1. Use the Remainder Theorem to prove that for any field F

   (i) $x^n - y^n$ is divisible by $x - y$ in $F[x, y]$ for $n \geq 1$,

   (ii) $x^3 + y^3 + z^3 - 3xyz$ is divisible by $x + y + z$ in $F[x, y, z]$.

   Express $2x^3 + 3x^2y - 3xy^2 - 2y^3$ as a product of three linear factors.

2. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$. Show that the quotient

$$s_\lambda = \begin{vmatrix} x^{\lambda_1+2} & x^{\lambda_2+1} & x^{\lambda_3} \\ y^{\lambda_1+2} & y^{\lambda_2+1} & y^{\lambda_3} \\ z^{\lambda_1+2} & z^{\lambda_2+1} & z^{\lambda_3} \end{vmatrix} \Bigg/ \begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix}$$

   is a symmetric polynomial in x, y, z. (It is called the **Schur function** associated to λ.) Show that $s_{(1,0,0)} = e_1$ and $s_{(1,1,1)} = e_3$, and evaluate $s_{(2,1,0)}$.

3. Using Lex order with $x > y > z$, show that the leading monomial of s_λ is $x^{\lambda_1} y^{\lambda_2} z^{\lambda_3}$. Deduce that the Schur functions s_λ form a vector space basis for the ring of symmetric functions S.

4. In the ring of coinvariants $C_4 = F[x, y, z, t]/I$, write down the relations in C_4 corresponding to the Gröbner basis for I given in Theorem 15.1. Write out the Artin basis and determine the number of elements in each degree.

   Express z^2 and z^3 in terms of the Artin basis. By multiplying your formula for z^3 by z, verify that $z^4 = 0$ in C_4.

5. Let α, β and γ be the complex roots of the cubic equation

$$x^3 + bx + c = 0,$$

   so that $\alpha + \beta + \gamma = 0$. Show that the symmetric polynomial

$$\Delta^2 = (\alpha - \beta)^2 (\alpha - \gamma)^2 (\beta - \gamma)^2$$

   is given in terms of the elementary symmetric functions in α, β and γ by

$$\Delta^2 = -4e_2^3 - 27e_3^2.$$

   Hence show that $x^3 + bx + c = 0$ has equal roots if and only if

$$(c/2)^2 = -(b/3)^3.$$