Embedding the Steinberg and the flag representations of $GL(n, \mathbb{F}_q)$ in the divided power algebra

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1 The Peterson problem

Problem (Frank Peterson, 1987) Find a minimal generating set for $P(n) = \mathbb{F}_2[t_1, \ldots, t_n]$ as a module over the mod 2 Steenrod algebra $\mathcal{A}_2$.

For topologists, the polynomial algebra is the cohomology $H^*(BV; \mathbb{F}_2)$ of the classifying space BV of an elementary abelian 2-group V of rank n, and Peterson's problem is seen as one aspect of the study of the structure of the classifying space BG of a finite group G. For representation theorists, V is a vector space of dimension n over $\mathbb{F}_2$, and $P(n) = \sum_{d \geq 0} P^d(n)$ is the symmetric algebra on the dual space V^*, with the group $G(n) = GL(n, \mathbb{F}_2)$ acting (on the right) by linear substitutions of the basis $t_1, \ldots, t_n$. The operation Sq^k is seen as a $\mathbb{F}_2 G(n)$-module map $P^d(n) \to P^{d+k}(n)$, and Peterson's problem is seen as a question about the symmetric power representations of $G(n)$.

Masaki Kameko conjectured that for all n and d

$$\dim Q^d(n) \leq 1 \cdot 3 \cdot 7 \cdots (2^n - 1)$$

where $Q^d(n) = P^d(n)/\mathcal{A}_2^+ P(n)$.

Equality has been proved (Tran Ngoc Nam, Reg Wood) in degrees

$$d = (2^{\lambda_1} - 1) + (2^{\lambda_2} - 1) + \ldots + (2^{\lambda_n} - 1),$$

which are 'generic' in the sense that the differences $\lambda_i - \lambda_{i+1}$ are sufficiently large.

By working with the dual homology version, Michael Crabb and John Hubbuck proved that in generic cases, $G(n)$ acts on $Q^d(n)$ by its permutation representation on complete flags, i.e. cosets of $L(n)$, the lower triangular subgroup. One of our main results develops their ideas further.

The Peterson problem generalises to all primes p, and even to prime powers $q = p^c$, using Larry Smith's algebra of Steenrod qth powers $\mathcal{A}_q \subset \mathcal{A}_p$. Since

$$[G(n) : L(n)] = (1+q)(1+q+q^2)\cdots(1+q+\ldots+q^{n-1})$$

we have an obvious generalization of Kameko's conjecture. But Martin Crossley has shown that for $p > 2$, $d = p^2 - 1$, $\dim Q^d(2) = 2p - 1 > p + 1$, so this is false. However, we can hope to extend some features of the case $p = 2$.

2 Two results for all n and q

Theorem 2.1 *Let $d = d_1 = ((q^n - 1)/(q - 1)) - n = (q^{n-1} - 1) + (q^{n-2} - 1) + \ldots + (q - 1)$, so that $\lambda = (n-1, \ldots, 2, 1, 0)$. Then $\dim Q^d(n) \geq q^{n(n-1)/2}$, and $Q^d(n)$ has a quotient space on which $G(n)$ acts by its Steinberg representation $St(n)$. In the case $q = 2$, equality holds, i.e. $Q^d(n) \cong St(n)$.*

It may be that equality holds for all q, but our method does not appear to generalize. We obtain a monomial basis for the quotient space which is related to the combinatorics of Young tableaux. It has been known since the 1980's (Mitchell-Priddy, Kuhn-Mitchell) that the first occurrence of $St(n)$ in $P(n)$ is in degree d_1.

Theorem 2.2 *Let $d = d_2 = ((q^{2n} - 1)/(q^2 - 1)) - n = (q^{2n-2} - 1) + (q^{2n-4} - 1) + \ldots + (q^2 - 1)$, so that $\lambda = (2n-2, \ldots, 4, 2, 0)$. Then $\dim Q^d(n) \geq [G(n) : L(n)]$, and $Q^d(n)$ has a quotient space on which $G(n)$ acts by its flag representation $Fl(n)$.*

Nam has claimed this result for $q = 2$ and all λ with $\lambda_i - \lambda_{i+1} \geq 2$ for all i. We have not been able to prove equality for $q = 2$. As mentioned above, this lower bound is not exact for odd primes.

3 The dual Peterson problem

For simplicity, assume in this section that $q = 2$. Then $P(n) = H^*(\mathbb{R}P^\infty \times \ldots \times \mathbb{R}P^\infty; \mathbb{F}_2)$ and its Hopf algebra dual is the divided power algebra $D(n) = H_*(\mathbb{R}P^\infty \times \ldots \times \mathbb{R}P^\infty; \mathbb{F}_2)$. Let $u_1, \ldots, u_n$ be the basis of $D^1(n)$ dual to $t_1, \ldots, t_n$.

Blocks and monomials (for $q = 2$): A monomial in $P(n)$ corresponds to a '$(0,1)$-block' whose rows are the reverse binary expansions of its exponents. Thus

$$t_1^{10} t_2^5 t_3^3 \longleftrightarrow \begin{array}{cccc} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & \\ 1 & 1 & & \end{array}$$

Similarly, a (divided) monomial in $D(n)$ corresponds to a $(0,1)$-block

$$u_1^{(10)}u_2^{(5)}u_3^{(3)} \longleftrightarrow \begin{matrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & \\ 1 & 1 & & \end{matrix}$$

where the divided power $u^{(r)}$ can be thought of as $u^r/r!$. Multiplication in $D(n)$ is by addition of blocks as matrices mod 2, so $1+1=0$ (with no 'carrying'). Thus $u_1^{(10)} = u_1^{(8)}u_1^{(2)}$, but $u_1^{(6)}u_1^{(4)} = 0$.

The Steenrod algebra $\mathcal{A}_2$ acts on $D(n)$ by dualizing the action on $P(n)$, so

$$Sq_k(u^{(r)}) = \binom{r-k}{k} u^{(r-k)}, \quad u \in D^1(n).$$

Dual Peterson problem: Give a basis for the Steenrod kernel

$$K(n) = \{f \in D(n) : Sq_k(f) = 0 \textbf{ for all } k > 0\}.$$

For all n and d, $K^d(n)$ and $Q^d(n)$ have the same dimension as vector spaces over $\mathbb{F}_2$, and are transpose duals as modules over $\mathbb{F}_2G(n)$. (The Steinberg representation $St(n)$ and the flag representation $Fl(n)$ are their own transpose duals.) $K(n)$ is a subring of $D(n)$. The ring structure was exploited by Crabb and Hubbuck (also by Joe Repka and Paul Selick) to study the lower bound on $K(n)$ generated by monomials given by blocks of the form

$$\begin{matrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & \cdots & 1 & \\ 1 & \cdots & 1 & & \end{matrix}$$

(called 'spikes' by Bill Singer). (The spikes give $n!$ linearly independent elements of $Q^d(n)$ or $K^d(n)$ for $d = d_1$ or d_2.)

Theorem (Crabb-Hubbuck) In 'generic' degrees d (genericity only requires the 'overlaps' $\lambda_i - \lambda_{i+1}$ to grow logarithmically), there is an injection $Fl(n) \hookrightarrow K^d(n)$.

4 Schubert cells

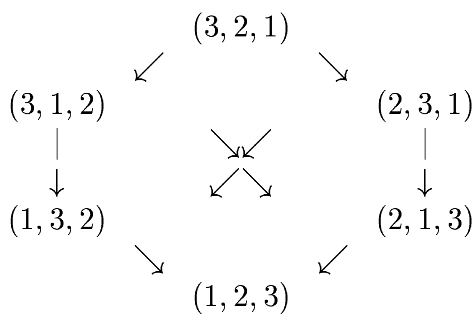
Here we work over $\mathbb{F}_q$. The set of (complete) flags has a well-known decomposition into **Schubert cells** $Sch(\rho)$ indexed by permutations ρ of $\{1, 2, \ldots, n\}$.

Let $g \in G(n)$ be a matrix and let W_i be the span of the first i rows, so that $W : W_1 \subset W_2 \subset \ldots \subset W_{n-1}$ is a flag. Addition of higher rows to lower ones does not change this flag, and so we may identify it with the coset $L(n)g$. Each coset has a unique row-reduced representative, such as

$$\begin{pmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

where the entries below the final 1 in each row are 0's and the $*$'s are arbitrary elements of $\mathbb{F}_q$. Reading the final 1's in row order gives ρ. The example shows $Sch(3,4,1,2)$.

$Sch(\rho)$ is a vector space over $\mathbb{F}_q$ of dimension $\ell(\rho)$, the length of ρ as a reduced word in the usual generating transpositions $\sigma_i = (i, i+1)$, $1 \leq i \leq n-1$, or (equivalently) the number of pairs (i,j) with $i < j$ and $\rho(i) > \rho(j)$. The Schubert cells have a natural partial order, the Bruhat order, shown below for $n = 3$.



We denote by ρ_0 the permutation $(n, n-1, \ldots, 1)$ of maximal length $n(n-1)/2$.

5 The Crabb-Hubbuck map

Let $\lambda : \lambda_1 > \lambda_2 > \ldots > \lambda_n \geq 0$ be a strictly decreasing partition and let $d = \sum_{i=1}^{n}(q^{\lambda_i} - 1)$. Following Crabb and Hubbuck, define $\phi^\lambda : Fl(n) \longrightarrow K^d(n)$ by

$$\phi^\lambda(W) = \prod_{i=1}^{n} w_i^{(q^{\lambda_i}-1)},$$

where the flag W is identified with the coset $L(n)g$, and w_i is the ith row of g. Standard relations in $D(n)$ ensure that ϕ^λ is well-defined.

The theorem of Crabb and Hubbuck states that, when λ is generic, the **flag polynomials** $\phi^\lambda(W)$ associated to flags W are linearly independent. We aim to prove that this is already true when $\lambda_i - \lambda_{i+1} = 2$ for all i. This is the minimal case in which ϕ^λ can be injective. If $\lambda_i - \lambda_{i+1} = 1$ and $\sigma = \sigma_i \circ \rho$, then each flag polynomial for $Sch(\rho)$ is the sum of q flag polynomials for $Sch(\sigma)$. This follows from the relation

$$u^{(q^{k+1}-1)}v^{(q^k-1)} + \sum_{a \in \mathbb{F}_q} (au+v)^{(q^{k+1}-1)}u^{(q^k-1)} = 0,$$

which holds for all k and all $u, v \in D^1(n)$. For example, for $n = 3$, $q = 2$, $\lambda = (2, 1, 0)$,

$$\phi^\lambda \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \phi^\lambda \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \phi^\lambda \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

which states that

$$u_2^{(3)}(u_1 + u_3) = (u_1 + u_3)^{(3)} u_2 + (u_1 + u_2 + u_3)^{(3)} u_2.$$

In this example, the relations imply that all the flag polynomials are linear combinations of polynomials for flags in $Sch(3, 2, 1)$. It is also easy to check that these eight polynomials are linearly independent.

For all n and q, the same situation arises for $\lambda = (n-1, \ldots, 2, 1, 0)$: the image of ϕ^λ is spanned by the flag polynomials for the top Schubert cell $Sch(\rho_0)$. It is not trivial to show that these $q^{n(n-1)/2}$ flag polynomials are linearly independent. To do this, we show that the image of ϕ^λ contains a copy of the Steinberg module $St(n)$ for $\mathbb{F}_q G(n)$. This is done using an easy inductive calculation on n, using the idempotent

$$e(n) = (-1)^n \overline{U}(n) \overline{S}(n) \in \mathbb{F}_q G(n)$$

where $\overline{U}(n)$ is the sum of upper triangular matrices and $\overline{S}(n)$ is the signed sum of permutation matrices in $G(n)$. Thus let W be the reference flag (the coset represented by the identity matrix), so that

$$\phi^\lambda(W) = u_1^{(q^{n-1}-1)} u_2^{(q^{n-2}-1)} \cdots u_{n-1}^{(q-1)}.$$

The inductive calculation shows that $\phi^\lambda(W)e(n) \neq 0$ (in fact, it is a signed sum of all $n!$ 'spikes'). Hence $St(n)$ appears in $\mathrm{Im}(\phi^\lambda)$, and so ϕ^λ is injective on $Sch(\rho_0)$. It follows that $\dim Q^{d_1}(n) \geq q^{n(n-1)/2}$.

6 Young tableaux and monomials

The argument sketched above shows that the quotient $Q^d(n)$ has dimension $\geq q^{n(n-1)/2}$ in degree $d = d_1$, the '1-overlap' case. The argument extends to all cases where λ is strictly decreasing: the image of the Crabb-Hubbuck map contains a copy of the Steinberg module. Inequalities in the other direction, giving upper bounds on $Q^d(n)$ or $K^d(n)$, seem to be much harder to obtain. However there is some glimmer of hope in the case $q = 2$: in particular, we can show that $Q^d(n) \cong St(n)$ in the 1-overlap case $d = d_1$. (A small amount of computational evidence suggests that this may hold for all q.)

Our argument uses the combinatorics of Young tableaux, which we associate with binary blocks representing monomials in $P(n)$ as follows. In the first column of a tableau of shape λ we record the positions of the 1's which appear in the first column of the block, and similarly for the other columns.

Example Let $n=3$, $q=2$, $\lambda=(2,1,0)$. Then $Q^4(3)$ has a basis of monomials corresponding to the 8 blocks

$$\begin{matrix}11\\1\\0\end{matrix},\quad \begin{matrix}11\\0\\1\end{matrix},\quad \begin{matrix}1\\11\\0\end{matrix},\quad \begin{matrix}1\\01\\1\end{matrix},\quad \begin{matrix}0\\11\\1\end{matrix},\quad \begin{matrix}0\\1\\11\end{matrix},\quad \begin{matrix}1\\1\\01\end{matrix},\quad \begin{matrix}1\\0\\11\end{matrix}.$$

We associate to these the 8 tableaux

$$\begin{matrix}11\\2\end{matrix},\quad \begin{matrix}11\\3\end{matrix},\quad \begin{matrix}12\\2\end{matrix},\quad \begin{matrix}12\\3\end{matrix},\quad \begin{matrix}22\\3\end{matrix},\quad \begin{matrix}23\\3\end{matrix},\quad \begin{matrix}13\\2\end{matrix},\quad \begin{matrix}13\\3\end{matrix}.$$

These particular Young tableaux are **semi-standard**, i.e. they increase strictly down columns and weakly along rows. The equation

$$Sq^1(t_1t_2t_3) = t_1^2t_2t_3 + t_1t_2^2t_3 + t_1t_2t_3^2$$

shows that the monomial $t_1^2t_2t_3$, whose block

$$\begin{matrix}01\\1\\1\end{matrix} \longleftrightarrow \begin{matrix}21\\3\end{matrix}$$

is not semi-standard, is not needed in a basis for $Q^4(3)$.

A basic step in studying the action of the Steenrod algebra $\mathcal{A}_2$ on $P(n)$ is to exploit the vector of column sums of a binary block, which we call the **ω-vector**. The action of $\mathcal{A}_2^+$ strictly lowers the ω-vector in left lexicographic order, so that we obtain a filtration of $P^d(n)$ by $\mathcal{A}_2$-submodules. We denote by $P^\omega(n)$ the filtration quotient with basis given by monomials m with ω-vector ω. Similarly we have a filtration quotient $D^\omega(n)$ of $D^d(n)$. When ω is weakly decreasing, $P^\omega(n)$ and $D^\omega(n)$ contain spikes.

The correspondence above matches binary blocks with decreasing ω-vector to Young tableaux which increase strictly on columns. The blocks which correspond to semi-standard tableaux are those which also satisfy the condition that the ω-vector of the subblock given by the first i rows is again (weakly) decreasing for all i. We call such blocks **semi-standard**, and obtain the following result.

Theorem 6.1 *When $P^d(n)$ has a unique weakly decreasing ω-vector, $Q^d(n)$ is spanned by semi-standard blocks.*

As a special case, this applies when $d=d_1$ is the 1-overlap degree. In this case, the number of semi-standard Young tableaux is given by the **hook formula** and is $2^{n(n-1)/2}$, the dimension of $St(n)$.

The proof of Theorem 6.1 exploits a standard combinatorial move called the 'χ-trick' together with careful control of the error terms which arise using inductions on both the left and right lexicographic orderings on ω-vectors. The details appear in the proceedings of the Hanoi 2004 conference on algebraic topology.

7 Truncated flag polynomials

Again consider the case $q = 2$, for which the block notation works well.

To prove Theorem 2.2 we want to show that all the flag polynomials are linearly independent in the '2-overlap' case $d = d_2$. The key observation here is that if we truncate them by omitting all terms with the 'wrong' ω-vector, then there is a simple relationship between the flag polynomials for $d = d_1$ and those for $d = d_2$. Working in $D(n)$, the terms we omit are those with **higher** ω-vectors in left lexicographic order. If $\phi^\lambda(W)$ is a flag polynomial, the lowest ω-vector occurring in it is the conjugate partition λ'. For general q, the lowest ω-vector in $\phi^\lambda(W)$ is $(q-1)\lambda'$. We denote the truncation of $\phi^\lambda(W)$ by $\overline{\phi}^\lambda(W)$.

We can separate the odd and even columns of a block with $\omega = \mu = (n-1, n-1, \ldots, 2, 2, 1, 1)$ to get two blocks with $\omega = \nu = (n-1, \ldots, 2, 1)$.

$$\begin{array}{cccccc} 1 & 1 & 0 & 1 & & \\ 0 & 1 & 1 & 1 & 1 & \\ 1 & 0 & 1 & & & \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array} \quad \mapsto \quad \left(\begin{array}{ccc} 1 & & \\ 0 & 1 & 1 \\ 1 & 1 & \\ 1 & & \end{array} \; , \; \begin{array}{ccc} 1 & 1 & \\ 1 & 1 & \\ 0 & & \\ 1 & 0 & 1 \end{array} \right).$$

Conversely, we can take two blocks and interlace their columns to obtain a single larger block. By extending this construction linearly, we obtain a vector space isomorphism

$$\pi : \; D^\nu(n) \otimes D^\nu(n) \longrightarrow D^\omega(n),$$

where $\nu = (n-1, \ldots, 2, 1)$, $\omega = (n-1, n-1, \ldots, 2, 2, 1, 1)$.

Proposition 7.1 *Let W be a flag in V, and let $\nu = (n-1, \ldots, 2, 1)$, $\lambda = (2n-2, \ldots, 4, 2, 0)$. Then the truncated flag polynomials for λ are the 'tensor squares' of the corresponding reduced flag polynomials for ν, i.e.*

$$\overline{\phi}^\lambda(W) = \pi(\overline{\phi}^\nu(W) \otimes \overline{\phi}^\nu(W)).$$

In particular, if $\overline{\phi}^\nu(W)$ has k terms then $\overline{\phi}^\lambda(W)$ has k^2 terms.

In the case $d = d_1$, we have seen that the flag polynomials for $Sch(\rho_0)$ form a basis for the image of ϕ^ν. This remains true after truncation. We can therefore write all truncated flag polynomials for $d = d_2$ as a linear combination of 'tensor products' of these $2^{n(n-1)/2}$ polynomials. We obtain a $[G(n) : L(n)] \times 2^{n(n-1)}$ matrix X over $\mathbb{F}_2$, with rows corresponding to the flags W, and with columns corresponding to ordered pairs of flags (W_1, W_2) in $Sch(\rho_0)$, giving the expansion of the truncated flag polynomials as a sum of the polynomials $\pi(\overline{\phi}^\nu(W_1) \otimes \overline{\phi}^\nu(W_2))$. Working downwards in the Bruhat order, it is possible to find a non-singular triangular submatrix of X. Hence X has linearly independent rows.

8 The 0-Hecke algebra $H(n)$

Given a permutation $\rho \in S(n)$, let $s(\rho) \in Fl(n)$ be the sum of the flags in the Schubert cell $Sch(\rho)$, regarded as a formal sum of row-reduced matrices. Matrix multiplication induces a product on the elements $s(\rho)$, under which they form a semigroup, which we might call the Schubert semigroup $Sch(n)$. The transpositions $\sigma_i = (i, i+1)$, $1 \leq i \leq n-1$ generate $Sch(n)$, subject to defining relations

$$\sigma_i^2 = \sigma_i, \quad \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1, \quad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ if } |i-j| = 1.$$

The corresponding semigroup algebra (with coefficients in an arbitrary field) is called the 0-Hecke algebra $H(n)$: it may be regarded as a deformation of the group algebra of $S(n)$ obtained by replacing each relation $\sigma_i^2 = e$ by $\sigma_i^2 = \sigma_i$.

The flag module $Fl(n)$ has a left action by $H(n)$, defined by identifying an element $\rho \in H(n)$ with the sum $s(\rho)$ of the flags in the corresponding Schubert cell $Sch(\rho)$, identifying flags with the corresponding row-reduced matrices, multiplying these matrices and, finally, row reducing the products to obtain a sum of flags, i.e. an element of $Fl(n)$. This action commutes with the action of $G(n)$ on the right of $Fl(n)$, and in fact $H(n)$ is the algebra of $G(n)$-endomorphisms of $Fl(n)$.

The structure of $H(n)$ was studied 30 years ago by P. N. Norton, a student of Roger Carter at Warwick. She obtained explicit decompositions of $H(n)$ as a direct sum of 2^{n-1} indecomposable left ideals, showed that all irreducible representations of $H(n)$ are 1-dimensional, and computed the Cartan matrix. From this we can obtain corresponding information about $Fl(n)$, i.e.

$$Fl(n) \cong \bigoplus_J Fl_J(n), \quad J \subseteq \{1, 2, \ldots, n-1\},$$

where the summands $Fl_J(n)$ are indecomposable $\mathbb{F}_q G(n)$-modules. For example, for $n = 3$ and $q = 2$ we have 4 summands

$$\begin{array}{ccc} & St & \\ & \oplus & \\ V & & V^* \\ | & \oplus & | \\ V^* & & V \\ & \oplus & \\ & T & \end{array}$$

where the trivial module T has dimension 1, the natural module V and its dual V^* have dimension 3 and the Steinberg module St has dimension 8. Michael Boardman's 1991 paper on the Peterson problem discusses this case in detail. When $n = 4$ and $q = 2$, the 8 summands have dimension 1, 14, 34, 14, 56, 76, 56 and 64, which fits well with Kameko's computed data on the Peterson problem.

9 Embedding the summands of $Fl(n)$

The decomposition of $Fl(n)$ and computations for $q = 2$ and $n \leq 4$ encourage us to try to extend our results to other degrees d.

Two summands of $Fl(n)$ are irreducible, namely $St(n)$, which corresponds to $J = \{1, 2, \ldots, n-1\}$, and the trivial module, which corresponds to $J = \emptyset$.

Conjecture 9.1 *Let $J \subseteq \{1, 2, \ldots, n-1\}$ and let λ be a partition such that*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 1, & \text{if } i \in J, \\ 2, & \text{if } i \notin J. \end{cases}$$

Then ϕ^λ is injective on $Fl_J(n)$.

These conditions describe 'generic' cases for injectivity of ϕ^λ on $Fl_J(n)$. There are also 'non-generic' cases where this happens. For example, the trivial module appears for $\lambda = (0)$.

The proposed line of proof for this conjecture is simpler in principle than the argument for Theorem 2.2 sketched above. The plan is to show that the socle of each summand $Fl_J(n)$ is irreducible, with a generator s_J given by an element of $H(n)$, regarded as the corresponding formal sum of flags in $Fl(n)$). One would then prove that $\phi^\lambda(s_J) \neq 0$ by identifying its leading monomial in left lexicographic order. These monomials appear to be determined combinatorially by J and λ in a simple way. For example in the Steinberg case $J = \{1, 2, \ldots, n-1\}$ they are the spikes, and for the generic case $J = \emptyset$ its block

$$\begin{matrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & \\ 1 & 1 & 0 & 1 & & \\ 1 & 1 & 1 & & & \end{matrix},$$

shown for $n = 4$, has $1, 2, \ldots, n-1$ 0's in alternate antidiagonals.

The case $q = 2$ is exceptional in that all the irreducible $G(n)$-modules appear in the socle of $Fl(n)$. This suggests that the Peterson problem is harder for $q > 2$, as is also clear by studying Crossley's work for $n = 2$. Even for $q = 2$, the upper bound given by Kameko's conjecture appears to be out of reach in the 2-overlap case. However, there appears to be enough that is systematic about the Peterson problem to provide an incentive for future work.