1. **(i)** Find a Gröbner basis of $I = \langle x^3 + y^3,\ x^4 + y^4 \rangle$ for Lex order with $x > y$.

   (ii) Write down a basis for the monomial ideal $\langle \mathrm{LT}\,(I) \rangle$. Hence find a basis for the quotient ring P/I as a vector space over $\mathbf{Q}$, where $P = \mathbf{Q}[x, y]$.

   (iii) Show that $x^5 + y^5 \in I$.

   (iv) Find the smallest value of k such that $x^k \in I$, and the smallest value of k such that $(x + y)^k \in I$.

2. Using DegLex order in $P = \mathbf{Q}[x, y]$ with $x > y$, find a reduced Gröbner basis for the ideal $I = \langle g_1, g_2 \rangle$, where

$$g_1 = x^3 - 2xy, \quad g_2 = x^2y - y^2.$$

   Show that every polynomial $f \in P$ can be written uniquely in the form

$$f = g + r$$

   where $g \in I$ and $\deg(r) \leq 2$. Hence write down a basis for the quotient ring P/I as a vector space over $\mathbf{Q}$.

3. Each of the following sets is a Gröbner basis for Lex order with $x > y > z$ in $\mathbf{Q}[x, y, z]$. Find a reduced Gröbner basis in each case.

   (i) $G = \{x^2 - xz^2,\ y^2 - yz^3,\ z^3 - z\}$;

   (ii) $G = \{x + y + z,\ y^2 + yz + z^2,\ z^3 - 1,\ xy + yz + xz,\ xyz - 1\}$;

   (iii) $G = \{x + yz^2,\ xz + y,\ yz^3 - y\}$.

4. Let J be the ideal $\langle xy + x + y^2,\ y^2 - 2y - 2 \rangle$ in $\mathbf{Q}[x, y]$. Show that

   (i) $J \cap \mathbf{Q}[y]$ is generated by $y^2 - 2y - 2$;

   (ii) $J \cap \mathbf{Q}[x]$ is generated by $x + 2$.

5. Using Lex order with $x > y$, find a Gröbner basis for the intersection $I \cap J$ of the ideals $I = \langle x, y^3 \rangle$ and $J = \langle x + y \rangle$ in $\mathbf{Q}[x, y]$.