 2$ only, all irreducible $\mathbb{F}_q G(n)$-modules are isomorphic to submodules of $Fl(n)$. This suggests that the Peterson problem is harder for $q > 2$. Even for $q = 2$, Kameko's upper bound conjecture appears to be out of reach for $d = d_2$.