MT30512 Polynomials Examples 2

Please note: Examples and solutions will be circulated in examples classes. They will also be posted on the web at

`http://www.maths.manchester.ac.uk/~grant`

at appropriate times. The course notes (for the whole semester) are also posted there. **The class test will be held at 1.00pm on Friday 9 March.**

Notation: $\mathbf{Q}$ is the field of rational numbers; $\mathbf{Z}$ is the ring of integers; F is an arbitrary field; $R[x]$ and $R[x, y]$ (etc) are the polynomial rings in variables x, x and y (etc) with coefficients in the ring R. The ring R will always be a commutative ring with an identity element denoted by 1. Recall that R is a field iff every nonzero element $a \in R$ has a multiplicative inverse a^{-1}, i.e. $a \cdot a^{-1} = a^{-1} \cdot a = 1$.

1. A theorem proved in class states that when the coefficient ring R is a field, every ideal in the polynomial ring $R[x]$ is a principal ideal.

   Give an example of an ideal in $R[x]$ which is **not** a principal ideal, in the case where $R = \mathbf{Z}$. (Hint: what are the ideals in $\mathbf{Z}$?)

2. Which of the following polynomials belong to the monomial ideal $I = \langle x^4, x^2y, y^2\rangle \subset \mathbf{Q}[x, y]$?

   (i) $(x + y)^4$;

   (ii) $(1 + x + y)^4$;

   (iii) $xy(1 + x + y)^2$.

3. Make a list of all monomials which do **not** belong to the monomial ideal I of Question 2. Hence write down a basis for the quotient ring $F[x, y]/I$ as a vector space over F. What is the dimension of this vector space?

4. In $F[x, y]$, let
   $$I = \langle x^4y^2, x^3y^4\rangle, \ J = \langle x^3y^4, x^2y^5\rangle.$$
   Find minimal generating sets for the ideals
   $$I + J, \ IJ \text{ and } I \cap J.$$

5. Determine whether the ideals I and J of $\mathbf{Q}[x, y]$ are equal, where

   (i) $I = \langle xy + y^2, x^2 + xy + y^2, x^2 + y^2\rangle$,

   (ii) $I = \langle x^2 + xy, xy - y^2, x^2 + y^2\rangle$,

   and, in both cases, $J = \langle x^2, xy, y^2\rangle$.