We will use the notation $\mathbf{Q}$ for the field of rational numbers, $\mathbf{R}$ for the field of real numbers and $\mathbf{C}$ for the field of complex numbers. For most of our work, the choice of coefficient field F will not matter very much, but it is important to be aware from the start that some things depend very strongly on the choice of coefficients. In this example sheet, notice that division with remainder works in exactly the same way over any field, while factorisation can behave in quite different ways over different fields. To emphasise this, some examples with the finite field $\mathbf{F}_p$ of remainders mod a prime number p have been included, but you will not be missing anything essential if you leave these out!

1. In each of the following cases, given the field F and $f, g \in F[x]$, find $q, r \in F[x]$ such that $f = qg + r$ and $\deg r < \deg g$ or $r = 0$.

   (i) $f = x^6 - x^5 + 3x^2 - 1$, $g = x^2 + x + 1$,

   (ii) $f = x^6 - x^5 + 3x^2 - 1$, $g = x^3 + x + 1$.

2. Repeat Question 1 for the following examples over finite fields. Interpret the coefficients as remainders mod the given prime, for example, over $\mathbf{F}_5$ we can write $-3 = 2$, $5 = 0$ etc.

   (i) $f = x^5 - 3x^4 + 2x^3 - x^2 + 4$, $g = x^2 + 2x + 1$, $F = \mathbf{F}_5$,

   (ii) $f = x^5 + x^4 + x^3 + x^2 + 1$, $g = x^2 + x + 1$, $F = \mathbf{F}_2$.

3. A polynomial is said to be **irreducible** over a field of coefficients F if it is not constant (i.e. not just an element of F) and if it cannot be factored as the product of two polynomials of lower degree with coefficients in F.

   - Write down necessary and sufficient conditions for the polynomial $ax^2 + bx + c \in \mathbf{R}[x]$ to be irreducible over $\mathbf{R}$. (Be careful to include all cases, e.g. $a = 0$.)
   - Explain why there are no irreducible polynomials of degree 3 over $\mathbf{R}$. Are there any of degree 4?

4. Factor **(i)** $x^4 + 1$ into irreducible polynomials over $\mathbf{C}$, $\mathbf{R}$ and $\mathbf{Q}$.

   Repeat for **(ii)** $x^4 + 2$ and **(iii)** $x^4 + 4$.

   Finally, if you really want to impress your friends, factor these polynomials over the field of 3 elements, $\mathbf{F}_3$.