MT30512 Polynomials   Examples 6

The questions are intended to be done by computer, using **Mathematica**. No computer lab classes will be held, so you should ask for help, if necessary, in the examples classes. After starting **Mathematica**, simply type the command

$$\textbf{GroebnerBasis}[\{x^3 + y^3, x^4 + y^4\}]$$

followed by 'shift+enter'. The program should respond with

$$\{y^4,\ xy^2 - y^3,\ x^2 + y^2\}.$$

The default ordering is Lexicographic with $x > y$. Use the Help system to find out how to use other orderings and options.

1. Let $I = \langle x^3y + xy^2 - x, x^2y^2 - y^3\rangle$ in $\mathbf{Q}[x, y]$. Show that

   (i) for Lex with $x < y$, $\{2x^5 - x, x^3 - xy, x^2 - 2y^3\}$ is a Gröbner basis;

   (ii) for Lex with $y < x$, $\{x^2 - 2y^3, 2xy^2 - x, 2y^5 - y^3\}$ is a Gröbner basis.

   Hence write down generators $f(y)$ for $I \cap \mathbf{Q}[y]$, and $g(x)$ for $I \cap \mathbf{Q}[x]$.

2. **(i)** Find a Gröbner basis of

   $$I = \langle x + y + z - 1,\ x^2 + y^2 + z^2 - 3,\ x^3 + y^3 + z^3 - 4\rangle$$

   in $\mathbf{R}[x, y]$ for Lex order with $x > y > z$.

   (ii) Use Mathematica to find the normal form of $x^4 + y^4 + z^4$ with respect to I, using the same monomial order.

   (iii) If $x, y, z \in \mathbf{C}$ satisfy the system of equations

   $$\begin{aligned} x + y + z &= 1, \\ x^2 + y^2 + z^2 &= 3, \\ x^3 + y^3 + z^3 &= 4, \end{aligned}$$

   show that $x^4 + y^4 + z^4 = 7$. Evaluate also $x^5 + y^5 + z^5$.

3. Compute a Gröbner basis for $I = \langle x^5 + y^4 + z^3 - 1,\ x^3 + y^2 + z^2 - 1\rangle$ using Lex, DegLex and DegRevLex orders with $x > y > z$. Repeat with $z > y > x$. What differences do you find in the results?

1. $f(y) = 2y^5 - y^3$ generates $I \cap \mathbf{Q}[y]$, and $g(x) = 2x^5 - x$ generates $I \cap \mathbf{Q}[x]$.

2. **(i)** **Mathematica** should give the reduced Gröbner basis $G = \{g_1, g_2, g_3\}$ where

$$g_1 = x + y + z - 1,\ g_2 = y^2 + yz - y + z^2 - z - 1,\ g_3 = z^3 - z^2 - z.$$

Note that **Mathematica** uses its internal syntax to output the polynomials, i.e. they are not put into standard form for the given monomial ordering.

(ii) The command **PolynomialReduce** produces the partial quotients and remainder. The syntax is

$$\textbf{PolynomialReduce}[f, \{g_1, g_2, g_3\}, \{x, y, z\}]$$

where $f = x^4 + y^4 + z^4$ is the polynomial to be reduced. We have $f = q_1 g_1 + q_2 g_2 + q_3 g_3 + r$ where (converting to standard form)

$$\begin{aligned} q_1 &= x^3 - x^2y - x^2z + x^2 + xyz - xy - xz + 2x + yz - 2y - 2z + 3, \\ q_2 &= x^2 - xz + x + y^2 - yz + y - 2z + 4, \\ q_3 &= x + y + z + 3, \\ r &= 7. \end{aligned}$$

(iii) The last output, 7, is the remainder, showing that the given equations imply that $x^4 + y^4 + z^4 = 7$. Similarly, we find $x^5 + y^5 + z^5 = 11$.

(It is not hard to find the solutions explicitly in this example, by solving $g_3 = 0$ for z and using symmetry in x, y, z.)

3. With $x > y > z$, the Gröbner basis for Lex order is horrendous! Using DegLex gives a considerable improvement, while DegRevLex is better still. You should obtain a Gröbner basis consisting of four polynomials in this case.

Reversing the order of the variables also gives simpler Gröbner bases. In particular, you should find that DegRevLex gives a Gröbner basis consisting of the two polynomials

$$g_1 = -1 + x^3 + y^2 + z^2,\ g_2 = -1 + x^2 - x^2y^2 + y^4 - x^2z^2 + z^3.$$

You should check this by hand, in order to confirm your complete mastery of the subject of Gröbner bases!