Heavy Traffic and Heavy Tails for Subexponential Distributions

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Abstract. Consider a family of random walks $S_n^{(a)} = X_1^{(a)} + \cdots + X_n^{(a)}$ with negative drift $E S_1^{(a)} = -a < 0$ and $E |S_1^{(a)}|^{2+\varepsilon} < \infty$ for some $\varepsilon > 0$. Let $M^{(a)} = \max_{n \geq 0} S_n^{(a)}$ be the maximum of the random walk. It is known that the probability $P(M^{(a)} > x)$ decays exponentially fast as $a \to 0$ (heavy traffic asymptotics, see e.g. [10] and [16]) and, for subexponential distributions $P(M^{(a)} > x)$ decays according the the integrated tail as $x \to \infty$ (heavy-tail asymptotics, see [17]). This paper presents a link between these two asymptotics and studies the probability $P(M^{(a)} > x)$ as $a \to 0$ for $x = x_a \to \infty$ as $a \to 0$ and identifies the regions of $x$ for which the heavy traffic asymptotics and the heavy tail asymptotics hold. Further, the distributions for which an intermediate zone between these two limits exists are identified and the exact limit in this zone is provided. Our approach is not based on an approach via geometric sums, like most of the results on the behaviour of $P(M^{(a)} > x)$ are. Instead we use martingale arguments and inequalities.

1. Introduction and statement of the results

Let $\{S_n^{(a)}, n \geq 0\}, a \in [0,a_0]$ with $a_0 > 0$ denote a family of random walks with increments $X_i^{(a)}$ and starting point zero, that is,

$$S_0^{(a)} := 0, \quad S_n^{(a)} := \sum_{i=1}^{n} X_i^{(a)}, \quad n \geq 1.$$  

We shall assume that $X_1^{(a)}, X_2^{(a)}, \ldots$ are independent copies of a random variable $X^{(a)} = X - a$ with distribution function $F$, negative drift $a := -E X^{(a)} > 0$, $E |X^{(a)}|^{2+\varepsilon} < \infty$ and denote $\sigma^2 = \text{Var} X^{(a)}$. The random walk $S_n^{(a)}$ drifts to $-\infty$ as $n \to \infty$ and the total maximum $M^{(a)} := \max_{k \geq 0} S_k^{(a)}$ is finite almost surely. However, $M^{(a)} \to \infty$ in probability as $a \to 0$.

The main motivation for studying the random variable $M^{(a)}$ comes from queueing theory, since its distribution coincides with the stationary distribution of the queue-length in a G/G/1 queue. Another important application comes from the insurance mathematics: Under some special restrictions on $X^{(a)}$ the quantity $P(M^{(a)} > u)$ is equal to the ruin probability in the so-called renewal arrivals model.

The asymptotic tail behaviour of the maximum of the random walk $S_n^{(a)}$ has been studied extensively in the literature. The first result goes back, apparently,
to Cramér and Lundberg (see, for example, Asmussen [1]). If $a$ is fixed,

$$E[e^{h_0 X^{(a)}}] = 1$$

for some $h_0 > 0$, \hspace{2cm} (1)

and, in addition, $E[X^{(a)} e^{h_0 X^{(a)}}] < \infty$, then there exists a constant $c_0 \in (0, 1)$ such that

$$P(M^{(a)} > x) \sim c_0 e^{-h_0 x} \text{ as } x \to \infty.$$ \hspace{2cm} (2)

If (1) is not fulfilled, i.e. $E[\exp(\epsilon X)] = \infty$ for all $\epsilon > 0$, then one should assume that the distribution of $X^{(a)}$ is regular in some sense. To specify what regular means we recall some definitions and known properties. For their proofs we refer to Foss, Korshunov and Zachary [9]. We assume that the distribution function $F$ is subexponential, that means $F$ has support $\mathbb{R}_+$, $F(x) > 0$ for all $x$ and

$$\lim_{x \to \infty} \frac{F^{*n}(x)}{F(x)} = n$$

for all $n \geq 2$, where $F^{*n}(x)$ is the $n$-fold convolution of $F$ with itself. For the subexponentiability it is sufficient to verify the equation (3) in the case $n=2$. All subexponential distributions are heavy-tailed.

For fixed $a$ a classical result concerning the maximum of a random walk for subexponential distributions is due to Veraverbeke [17], who showed that if the integrated tail of $F$, that is $F^I(x) = \int_x^\infty F(u)du$ with $F(x) = 1 - F(x)$, is subexponential, then

$$P(M^{(a)} > x) \sim a^{-1} F^I(x) \text{ as } x \to \infty.$$ \hspace{2cm} (4)

Let us turn now to the case of $a \to 0$. This case is very interesting from the point of view of queueing theory as it describes the behaviour of a system in heavy traffic. There is a large volume of literature examining this case. These studies were initiated by Kingman [10], who considered the case when $|X^{(a)}|$ has an exponential moment, proved that for fixed $x$,

$$P(M^{(a)} > x/a) \sim e^{-2x/\sigma^2} \text{ as } a \to 0.$$ \hspace{2cm} (5)

Prohorov [16] extended this result to the case that the increments have finite variance.

One can see that (5) has a form similar to (2). Indeed, if Cramer condition (1) holds then letting $a \to 0$ one can see that $h_0 \to 0$ and in the limit (2) becomes (5). However it is not immediately clear what happens when the Cramer condition does not hold and the distribution of $X$ is heavy-tailed. The problem is the following: the heavy traffic theory predicts an exponential decay, whereas the heavy traffic asymptotics predicts a decay according to the integrated tail of the distribution. This fact raises an interesting mathematical issue, how exponential asymptotics turn into the integrated tail asymptotics.

Suppose we let $a \to 0$ and $x \to \infty$ simultaneously. On one hand, if $a \to 0$ much faster than $x \to \infty$, probability $P(M^{(a)} > x)$ should still behave like in the heavy traffic approximation (5). On the other hand, if $a \to 0$ much slower than $x \to \infty$, the heavy tail approximation from (4) should still be valid. In particular, the question is whether there exists a transition point, at which the transition from (5) to (4) takes place. Or, otherwise, the question is whether there is a region in which neither the heavy traffic nor the heavy tail asymptotics holds and what the asymptotical behaviour of $P(M^{(a)} > x)$ will be like in this region. Answers to these questions depend on the distributions of the increment of the random walk.
Kugler and Wachtel \cite{12} and Kugler \cite{13} have shown that if the increments are regularly varying of index \( r > 2 \), that is,
\[
P(X > u) = u^{-r} L(u)
\]
where \( L \) is a slowly varying function, then there exists a sharp transition point
\[
x_{RV}(a) \approx \frac{\sigma^2 (r - 2)}{2 a} \log \frac{1}{a}. \tag{6}
\]
This generalized a result from Olvera-Cravioto, Blanchet and Glynn \cite{14}, who derived this critical value in the setting of an M/G/1 queue. Kugler and Wachtel \cite{12} showed also that if only \( \mathbb{E}(\min\{X, 0\})^2 < \infty \), but the variance is in general not finite, i.e \( r \in (1, 2) \), then the heavy tail approximation (4) holds for all \( x/x_{RV}(a) > 1 + o(1) \).

If the increments have Weibull distribution, that is \( P(X > u) = e^{-u^\gamma}, \gamma \in (0, 1) \), one could believe that there is still a sharp transition point. Equating the right hand sides of (4) and (5), one could guess that the critical point would be
\[
x_W(a) \approx \left( \frac{\sigma^2}{2a} \right)^{1/(1-\gamma)} - \frac{\sigma^2}{2a(1-\gamma)} \log \frac{1}{a}. \tag{7}
\]
There are also some recent results that coped with the case of Weibull-type distributions, but only in the case of an M/G/1 queue. In \cite{15}, Olvera-Cravioto and Glynn conjecture that for Weibull type distributions there is a third region in which neither the heavy traffic nor the integrated tail asymptotic is valid if and only if \( 1/2 < \gamma < 1 \). This is not the case and surprisingly this third region exists for a larger region, that is for all \( \gamma \in (0, 1) \), see Examples below. Another remarkable result, which covers various subexponential distributions (including Regular varying and Weibullian), is due to Blanchet and Lam. They have recently found a uniform, explicit representation for the probability \( P(M(a) > x) \), which consists of the exponential term from Kingman’s asymptotics, the integrated tail term and a convolution term. For further discussion of this result see Remark 4 and \cite{13}. The reason why all these results only work in the setting of an M/G/1 queue is that their approach is based on the representation of \( M(a) \) as a geometric sum of independent random variables:
\[
P(M(a) > x) = \sum_{k=0}^{\infty} q(1-q)^k P(\chi_1^+ + \chi_2^+ + \cdots + \chi_k^+ > x),
\]
where \( \{\chi_i^+\} \) are independent random variables and \( q = P(M = 0) \). The main difficulty in this approach is the fact that one has to know the distribution of \( \chi_i^+ \) and the parameter \( q \). However, the distribution of \( q \) and \( P(\chi_i^+ > x) \) is only known if the ladder epochs of the random walk are exponentially distributed, which corresponds to the case of an M/G/1 queue. But in general one has to obtain appropriate estimates for \( q \) and \( P(\chi_i^+ > x) \). Therefore this approach may be unsuitable for general distribution of the ladder epochs (which corresponds to the case of a G/G/1 queue). In the present work we use a different approach which relies on martingale methods. Appearance of martingales is due to the equation \( (M(a) + X(a))^+ \overset{d}= M(a) \).
Before we state our main result we define two sequences \( \theta_a \) and \( t(a) \) and introduce assumptions on the distribution of \( X \). Let \( \theta_a \) be the solution to the equation

\[
E \left[ e^{\theta_a X(a)} ; X(a) \leq 1/a \right] = 1.
\] (8)

One can easily see by Taylor expansion, that \( \theta_a \sim 2a/\sigma^2 \) as \( a \to 0 \). Assume, as \( x \to \infty \),

\[
F(x) \sim e^{-g(x)}.
\] (9)

Without loss of generality we can assume that \( g \) is differentiable and positive. Further, assume that there exists some \( \gamma_0 < 1 \) such that

\[
g(x) x^{\gamma_0} \downarrow, \quad x \geq x_0,
\] (10)

for some \( x_0 \) large enough. The possible choices of \( \gamma_0 \) depend on the exact definition of \( g \) and since the minimal choice of \( \gamma_0 \) will play an important role in our results we further define

\[
\gamma^* = \inf \{ \gamma > 0 \mid \exists x_0 : g(x)/x^\gamma \text{ is decreasing for all } x \geq x_0 \} \in [0, \gamma_0].
\] (11)

If \( g(t) \gg t^{\varepsilon_1} \) for some \( \varepsilon_1 > 0 \), one can equivalently define \( \gamma^* \) by

\[
\gamma^* = \sup \{ \gamma > 0 \mid \exists x_0 : g(x)/x^\gamma \text{ is increasing for all } x \geq x_0 \}.
\] (12)

The boundary sequence \( x(a) \) is introduced in the following Lemma.

**Lemma 1.** There exists an increasing solution \( x(a) \geq 1/a \) to the equation

\[
\theta_a x - g(x) - \ln(a \theta_a) = o(1)
\] (13)

for and \( 0 < a < a_0 \), where \( a_0 \) is sufficiently small. For this solution and any other \( \tilde{x}(a) = x(a) + o(1/a) \),

\[
e^{-\theta_a x} \sim a^{-1} \int_x^\infty e^{-g(u)} du.
\] (14)

This solution is asymptotically unique in the sense that any other solution \( \tilde{x}(a) \) satisfies \( \tilde{x}(a) = x(a) + o(1/a) \) as \( a \to 0 \).

Let \( c \) be a positive number,

\[
G_c(x) = \begin{cases} 1, & \text{if } x \leq 0, \\ \exp\{-(\theta_a + c)x\}, & \text{if } x > 0 \end{cases}
\] (15)

and define further,

\[
\hat{G}_c(x) = \begin{cases} \frac{1}{\alpha} \tilde{F}(x), & \text{if } x \geq \delta x(a), \\ 0, & \text{if } x < \delta x(a) \end{cases}
\] (16)

Note that \( \hat{G}_c(x) = 0 \) rather then 1 as in [5]. This is due to the fact that we are going to consider the sum of two functions and 1 will come from another term as \( e^{-2a^2z/\sigma^2} \) is approximately 1 for small \( a \) and \( z \) in a fixed interval. Further, define for some \( \alpha > 0 \)

\[
\tilde{G}_c(x) = \begin{cases} e^\alpha, & \text{if } x \leq 0, \\ \exp\{-(\theta_a + c)x\}, & \text{if } x > 0 \end{cases}
\] (17)
**Proposition 2.** Suppose $E|X|^2 < \infty$ for some $\varepsilon > 0$. Assume that $x = x(a)$ is such that $g(x)/\theta_x$ converges as $a \to 0$. If $g(t) \gg t^{\varepsilon_1}$ for some $\varepsilon_1$, also assume that $E \min\{0, X\}^{1+1/(1-\gamma)} < \infty$. Let $\mu_y = \inf\{k \geq 1 : S_k \geq y\}$ and

$$L = \begin{cases} 
1, & \text{if } g(t) \ll t^{\varepsilon_1} \forall \varepsilon_1 > 0, \\
\lim_{a \to 0} \left( \frac{1}{1-g(a)/(\theta_x)} \right)^2, & \text{if } \exists \varepsilon_1 > 0 : g(t) \gg t^{\varepsilon_1}.
\end{cases} \quad (18)$$

Suppose $\delta \in (0, 1)$ small enough and $c_\alpha = 1/(\ln(1/a)t(a))$. Then,

$$Y_n^{(1)} = G_{-c_\alpha}(x - S^{(a)}_{n\wedge \mu_x}) + L\hat{G}_{1-\varepsilon}(x - S_{n\wedge \mu_x - \varepsilon(x)}) \quad (19)$$

is a nonnegative supermartingale for all a small enough. On the other hand,

$$Y_n^{(2)} = \hat{G}_{c_\alpha}(x - S^{(a)}_{n\wedge \mu_x}) + L\hat{G}_{1+\varepsilon}(x - S_{n\wedge \mu_x + \varepsilon(x)}) \quad (20)$$

with $\alpha > 0$ arbitrary is a nonnegative submartingale for all a small enough.

With these super- and submartingales we can derive subexponential asymptotics for the maximum of the random walk.

**Theorem 3.** Suppose $E|X|^2 < \infty$ for some $\varepsilon > 0$. If $g(t) \gg t^{\varepsilon_1}$ for some $\varepsilon_1$, also assume that $E \min\{0, X\}^{1+1/(1-\gamma)} < \infty$. Then,

$$P(M^{(a)} > x) \sim e^{-\theta_x x} + \frac{L}{a} F'(x) \quad (21)$$

Since $L \equiv 1$ for all $g$ such that $g(t) \ll t^{\varepsilon_1}$ for all $\varepsilon_1 > 0$, this theorem implies that the intermediate region, in which neither the heavy traffic nor the heavy tail asymptotic is valid, appears if and only if the right tail of the increments decreases at least as fast as it does for the Weibull distribution, that is

$$F(x) \ll e^{-x^{\varepsilon_1}}$$

for some $\varepsilon_1 > 0$. Consequently, there is no intermediate region for Regularly varying and Lognormal-type tail distributions. However, for those $x$ such that

$$\frac{1}{a} F'(x) \asymp e^{-\theta_x x},$$

there is some mixing of the two terms.

**Remark 4.** Blanchet and Lam [2] have shown that if the ladder epochs of the random walk are exponentially distributed (which corresponds to the case of and M/G/1 queue), then, as $a \to 0$,

$$P(M > x) \sim e^{-\theta_x x} + \left( \frac{F'(x)}{a} + \int_{1/a}^x \left( \frac{1}{a} + 2(x - y) \right) e^{-\theta_x(x-y)} dF'(y) \right) 1\{x \geq \frac{1}{a}\}. \quad (22)$$

One can show with very similar techniques as used in the proof of Proposition 2, that if the tails possesses a Weibull distribution, or even more general, a semiexponential distribution, then the integral term from (22) is exactly $(L/a) F'$. For details, see [13]. The only reason that we need a specific assumption on the distribution when comparing (22) and (21) is to compare $F'$ with $F$. That means that if one assumes that $F'(x) = e^{-g(x)}$ instead of $F'(x) = e^{-g(x)}$, one can easily reestablish our result (21) from (22) for a general subexponential distribution.
For the proof of Proposition 2 and Theorem 3 it turns out that it is not necessary to have an exact solution in (8), but it is sufficient to have an equation of the form
\[ E\left[e^{\theta X^{(a)}} ; X^{(a)} \leq 1/a \right] = 1 + o(a c), \] (23)
where \( c = o(a) \) reflects the required precision and depends on the distribution of the right tail. It turns out that a suitable choice is \( c = 1/(x(a) \ln(1/a)) \). The definition of \( x(a) \) implies that \( c \gg a^k \) for \( k \) large enough, that means \( k^* := \max\{k \in \{1, 2, \ldots\} : c \ll a^k\} \) is well defined. Then, by considering an asymptotic equation of the form (23) one can pick a solution \( \theta \) of the form
\[ \theta = 2a + C_2 a^2 + \cdots + C_{k^*} a^{k^*}. \] (24)
The expansion is valid up to the order of the moment existence of \( X^{(a)} \) and the constants \( C_2, C_3, \ldots, C_{k^*} \) can be defined by expansion and using these moments. For Weibulls with parameter strictly less than 1/2, it is \( \theta = (2ax(1))/\sigma^2 + o(1) \), see Example 4, so the result from (21) simplifies to
\[ P(M^{(a)} > x) \sim e^{-2ax}/\sigma^2 + \frac{L}{a} F(x). \]
For even lighter tails one needs more moments to expand \( \theta \) like in (24).

**Remark 5.** The assumption \( X^{(a)} = X - a \) is needed only for the expansion of \( \theta \) and can be generalized to the assumption of moment equivalence, that is \( \lim_{a \to 0} E(X^{(a)})^k = E X^k \). If the tails decay slower than \( e^{-\sqrt{x}} \) we only need to assume \( \lim_{a \to 0} E(X^{(a)})^2 = \sigma^2 \). For increments of type \( X - a \) we always have an expansion in powers, so the coefficient will change but still depend only on first cumulants.

1.1. **Examples.** Let us consider different kinds of distribution functions and outline the result that Theorem 3 gives for these distributions. Especially, we shall see that - depending on the distribution - there may be a mixing area around the transition zone, in which the order of the exponential term and the order of the integrated tail term are the same.

**Example 1: Regular variation.** Suppose the right tail of \( X^{(a)} \) is regularly varying with index \( r > 2 \), that is \( g(x) = r \ln x - \ln L_1(x) \), where \( L_1 \) is some slowly varying function. Then, Kugler and Wachtel [12] (see also Olvera-Cravioto, Blanchet and Glynn [14]) have shown that the transition point is
\[ x_{RV}(a) \approx \frac{(r-2)\sigma^2}{2a} \ln \frac{1}{a}. \]
Since \( \theta \) possesses the representation \( \theta = 2a/\sigma^2 + O(a) \) it is \( \theta x = 2ax/\sigma^2 + o(1) \) for \( x \) such that \( x/x_{RV}(a) = O(1) \). On the other side, it is known that \( F'(x) \sim x^{1-r} L_1(x)/(r-1) \) for all \( x \) such that \( x \to \infty \) as \( a \to 0 \) and obviously \( F'(x)/a \gg e^{-2ax}/\sigma^2 \) for all \( x \gg x_{RV}(a) \). Hence, Theorem 3 can be rewritten as
\[ P(M^{(a)} > x) \sim e^{-2ax}/\sigma^2 + \frac{x^{1-r} L_1(x)}{(r-1)a} \] (25)
One can easily that \( e^{-2ax}/\sigma^2 \gg F'(x)/a \) if \( x < c x_{RV}(a)(1 + o(1)) \) with \( c < 1 \) and that \( e^{-2ax}/\sigma^2 \ll F'(x)/a \) if \( x > c x_{RV}(a)(1 + o(1)) \) with \( c > 1 \). In the case \( c = 1 \), it depends on the exact localization of \( x \), the order of the slowly varying function \( L_1 \)
and the exact value of \( x \), whether the exponential term or the integrated tail term dominates or if they even have the same order. Hence, Theorem 1 states that

\[
P(M^{(a)} > x) = \begin{cases} 
  e^{-2ax/\sigma^2} & : \lim_{a \to 0} x/x_{RV}(a) < 1 \\
  \frac{1}{a} F'(x) & : \lim_{a \to 0} x/x_{RV}(a) > 1 \\
  e^{-2ax/\sigma^2} + \frac{1}{a} F'(x) & : \lim_{a \to 0} x/x_{RV}(a) = 1 
\end{cases}
\] (26)

Let us discuss the region \( \lim_{a \to 0} x/x_{RV}(a) = 1 \) a little bit more. It is easy to see that \( F'(x) \sim F'(x_{RV}(a)) \) in this region and for \( x = x_{RV}(a) + O(1/a) \), \( e^{-2ax/\sigma^2} \sim e^{-2ax_{RV}(a)/\sigma^2} \). On the other side, \( F'(x_{RV}(a))/a \gg e^{-2ax_{RV}(a)/\sigma^2} \) for \( L_1(x) \gg (\ln(x))^{r-1} \) and \( F'(x_{RV}(a))/a \ll e^{-2ax(\sigma)/\sigma^2} \) for \( L_1(x) \ll (\ln(x))^{r-1} \). Hence, if \( x = x_{RV}(a) + O(1/a) \), \( F'(x)/a \gg e^{-2ax/\sigma^2} \) for \( L_1(x) \gg (\ln(x))^{r-1} \) and \( F'(x)/a \ll e^{-2ax/\sigma^2} \) for \( L_1(x) \ll (\ln(x))^{r-1} \).

**Example 2: Pareto distribution.** Suppose the right tail of \( X^{(a)} \) is pareto-distributed with index \( r > 2 \), that is \( g(x) = r \ln x \). This is the same as in Example 1, but with \( L_1(x) \equiv 1 \). One can refine the region \( x \sim x_{RV}(a) \) from Example 1 and show that the critical value is

\[
x_{P}(a) \approx \frac{(r-2)\sigma^2}{2a} \ln(1/a) + \frac{(r-1)\sigma^2}{2a} \ln(1/a)) = t_{P1}(a) + t_{P2}(a).
\]

Hence, Theorem 3 states that

\[
P(M^{(a)} > x) = \begin{cases} 
  e^{-2ax/\sigma^2} & : \lim_{a \to 0} x/x_{P}(a) < 1 \\
  \frac{1}{a} F'(x) & : \lim_{a \to 0} x/x_{P}(a) > 1 \\
  e^{-2ax/\sigma^2} + \frac{1}{a} F'(x) & : \lim_{a \to 0} x/x_{P}(a) = 1 
\end{cases}
\] (27)

In the third region of the latter result there may occur some mixing between the two terms, for example if \( x = \frac{(r-2)\sigma^2}{2a} \ln(1/a) + \frac{(r-1)\sigma^2}{2a} \ln(1/a)) + o(1/a) \), then

\[
P(M^{(a)} > x) \sim (r-1) \left( \frac{(r-2)\sigma^2}{2} \right)^{r-1} \frac{1}{a} F'(x)
\]

**Example 3: Lognormal-type distribution.** Let \( g(x) = r \ln^\beta x \) with \( \beta > 1 \) and \( r > 0 \) such that \( \mathbb{E}|X^{(a)}|^{2+\varepsilon} > \infty \). Then, as one can easily see,

\[
F'(x) \sim \frac{x}{\beta(\ln x)^{\beta-1}} e^{-r \ln^\beta x}
\]

and by equating the integrated tail term and the exponential term it is not too hard to see that the critical value is

\[
x_{LN}(a) \approx \frac{r}{\beta_a} \ln^\beta(1/\theta_a)
\]

Hence, Theorem 3 states that

\[
P(M^{(a)} > x) = \begin{cases} 
  e^{-2ax/\sigma^2} & : \lim_{a \to 0} x/t_{LN}(a) < 1 \\
  \frac{1}{a} F'(x) & : \lim_{a \to 0} x/t_{LN}(a) > 1 \\
  e^{-2ax/\sigma^2} + \frac{1}{a} F'(x) & : \lim_{a \to 0} x/t_{LN}(a) = 1 
\end{cases}
\] (28)

In the region \( x \sim x_{LN}(a) \), one can easily see that for \( x = r \ln^\beta(1/\theta_a)/\theta_a \), \( F'(x)/a \gg e^{-\theta_a x} \) if \( \beta \in (1,2) \), and \( F'(x)/a \ll e^{-\theta_a x} \) if \( \beta \geq 2 \).
Example 4: Weibull distribution. Suppose the right tail of $X^{(a)}$ possesses a Weibull distribution, that is $g(x) = x^\gamma$ with $\gamma \in (0, 1)$. Then, one can easily see by substitution and using asymptotical properties of the incomplete gamma function, that

$$F'(x) \sim \frac{1}{\gamma} x^{1-\gamma} F(x)$$

as $x \to \infty$. With this result one can see by equating the exponential and the integrated tail term that the critical value is

$$x_W(a) = \left(\frac{1}{\theta_a}\right)^{1/(1-\gamma)} - \frac{2}{\theta_a(1-\gamma)} \ln \frac{\sqrt{2(\gamma \sigma^2)}}{\theta_a} \quad (29)$$

By the definition of $x(a)$,

$$e^{-\theta_a x(a)} \sim \frac{1}{a} F'(x(a)) \sim \frac{1}{a} x(a)^{1-\gamma} e^{-x(a)\gamma} \sim \frac{2}{\sigma^2} e^{-x(a)\gamma} \quad (30)$$

and for $z \ll x(a)^{1-\gamma}/2$ it is

$$e^{-(x(a)-z)\gamma} \sim e^{-x(a)\gamma + \gamma z/a} x(a)^{1-\gamma} \quad (31)$$

With these results, one can easily see that

$$e^{-\theta_a x} \gg \frac{1}{a} F'(x)$$

for all $x \leq x(a) - z$ with $1/\theta_a \ll z \ll x(a)$ and

$$e^{-\theta_a x} \ll \frac{1}{a} F'(x)$$

for all $x \geq x(a) + z$ with $1/\theta_a \ll z \ll x(a)$. The relations (30) and (31) imply that for $x = x(a) + K/\theta_a + o(1/a)$, where $K$ is a fixed constant,

$$e^{-\theta_a x} \sim \frac{e^{-K/(1-\gamma)}}{a} F'(x(a)) \sim \frac{e^{-K/(1-\gamma)}}{a} F'(x). \quad (34)$$

Analogously, for $x = x(a) - K/\theta_a + o(1/a)$

$$\frac{1}{a} F'(x) \sim e^{-K/(1-\gamma)} e^{-\theta_a x}. \quad (35)$$

Further, $L \sim 1$ for $x \gg x(a)$ and $L \sim 1/(1-\gamma)$ for $x \sim x(a)$. Combining all the above results we arrive at

$$\mathbb{P}(M^{(a)} > x) \sim \begin{cases} 
  e^{-\theta_a x}, & \text{if } x \ll x(a) - 1/\theta_a \\
  \left(1 + e^{-K/(1-\gamma)}ight) e^{-\theta_a x}, & \text{if } x = x(a) - K/\theta_a + o(1/a), K > 0 \\
  \left(e^{-K/(1-\gamma)} + \frac{1}{1-\gamma}\right) \frac{F'(x)}{a}, & \text{if } x = x(a) + K/\theta_a + o(1/a), K > 0 \\
  \frac{L}{a} F'(x), & \text{if } x \gg x(a) - 1/\theta_a, x \not\sim x(a) \\
  \frac{L}{a} F'(x), & \text{if } x \gg x(a) 
\end{cases} \quad (36)$$

where $K$ is a fixed positive constant.

Example 5: Semiexponential Distributions. Semiexponential distributions are distributions for which the right tail $F(x)$ is of the form

$$F(x) = e^{-x^L L_1(x)}, \quad (37)$$
where $\gamma \in (0,1)$ and $L_1$ is a slowly varying function. Assume without loss of generality that $L_1$ is differentiable. It is known that

$$F'(x) \sim \frac{F(x)}{g'(x)} \sim \frac{F(x)}{e^{\gamma x^{-1}}L_1(x) + x^\gamma L_1'(x)}$$

in this case. One can easily see that for all $L_1$ such that $L_1(x) \to \infty$ as $x \to \infty$,

$$e^{-\theta a x} \gg \frac{1}{\alpha} F'(x)$$

for $x$ such that $x = O((1/\theta a)^{1/(1-\gamma)})$ and

$$e^{-\theta a x} \ll \frac{1}{\alpha} F'(x)$$

for $x \gg (1/\theta a)^C$ with $C > 1/(1-\gamma)$. So the critical value is $(1/\theta a)^{1/(1-\gamma)}$ and Theorem 3 states that

$$P(M^{(a)} > x) \sim \begin{cases} 
  e^{-\theta a x}, & \text{if } x = O((1/\theta a)^{1/(1-\gamma)}) \\
  \frac{1}{\alpha} F'(x), & \text{if } x \gg (1/\theta a)^{\delta+1/(1-\gamma)}.
\end{cases}$$

(38)

for an arbitrary $\delta > 0$.

2. Proofs

2.1. Proof of Lemma 1. We will prove that there exists $x(a)$ such that we have exact equality $\theta a x - g(x) + \ln(a \theta a) = 0$. The latter follows from the continuity of $g(x)$. Indeed on one hand for $x = 1/a$

$$\theta a x - g(x) - \ln(a \theta a) \sim 2/\sigma^2 + 2 \ln(1/a) - g(1/a)$$

$$\leq (1 + o(1))(2 \ln(1/a) - (2 + \varepsilon) \ln(1/a)) < 0,$$

for $a \leq a_0$. Here we used the existence of the $E[|X|^{2+\varepsilon}]$. On the other hand, it follows from (10) that for some $\gamma_0 + \delta < 1$ we have $g(x) = o(x^{\gamma_0+\delta})$. Hence, for

$$x = (1/\theta a)^C,$$

$$\theta a x - g(x) - \ln(a \theta a) \sim \theta a^{1-C} - \theta a^{C(\gamma_0+\delta)} > 0,$$

provided $C > (1-\gamma_0 - \delta)^{-1}$. Asymptotics (14) follows from (13) by the L’Hopital rule.

Moreover, the function $x(a)$ is monotone increasing in $a$. Indeed, using $\theta a = 2a/\sigma^2 + o(a)$

$$(\theta a x - g(x) + \ln a)'_a = (2/\sigma^2 + o(1))x + 1/a \geq 0,$$

which means that when $a$ decreases, $x \theta a - g(x) + \ln a$ is decreasing and since $x \gg g(x)$ for $x \to \infty$ this implies that the solution $x(a)$ to $\theta a x - g(x) + \ln a = 0$ increases.

Suppose now that $x(a)$ is another solution to (13). Let us first consider distribution functions such that $g(x) \ll x_1^e$ for all $e > 0$. Then, there exists some $x_0$ such that

$$\frac{g(x)}{x^{e+1}}, \ x \geq x_0,$$

for all $e > 0$. Since $(1 - w/x)^{e+1/2} \geq 1 - \varepsilon w/x$ for all $w \leq x/2$, it is not too hard to see that

$$g(x) - g(x-w) \leq \varepsilon w \frac{g(x)}{x}, \ w \leq x/2,$$  (39)
for all \(\varepsilon_1 > 0\). It is known that if \(g(x) = (2 + \varepsilon) \ln(x)\),
\[
\theta_a x(a) / \ln(1/a) \neq o(1),
\]
see [12]. Consequently, by additionally regarding (14), the definition of \(x(a)\) and
(24),
\[
g(x(a)) / \theta_a x(a) = 1 + \ln(1/(a\theta_a)) + o(1) = O(1). \tag{41}
\]
This means that there exists a constant \(C_1\) such that \(g(x(a))/(\theta_a x(a)) \leq C_1\). Hence,
for all \(\varepsilon_1 > 0\),
\[
|\tilde{x}(a) - x(a)| = \left| \frac{g(\tilde{x}(a)) - g(x(a))}{\theta_a} + o(1/\theta_a) \right|
\leq \varepsilon_1 |\tilde{x}(a) - x(a)| \frac{g(x(a))}{\theta_a x(a)} + o(1/\theta_a) \leq \varepsilon_1 C_1 |\tilde{x}(a) - x(a)| + o(1/\theta_a).
\]
For \(\varepsilon_1 < 1/C_1\) this implies \(\tilde{x}(a) = x(a) + o(1/a)\). Now suppose there exists some
\(\varepsilon_1 > 0\) such that \(g(x) \gg x\). By the definition of \(t(a)\),
\[
x(a) = \frac{g(x(a))}{\theta_a} - \frac{1}{\theta_a} \ln \frac{1}{\theta_a} + o \left( \frac{1}{\theta_a} \right) \tag{42}
\]
Therefore, there exists some \(\varepsilon_2 > 0\) such that
\[
x(a) \gg \left( \frac{1}{\theta_a} \right)^{1+\varepsilon_2} \tag{43}
\]
and hence, by (52),
\[
g(x(a)) / \theta_a x(a) = 1 + o(1). \tag{44}
\]
Combining this result with (10) we can see that we can choose \(\hat{\gamma}\) such that \(0 < \gamma_0 < \hat{\gamma} < 1\) with
\[
|\tilde{x}(a) - x(a)| = \left| \frac{g(\tilde{x}(a)) - g(x(a))}{\theta_a} + o(1/\theta_a) \right|
\leq \gamma_0 |\tilde{x}(a) - x(a)| \frac{g(x(a))}{\theta_a x(a)} + o(1/\theta_a) \leq \hat{\gamma} |\tilde{x}(a) - x(a)| + o(1/\theta_a)
\]
Consequently, we obtain \(\tilde{x}(a) = x(a) + o(1/a)\).

2.2. Proof of Proposition 2. During the whole proof we assume \(a\) to be sufficiently small, even if not explicitly mentioned. To prove that \(Y_n^{(1)}\) is a supermartingale, it is sufficient to show that for all \(y \leq x - \delta x(a)\),
\[
E[G_{-\varepsilon_0}(x-y-X^{(a)})] + LE[\tilde{G}_{1-\varepsilon}(x-y-X^{(a)})] \leq G_{-\varepsilon_0}(x-y) + L\tilde{G}_{1-\varepsilon}(x-y) \tag{44}
\]
and that
\[
E[G_{-\varepsilon_0}(x-y-X^{(a)})] \leq G_{-\varepsilon_0}(x-y) \tag{45}
\]
for all \(y \in (x - \delta x(a), x]\). Put \(t := x - y\) and remark that \(x(a)\) does not depend on \(x\), but only on \(a\). Then, (39) is equivalent to
\[
E[G_{-\varepsilon_0}(t-X^{(a)})] + LE[\tilde{G}_{1-\varepsilon}(t-X^{(a)})] \leq G_{-\varepsilon_0}(t) + L\tilde{G}_{1-\varepsilon}(t) \tag{46}
\]
for all \( t \geq \delta t(a) \), where we wrote \( t(a) \) instead of \( x(a) \) due to the change of variables. In addition, (40) is equivalent to
\[
E[G_{-c_a}(t - X(a))] \leq G_{-c_a}(t)
\]
for all \( 0 \leq t \leq \delta t(a) \). Let us bound the expectation on the left side of the latter inequality. Put \( \kappa_a = \theta_a - c_a \) for brevity, then
\[
E[G_{-c_a}(t - X(a))] \leq e^{-\kappa_a t}P(X(a) \leq -1/\kappa_a) + e^{-\kappa_a t}E[e^{\kappa_a X(a)}; X(a) \in (-1/\kappa_a, 1/\kappa_a)]
\]
\[
+ e^{-\kappa_a t}E[e^{\kappa_a X(a)}; X(a) \in (1/\kappa_a, t)] + \overline{F}(t)
\]
\[
=: G_1 + G_2 + G_3 + G_4.
\]

To bound the second term we use the estimate \( e^x \leq 1 + x + x^2 \), which is valid for \( x \leq 1 \). Then, \( c_a = o(a) \) implies
\[
e^{\kappa_a X(a)} = e^{\theta_a X(a) - c_a X(a)} \leq e^{\theta_a X(a)} \left( 1 - c_a X(a) + (c_a X(a))^2 \right)
\]
for \( X(a) \in [-1/\kappa_a, 1/\kappa_a] \). Using \( c_a = o(a) \) again, we obtain
\[
G_2 \leq e^{-\kappa_a t}E\left[e^{\theta_a X(a)} \left( 1 - c_a X(a) + (c_a X(a))^2 \right); X(a) \in (-1/\kappa_a, 1/\kappa_a)\right]
\]
\[
\leq e^{-\kappa_a t}E\left[e^{\theta_a X(a)} \left( 1 - c_a X(a) \right); X(a) \in (-1/\kappa_a, 1/\kappa_a)\right] + \sigma^2 \gamma c_a^2 e^{-\kappa_a t}.
\]
Further, since \( \theta_a = 2a/\sigma^2 + o(a) \) and \( E[X(a)^2 + o(1)] < \infty \),
\[
E\left[X(a)e^{\theta_a X(a)}; X(a) \in [-1/\kappa_a, 1/\kappa_a]\right]
\]
\[
= E\left[X(a); X(a) \in [-1/\kappa_a, 1/\kappa_a]\right] + \theta_a E\left[X(a)^2; X(a) \in [-1/\kappa_a, 1/\kappa_a]\right] + o(a)
\]
\[
= -a + \sigma^2 \theta_a + o(a) = a + o(a).
\]
If \( g \) is such that \( g(t) \ll t^{\gamma_1} \) for all \( \gamma_1 > 0 \), one can easily see by the definition of \( t(a) \), that \( t(a) \ll a^{-1/\gamma_1} \). Hence, \( c_a \gg a^{1+\varepsilon} \) and therefore by the assumption \( E[\min\{0, X(a)\}]^{2+\varepsilon} < \infty \),
\[
P(|X(a)| > 1/\kappa_a) = o(ac_a).
\]
On the other side, if \( g(t) \gg t^{\gamma_1} \) for some \( \gamma_1 > 0 \), then the assumption \( E[X(a)^{1+1/(1-\gamma_1)}] < \infty \) implies that (44) is also valid in this case and therefore
\[
E\left[e^{\theta_a X(a)}; X(a) \in (1/\kappa_a, 1/a]\right] = o(ac_a).
\]
Combining the latter calculations with the definition of \( \theta_a \) and \( c_a = o(a) \), we obtain
\[
G_2 \leq e^{-\kappa_a t} - c_a e^{-\kappa_a t}E\left[X(a)e^{\theta_a X(a)}; X(a) \in [-1/\kappa_a, 1/\kappa_a]\right] + \sigma^2 c_a^2 e^{-\kappa_a t}
\]
\[
\leq e^{-\kappa_a t} - ac_a e^{-\kappa_a t} + o(ac_a e^{-\kappa_a t}).
\]
Next, integrating by parts,
\[
G_3 = e^{-\kappa_a t} \int_{1/\kappa_a}^{t} e^{\kappa_a y} P(X(a) \in dy)
\]
\[
= -\overline{F}(t) + e^{1-\kappa_a t} \overline{F}(1/\kappa_a) \kappa_a e^{-\kappa_a t} \int_{1/\kappa_a}^{t} e^{\kappa_a y} \overline{F}(y) dy.
\]
Here we used the fact that the edges of the interval $[1/c_a, 1]$ for $y$ is increasing for $t > 0$. To calculate the first term on the right hand side of (47) let us examine two and $g$. Due to (49), the definition of $t(a)$, (51) and the fact that $t(a)$ is increases as the order of $g$ increases,

$$
\int_{1/c_a}^t e^{\kappa_a y} F(y) dy \leq \int_{1/c_a}^{t(a) - C \ln (1/a)/\theta_a} e^{\theta_a y - g(y) - c_a y} dy
$$

for all $t \geq 0$. If $0 < t \leq 1/c_a$, the integral term doesn’t give any positive contribution and we attain

$$
\int_{1/c_a}^t e^{\kappa_a y} F(y) dy \leq \int_{1/c_a}^{t(a) - C \ln (1/a)/\theta_a} e^{\theta_a y - g(y) - c_a y} dy \leq \left( e^{\theta_a t(a) - C \ln (1/a)/\theta_a} - g(t(a) - C \ln (1/a)/\theta_a) + e^{\theta_a/c_a - g(1/c_a)} \right) \int_{1/c_a}^{t(a)} e^{-c_a y} dy.
$$

(52)

Here we used the fact that $\theta_y y - g(y)$ is convex and takes its maximum at one of the edges of the interval $[1/c_a, t(a) - C \ln (1/a)/\theta_a]$. This is true since $\theta_y y - g(y)$ is increasing for $y$ such that $g'(y) < \theta_a$ and decreasing for $y$ such that $g'(y) > \theta_a$ and $g$ is concave. Now, regard that $c_a$ is such that $c_a = o(a)$. Hence,

$$
\int_{1/c_a}^{t(a)} e^{-c_a y} dy = \frac{1}{c_a} \left( e^{-c_a/c_a} - e^{-c_a t(a)} \right) \leq \frac{1}{c_a}.
$$

(53)

To calculate the the first term on the right hand side of (47) let us examine two different classes of tails separately. First, let $g(t)$ be such that $g(t) \ll t^{\varepsilon_1}$ for all $\varepsilon_1$. Due to (49), the definition of $t(a)$, (51) and the fact that $t(a)$ is increases as the order of $g$ increases,

$$
\int_{1/c_a}^t e^{\kappa_a y} F(y) dy = \frac{1}{\theta_a t(a)} \left( e^{-c_a/c_a} - e^{-c_a t(a)} \right) \leq \frac{1}{\theta_a t(a)}.
$$

(53)

for $C$ small enough. Consequently, by again using (51), the definition of $t(a)$ and the fact that that $t(a)$ increases as the order of $g$ increases,

$$
\frac{g(t(a))}{\theta_a t(a)} = 1 + \frac{\ln (1/(\theta_a t(a)))}{\theta_a t(a)} + o(1) = O(1).
$$

(55)

This means that there exists a constant $C_1$ such that $g(t(a))/(\theta_a t(a)) \leq C_1$. Plugging this result into (50), we attain

$$
\int_{1/c_a}^{t(a) - C \ln (1/a)/\theta_a} e^{\theta_a t(a) - C \ln (1/a)/\theta_a} - g(t(a) - C \ln (1/a)/\theta_a) \leq (1 + o(1))\theta_a t(1+C(1-\varepsilon_1)C_1)
$$

(56)

Since $\varepsilon_1$ was arbitrary one can choose $\varepsilon_1 < 1/C_1$. Further, (14) and the definition of $t(a)$ gives

$$
t(a) = \frac{g(t(a))}{\theta_a} - \frac{1}{\theta_a} \ln \frac{1}{\theta_a} + o \left( \frac{1}{\theta_a} \right)
$$

(57)
and one can easily see this implies that
\[ t(a) \ll \left( \frac{1}{\theta_a} \right)^{1+\varepsilon_2} \]  
for all \( \varepsilon_2 > 0 \), if \( g \) is such that \( g(t) \ll t^{\varepsilon_1} \) for all \( \varepsilon_1 > 0 \). Thus, (53), (55) and the
\begin{align*}
deinition of \( c_a \) imply
\begin{equation}
e^{\theta_a t(a) - C \ln(1/a)/\theta_a - g(t(a) - C \ln(1/a)/\theta_a)} = O(a^{2+C(1-\varepsilon_1 C_3)}) = o(c_a^2).
\end{equation}
\end{align*}
On the other side, it follows from \( \mathbb{E}|X(a)|^{2+\varepsilon} < \infty \) that
\[ \mathbb{F}(1/\theta_a) = o(a^{2+\varepsilon}) = o(c_a^2). \]
Plugging the results from (48), (56) and (57) into (47) we finally obtain
\[ x_a e^{\kappa a t} \int_{1/x_a}^{t} e^{\kappa a y} \mathbb{F}(y) dy = o(ac_a e^{-\kappa t}) \]
for \( t \leq t(a) - C \ln(1/a)/\theta_a \), if \( g \) is such that \( g(t) \ll t^{\varepsilon_1} \) for all \( \varepsilon_1 > 0 \).

Next, assume that there exists some \( \varepsilon_1 \) such that \( g(t) \gg t^{\varepsilon_1} \). In this case, \( \gamma^* \) is well defined. Choose \( \delta_2 > 0 \) such that \( \gamma^* + \delta_2 < 1 \). Due to our assumptions from (11),
\begin{equation}
\frac{g(t)}{t^{\gamma^* + \delta_2}} \downarrow, \quad t \geq t_0.
\end{equation}
for arbitrary \( \delta_2 > 0 \). One can easily see that for every \( \gamma > \gamma^* + \delta_2 \),
\begin{equation}
g(t) - g(t - w) \leq (\gamma^* + \delta_2) w \frac{g(t)}{t} + o\left( w \frac{g(t)}{t} \right) \leq \gamma \left( 1 + \varepsilon_2 \right) w \frac{g(t)}{t}.
\end{equation}

for \( w \ll t \). Consequently, due to the latter inequality and the definition of \( t(a) \),
\begin{align*}
e^{\theta_a t(a) - C \ln(1/a)/\theta_a - g(t(a) - C \ln(1/a)/\theta_a)} & \leq e^{\theta_a t(a) - g(t(a) - C \ln(1/a) + \gamma \ln(1/a) g(t(a))/\theta_a t(a))} \\
& \sim a \theta_a e^{C \ln(1/a)(-1 + \gamma g(t(a))/\theta_a t(a))}.
\end{align*}

One can easily see that (54) implies that
\begin{equation}
t(a) \gg \left( \frac{1}{\theta_a} \right)^{1/(1-\varepsilon_1)} \gg 1/\theta_a \ln(1/a)
\end{equation}
and by (52) we conclude that there exists \( \varepsilon_2 > 0 \) such that \( \gamma(1+\varepsilon_2) < 1 \) and
\begin{equation}
\frac{\hat{\gamma}}{\theta_a t(a)} = \hat{\gamma} + o(1) \leq \gamma(1+\varepsilon_2).
\end{equation}

Plugging the latter result into (61) we attain
\begin{equation}
e^{\theta_a t(a) - C \ln(1/a)/\theta_a - g(t(a) - C \ln(1/a)/\theta_a)} \leq (1 + o(1)) \theta_a a^{1+C(1-\gamma(1+\varepsilon_2))}.
\end{equation}
Since we assumed that there exists some \( 0 < \gamma < 1 \) such that \( g(t)/t^{\gamma} \downarrow \), one can easily see that (54) implies that
\begin{equation}
t(a) \ll \left( \frac{1}{\theta_a} \right)^{C_3}
\end{equation}
for \( C_3 \) large enough. Consequently, for \( C \) large enough,
\begin{equation}
e^{\theta_a t(a) - C \ln(1/a)/\theta_a - g(t(a) - C \ln(1/a)/\theta_a)} = O(c_a^2).
\end{equation}
On the other hand, by (65),
\[
F(1/\theta_a) = e^{-\theta_a^{-1}} = o(e_a^2) \tag{70}
\]
in this case, too. Plugging the results from (48), (66) and (67) into (47) we finally obtain
\[
\int_{1/\kappa_a}^t e^{\kappa_a y} F(y) dy = o(ac_a e^{-\kappa_a t}) \tag{71}
\]
for \( t \leq t(a) - C \ln(1/a)/\kappa, \) if \( C \) large enough and \( g(t) \gg t^{\varepsilon_1} \) for some \( \varepsilon_1 > 0. \)

Next, consider the case \( t > t(a) - C \ln(1/a)/\theta_a. \) In this case we split the integral from (45) in two parts
\[
\int_{1/\kappa_a}^t = \int_{1/\kappa_a}^{t-C \ln(1/a)/\kappa} + \int_{t-C \ln(1/a)/\kappa}^t,
\]
where we choose the constant \( C \) without loss of generality in such a way that \( t(a) > C \ln(1/a)/\theta_a. \) The first integral can be estimated similar to the case \( t \leq t(a) - C \ln(1/a)/\theta_a. \)

\[
\int_{1/\kappa_a}^{t-C \ln(1/a)/\kappa} e^{\kappa_a y} F(y) dy = \int_{1/\kappa_a}^{t-C \ln(1/a)/\kappa} e^{\kappa_a y - g(y)} dy \\
\leq \left( e^{\kappa_a (t-C \ln(1/a)/\kappa_a)} - g(t-C \ln(1/a)/\kappa_a) + e^{1-g(1/\kappa_a)} \right) \int_{1/a}^t e^{-\kappa_a y} dy \\
\leq \frac{1}{c_a} \left( C_5 a^{C_\varepsilon t - g(t-C \ln(1/a)/\kappa_a)} + e^{1-g(1/\kappa_a)} \right).
\tag{73}
\]
with a suitable constant \( C_5. \) Here we again used that \( \theta_a y - g(y) = y(\theta_a - g(y)/y) \) is convex and takes its maximum at one of the edges. For further investigation of the terms from the latter inequality, we again consider two different cases of distribution functions separately.

First, let \( g \) be such that \( g(t) \ll t^{\varepsilon_1} \) for all \( \varepsilon_1 > 0. \) Let \( C \) be so small that \( C \ln(1/a)/\kappa_a \leq t(a)/3, \) which is possible due to (51). Then, \( C \ln(1/a)/\kappa_a \leq t/2 \) for all \( t > t(a) - C \ln(1/a)/\kappa_a \) and therefore
\[
g(t(a) - C \ln(1/a)/\kappa_a) \leq 2 g(t(a)) \frac{t(a)}{t(a)} \tag{74}
\]
Since \( g(t)/t \downarrow \) in \( t \) the latter result plus (52) imply that \( g(t)/(\kappa_a t) = O(1) \) for \( t \geq t(a) - C \ln(1/a)/\kappa_a \). Consequently, (49) implies that there exists a constant \( C_4 \) such that
\[
g(t) - g(t - C \ln(1/a)/\kappa_a) \leq \varepsilon_1 C \ln(1/a) \frac{g(t)}{\kappa_a t} \leq \varepsilon_1 C_4 \ln(1/a) \tag{75}
\]
and hence
\[
e^{-g(t-C \ln(1/a)/\kappa_a) \theta_a} \leq a^{-\varepsilon_1 C_4} F(t). \tag{76}
\]
Since \( \varepsilon_1 \) was arbitrary, we can choose \( \varepsilon_1 \) such that \( \varepsilon_1 C_4 < 1. \) Consequently, plugging (57) and (76) into (70) and using the definition of \( c_a, \kappa_a \) and \( \theta_a \) we obtain
\[
\kappa_a e^{-\kappa_a t} \int_{1/\kappa_a}^{t-C \ln(1/a)/\kappa_a} e^{\kappa_a y} F(y) dy \leq C_5 \frac{\kappa_a}{c_a} a^{C(1-\varepsilon_1 C_4)} F(t) + o(ac_a e^{-\kappa_a t}) \\
= o \left( F(t) \right) + o(ac_a e^{-\kappa_a t}). \tag{77}
\]
Let us now consider distribution functions such that \( g(t) \gg t^{\varepsilon_1} \) for some \( \varepsilon_1 > 0 \).
By regarding (62), one can easily see that for all \( C > 0 \) and \( t \geq t(a) - C \ln(1/a)/\theta_a \),
\( C \ln(1/a)/\kappa_a \leq t/2 \). Hence, equation (60) gives us
\[
g(t) - g(t - C \ln(1/a)/\kappa_a) \leq \hat{\gamma} C \ln(1/a) \frac{g(t)}{\kappa_a} \quad (78)
\]
with \( \hat{\gamma} < 1 \). Let us show that \( g(t)/(\kappa_a t) = 1 + o(1) \) for \( t \geq t(a) - C \ln(1/a)/\theta_a \) in
this case. We know from (52) and (62) that
\[
\frac{g(t(a))}{\theta_a t(a)} \sim 1. \quad (79)
\]
Hence, by again using (62),
\[
\frac{g(t(a) - C \ln(1/a)/\theta_a)}{\theta_a t(a) - C \ln(1/a)/\theta_a} \leq \frac{g(t(a))}{\theta_a t(a)} \sim \frac{g(t(a))}{\theta_a t(a)} \sim 1 \quad (80)
\]
and consequently, since \( g(t)/t \) is decreasing in \( t \), there exists \( \varepsilon_5 > 0 \) such that
\[
\hat{\gamma}(1 + \varepsilon_5) < 1 \quad \text{and} \quad g(t)/(\kappa_a t) \leq 1 + \varepsilon_5
\]
for \( t \geq t(a) - C \ln(1/a)/\theta_a \) and \( \varepsilon_5 \) small enough. Combining this result with (78),
we attain
\[
g(t) - g(t - C \ln(1/a)/\kappa_a) \leq \hat{\gamma}(1 + \varepsilon_5) C \ln(1/a). \quad (81)
\]
Therefore, by plugging (67) and the latter result into (70),
\[
x_a e^{-\kappa_a t} \int_{1/x_a}^{t-C \ln(1/a)/x_a} e^{\kappa_a y} \mathcal{F}(y)dy \leq \frac{x_a}{\theta_a} a^{C(1-\hat{\gamma}(1+\varepsilon_5))} \mathcal{F}(t) + o(ac_a e^{-\kappa_a t}). \quad (82)
\]
Using (65), the latter inequality implies that for \( C \) large enough,
\[
x_a e^{-\kappa_a t} \int_{1/x_a}^{t-C \ln(1/a)/x_a} e^{\kappa_a y} \mathcal{F}(y)dy = o(\mathcal{F}(t)) + o(ac_a e^{-\kappa_a t}) \quad (83)
\]
in the case that there exists some \( \varepsilon_1 > 0 \) such that \( g(t) \gg t^{\varepsilon_1} \).
Let us now examine the second integral from the right hand side of (69). First, regard \( g \) such that \( g(t) \ll t^{\varepsilon_1} \) for all \( \varepsilon_1 > 0 \). In this case, \( \varepsilon_1 g(t)/(\theta_a t) < 1 \) for \( \varepsilon_1 \) small enough and \( t \geq t(a) - C \ln(1/a)/\theta_a \). Hence, by applying (49) with \( C \) and \( \varepsilon_1 \) small enough, we attain
\[
x_a e^{-\kappa_a t} \int_{t-C \ln(1/a)/x_a}^{t} e^{\kappa_a y} \mathcal{F}(y)dy = x_a \int_{0}^{C \ln(1/a)/x_a} e^{-\kappa_a w-g(t-w)}dw
\]
\[
\leq x_a e^{-g(t)} \int_{0}^{C \ln(1/a)/x_a} e^{-\kappa_a w-\varepsilon_1 g(t)/(\kappa_a t)}dw \sim \frac{\mathcal{F}(t)}{1-\varepsilon_1 g(t)/(\kappa_a t)} \quad (84)
\]
for all \( t \geq t(a) - C \ln(1/a)/\theta_a \).
Now, regard $g$ such that $g(t) \gg t^{\varepsilon_1}$ for some $\varepsilon_1 > 0$. In this case $g(t)/(x_a t) \leq 1 + o(1)$ for all $t \geq t(a) - C \ln(1/a)/\theta_a$. Hence, by (60) and the definition of $c_a$,

$$
X_a e^{-x_a^2 t} \int_{t-C \ln(1/a)/\theta_a}^{\infty} e^{x_a y} \mathcal{F}(y) \, dy = X_a \int_{0}^{C \ln(1/a)/\theta_a} e^{-x_a w - g(t-w)} \, dw 
$$

$$
\leq X_a e^{-g(t)} \int_{0}^{C \ln(1/a)/\theta_a} e^{-w x_a (1-(\gamma^* + \delta_2)g(t)/(x_a t) + o(1))} \, dw 
$$

$$
\sim \frac{\mathcal{F}(t)}{1 - (\gamma^* + \delta_2)g(t)/(x_a t) + o(1)} \sim \frac{\mathcal{F}(t)}{1 - (\gamma^* + \delta_2)g(t)/(\theta_a t)}
$$

(85)

for all $t \geq t(a) - C \ln(1/a)/\theta_a$.

By plugging (77) and (85) into (45) we obtain

$$
\mathbf{E} G_{-c_a}(t - X^{(a)}) - G_{-c_a}(t) 
$$

$$
\leq -ac_a e^{-xt} + \frac{\mathcal{F}(t) \{ t \geq t(a) - C \ln(1/a)/\theta_a \}}{1 - \varepsilon_1 g(t)/(\theta_a t)} + o(ac_a e^{-xt}) + o(\mathcal{F}(t)),
$$

(86)

if $g(t) \ll t^{\varepsilon_1}$ for all $\varepsilon_1 > 0$ and by plugging (84) and (86) into (45),

$$
\mathbf{E} G_{-c_a}(t - X^{(a)}) - G_{-c_a}(t) 
$$

$$
\leq -ac_a e^{-xt} + \frac{\mathcal{F}(t) \{ t \geq t(a) - C \ln(1/a)/\theta_a \}}{1 - (\gamma^* + \delta_2)g(t)/(\theta_a t)} + o(ac_a e^{-xt}) + o(\mathcal{F}(t)),
$$

(87)

if $g(t) \ll t^{\varepsilon_1}$ for some $\varepsilon_1 > 0$. Let us show that the last two inequalities imply (42). If $g(t) \ll t^{\varepsilon_1}$ for all $\varepsilon_1 > 0$, one can easily see that (51) and the fact that $t(a)$ increases with the order of $g$ imply that for all $0 < \delta < 1$ and $C$ small enough,

$$
\delta t(a) < t(a) - C \ln(1/a)/\theta_a.
$$

Consequently, for $0 \leq t \leq \delta t(a)$ and $a$ small enough,

$$
\mathbf{E} G_{-c_a}(t - X^{(a)}) - G_{-c_a}(t) \leq -ac_a e^{-xt} + o(ac_a e^{-xt}) \leq 0.
$$

(88)

If, on the other side, $g(t) \gg t^{\varepsilon_1}$ for some $\varepsilon_1 > 0$, then $t(a) \gg C \ln(1/a)/\theta_a$, see (62). Hence, $t(a) - C \ln(1/a)/\theta_a \sim t(a) > \delta t(a)$ and consequently also

$$
\mathbf{E} G_{-c_a}(t - X^{(a)}) - G_{-c_a}(t) \leq -ac_a e^{-xt} + o(ac_a e^{-xt}) \leq 0
$$

(89)

in this case for $0 \leq t \leq \delta t(a)$.

It remains to show (41) for $t \geq \delta t(a)$ and therefore we need to examine $\widehat{G}_{1-\varepsilon}$.

By the definition of $\widehat{G}_{1-\varepsilon}$,

$$
a(1 - \varepsilon) \mathbf{E} \widehat{G}_{1-\varepsilon}(t - X) = \int_{-\infty}^{t - \delta t(a)} F(dz) \mathcal{F}'(t - z) 
$$

$$
= \left( \int_{0}^{t - \delta t(a)} + \int_{-\infty}^{0} \right) F(dz) \mathcal{F}'(t - z).
$$
Integrating the first integral by parts, we obtain
\[
\int_0^{t-\delta(t)} F(dz) \overline{F}'(t-z) dz = \overline{F}(0) \overline{F}'(t) - \overline{F}(t-\delta(t)) \overline{F}'(\delta(t)) + \int_0^{t-\delta(t)} \overline{F}(z) \overline{F}'(t-z) dz
\]
and by integrating the second integral by parts,
\[
\int_0^0 F(dz) \overline{F}'(t-z) = \overline{F}(0) \overline{F}'(t) - \overline{F}(t-z) F(z) dz.
\]
Combining the above equalities, we get
\[
a(1-\varepsilon) \mathbb{E} \hat{G}_{1-\varepsilon}(t - X) = \overline{F}'(t) - \overline{F}(t-\delta(t)) \overline{F}'(\delta(t)) + \int_0^{t/2} \overline{F}(z) \overline{F}'(t-z) dz + \int_0^{t/2} \overline{F}(z) \overline{F}'(t-z) dz - \int_{-\infty}^0 \overline{F}(t-z) F(z) dz.
\]
Hence,
\[
\mathbb{E} \hat{G}_{1-\varepsilon}(t - X) - \hat{G}_{1-\varepsilon}(t)
\]
\[
= \frac{\overline{F}(t)}{a(1-\varepsilon)} \left( - \frac{\overline{F}(t-\delta(t)) \overline{F}'(\delta(t))}{\overline{F}(t)} + \int_0^{t/2} \frac{\overline{F}(z) \overline{F}(t-z)}{\overline{F}(t)} dz \right)
\]
\[
= \int_0^{t/2} \overline{F}(z) \overline{F}'(t-z) dz - \int_{-\infty}^0 \frac{F(z) \overline{F}(t-z)}{\overline{F}(t)} dz.
\]
To examine the terms on the right hand side of the latter inequality let us again consider two different types of distribution functions. First, let \( g \) be such that there exists some \( \varepsilon_1 \) such that \( g(t) \gg t^{\varepsilon_1} \). Consider \( \nu \) such that \((t/g(t))^{1-\delta} \ll \nu \ll t/g(t))\) with a small constant \( \delta \). We will see later what small means in this context. Then, by (60),
\[
\int_0^{t/2} \overline{F}(z) \overline{F}'(t-z) dz \leq \int_0^{t/2} \overline{F}(z) \exp \left\{ (\gamma^* + \delta_z) \frac{g(t)}{t} + o \left( \frac{zt}{t} \right) \right\} dz
\]
\[
= \int_0^{t/2} \overline{F}(z) dz + (\gamma^* + \delta_z) \frac{g(t)}{t} \int_0^{t/2} z \overline{F}(z) dz + o(g(t)/t),
\]
where we used Taylor approximation \( \mathbb{E} \lvert X^{(a)} \rvert^{2+\varepsilon} < \infty \) with \( \varepsilon > 0 \) in the last equation. Due to our assumptions the function \( g \) is concave and increasing, consequently it is
\[
g(z) - g(\nu) \geq g(t - \nu) - g(t - z)
\]
for all \( z \in (\nu, t/2) \). Hence,
\[
\int_0^{t/2} \overline{F}(z) \overline{F}'(t-z) dz \leq t \exp \{ g(t) - g(\nu) - g(t - \nu) \}
\]
\[
\leq t \exp \{ -g(\nu) + \gamma_{\nu} g(t)/t \} \sim t \exp \{ -g(\nu) \},
\]
where we again used (60) and that \( \nu \ll t/g(t) \). Since \( g(\nu) \gg \nu^{\varepsilon_1} \) and \( g(t) = o(t^{\gamma^* + \delta_2}) \) for \( \delta_2 > 0 \) and \( \gamma^* + \delta_2 < 1 \), we obtain

\[
\int_{\nu}^{\nu + \varepsilon_1} \frac{F(z) - F(t - z)}{F(t)} \, dz \leq (1 + o(1)) t \exp\{-g(\nu)\} \leq (1 + o(1)) t \exp\{-\nu^{\varepsilon_1}\}
\]

\[
o(t \exp\{-t/g(t)^{\gamma^*(1-\delta_1)}\}) = o(g(t)/t)
\]

for all \( \nu \gg (t/g(t))^{1-\delta_1} \). Analogously to (92) one can show

\[
\int_{b_{\varepsilon}(a)}^{\nu} \frac{F(z) - F(t - z)}{F(t)} \, dz \leq \int_{b_{\varepsilon}(a)}^{\nu} \frac{F(z) + (\gamma^* + \delta_2) g(t)/t}{F(t)} \, dz + o(g(t)/t)
\]

and since \( t(a) \gg 1/a \) and \( \mathbb{E}[|X^{(a)}|^{2+\varepsilon}] < \infty \),

\[
\int_{b_{\varepsilon}(a)}^{\nu} F(z) \, dz = o(a) \quad \text{and} \quad \int_{b_{\varepsilon}(a)}^{\nu} zF(z) \, dz = o(1).
\]

Consequently,

\[
\int_{b_{\varepsilon}(a)}^{\nu} F(z) \frac{F(t - z)}{F(t)} \, dz = o(a) + o(g(t)/t).
\]

Further, (60) implies

\[
g(t - z) - g(t) \leq - (\gamma^* + \delta_2) z \frac{g(t - z)}{t - z} + o \left( -z \frac{g(t - z)}{t - z} \right)
\]

\[
\leq - (\gamma^* + \delta_2) z \frac{g(t)}{t} + o \left( -z \frac{g(t)}{t} \right)
\]

for all \( z < 0 \) such that \( -z \leq t/2 \). Thus, since \( \nu \ll t/g(t) \) and \( \mathbb{E}[|X^{(a)}|^{2+\varepsilon}] < \infty \),

\[
\int_{-\nu}^{0} F(z) \frac{F(t - z)}{F(t)} \, dz \geq \int_{-\nu}^{0} F(z) \exp \left\{ (\gamma^* + \delta_2) z \frac{g(t)}{t} + o \left( -z \frac{g(t)}{t} \right) \right\} \, dz
\]

\[
= \int_{-\nu}^{0} F(z) \, dz + (\gamma^* + \delta_2) g(t)/t \int_{-\nu}^{0} zF(z) \, dz + o \left( \frac{g(t)}{t} \right).
\]

Plugging (92), (95), (98) and (100) into (91) with \( (t/g(t))^{1-\delta_1} \ll \nu \ll t/g(t) \), where \( \delta_1 \) is small enough, we obtain

\[
\mathbb{E}[\hat{G}_{1-\varepsilon}(t - X)] - \hat{G}_{1-\varepsilon}(t)
\]

\[
\leq \frac{\mathcal{F}(t)}{a(1-\varepsilon)} \left[ \left( \int_{\nu}^{\nu + \varepsilon_1} F(z) \, dz - \int_{-\nu}^{0} F(z) \, dz \right) + \frac{(\gamma^* + \delta_2) g(t)}{t} \left( \int_{0}^{\nu} zF(z) \, dz - \int_{-\nu}^{0} zF(z) \, dz \right) + o(a) + o(g(t)/t) \right].
\]

Choose \( \delta_1 \) so small that

\[
\int_{\nu}^{\infty} F(z) \, dz = o(g(t)/t) \quad \text{and} \quad \int_{-\infty}^{-\nu} F(z) \, dz = o(g(t)/t).
\]

Then,

\[
\int_{0}^{\nu} F(z) \, dz - \int_{-\nu}^{0} F(z) \, dz = -a + o(g(t)/t)
\]

(102)
and, as one can easily see,
\[ \int_0^\nu zF(z)dz - \int_{-\nu}^0 zF(z)dz = \frac{\sigma^2}{2} + o(1). \] (103)
Hence, for \( t \geq \delta t(a) \),
\[ E[\tilde{G}_{1-\varepsilon}(t - X)] - \tilde{G}_{1-\varepsilon}(t) \leq \frac{F(t)}{a(1 - \varepsilon)} \left( -a + \frac{(\gamma^* + \delta_2)a^2g(t)}{2t} + o(a) + o\left( \frac{g(t)}{t} \right) \right) \]
\[ = \frac{F(t)}{1 - \varepsilon} \left( -1 + \frac{(\gamma^* + \delta_2)a^2g(t)}{\theta_at} + o(1) + o\left( \frac{g(t)}{\theta_at} \right) \right), \] (104)
where we used \( \theta_a \sim 2a/\sigma^2 \). Since \( g(t) = o(t) \), we obtain for all \( t \geq \delta t(a) \),
\[ \frac{g(t)}{\theta_at} \leq \frac{g(\delta t(a))}{\theta_at}(a) \leq \frac{g(t(a))}{\theta_at}(a) \sim \delta^{-1} < \infty, \] (105)
and consequently we finally obtain
\[ E[\tilde{G}_{1-\varepsilon}(t - X)] - \tilde{G}_{1-\varepsilon}(t) \leq \frac{F(t)}{1 - \varepsilon} \left( -1 + \frac{(\gamma^* + \delta_2)a^2g(t)}{\theta_at} + o(1) \right). \] (106)
Now, suppose that \( g(t) \ll t^{\varepsilon_1} \) for all \( \varepsilon_1 > 0 \). Then, (94) implies
\[ \int_{-\nu}^{t/2} \frac{F(z)}{F(t)}dz \leq (1 + o(1))t \exp\{-g(\nu)\} \leq (1 + o(1))t\nu^{-2+(2+\varepsilon)} = o(g(t)/t) \] (107)
for \( \nu \gg (t/g(t))^{1-\delta_1} \) with \( \delta_1 \) small enough. Further, by using inequality (49) instead of (60), one can totally analogously derive the results (92) and (100) with \( \varepsilon_1 \) instead of \( \gamma^* + \delta_2 \). Hence, in this case the bound from (107) can be improved to
\[ E[\tilde{G}_{1-\varepsilon}(t - X)] - \tilde{G}_{1-\varepsilon}(t) \leq \frac{F(t)}{1 - \varepsilon} \left( -1 + \varepsilon_1 \frac{g(t)}{\theta_at} + o(1) \right) \] (108)
for \( t \geq \delta t(a) \), where \( \varepsilon_1 \) is an arbitrary positive number.

With the results from (87) and (88) and the inequalities (107) and (109) we can now show (41). Let us first consider distributions such that \( g(t) \ll t^{\varepsilon_1} \) for all \( \varepsilon_1 > 0 \). In this case, for \( \delta t(a) \leq t \leq t(a) - C \ln(1/a)/\theta_a, \) (87) and (109) imply
\[ E[G_{-c_\alpha}(t - X^{(a)})] + E[\tilde{G}_{1-\varepsilon}(t - X^{(a)})] - G_{-c_\alpha}(t) - \tilde{G}_{1-\varepsilon}(t) \leq -ac_\alpha e^{-\varepsilon t} + \frac{F(t)}{1 - \varepsilon} \left( -1 + \varepsilon_1 \frac{g(t)}{\theta_at} \right) + o(ac_\alpha e^{-\varepsilon t}) + o(F(t)) \] (109)
for \( a \) small enough. For \( \varepsilon_1 < \delta \) it follows from \( g(t(a))/\theta_at(a) = O(1) \) that
\[ \frac{\varepsilon_1 g(\delta t(a))}{\theta_at(a)} \leq \frac{\varepsilon_1 g(t(a))}{\theta_at(a)} \leq \frac{\varepsilon_1 g(t(a))}{\delta \theta_at(a)} < 1 \] (110)
for \( \varepsilon_1 \) small enough. Hence,
\[ E[G_{-c_\alpha}(t - X^{(a)})] + E[\tilde{G}_{1-\varepsilon}(t - X^{(a)})] - G_{-c_\alpha}(t) - \tilde{G}_{1-\varepsilon}(t) \leq 0 \] (111)
for a small enough. Now, let $t > t(a) - C \ln(1/a)/\theta_a$. Then, again by virtue of (87) and (109),

$$ \mathbf{E}[G_{-c_a}(t - X^{(a)})] + \mathbf{E}[\hat{G}_{1-\varepsilon}(t - X^{(a)})] - G_{-c_a}(t) - \hat{G}_{1-\varepsilon}(t) $$

$$ \leq -ac_a e^{-\varepsilon t} + \frac{\mathcal{F}(t)}{1 - \varepsilon} \left( -1 + \varepsilon_1 \frac{g(t)}{\theta_a t} \right) \left( 1 + \varepsilon_1 \frac{g(t)}{\theta_a t} \right) $$

$$ + o(ac_a e^{-\varepsilon t}) + o(\mathcal{F}(t)). \quad (112) $$

One can easily see, that it follows from $g(t(a))/\theta_{a t(a)} = O(1)$, that for $\varepsilon_1$ sufficiently small

$$ 1 - \varepsilon < \left( 1 - \varepsilon_1 \frac{\sigma^2 g(t)}{\theta_a t} \right)^2. $$

Plugging this result into (113) we obtain

$$ \mathbf{E}[G_{-c_a}(t - X^{(a)})] + \mathbf{E}[\hat{G}_{1-\varepsilon}(t - X^{(a)})] - G_{-c_a}(t) - \hat{G}_{1-\varepsilon}(t) \leq 0 \quad (113) $$

for $t > t(a) - C \ln(1/a)/\theta_a$.

Now, consider $g$ such that $g(t) \gg t^{\varepsilon_1}$ for some $\varepsilon_1 > 0$. Then, for $\delta t(a) \leq t \leq t(a) - C \ln(1/a)/\theta_a$, (88) and (107) give

$$ \mathbf{E}[G_{-c_a}(t - X^{(a)})] + LE[\hat{G}_{1-\varepsilon}(t - X^{(a)})] - G_{-c_a}(t) - L\hat{G}_{1-\varepsilon}(t) $$

$$ \leq -ac_a e^{-\varepsilon t} + \frac{L\mathcal{F}(t)}{1 - \varepsilon} \left( -1 + (\gamma^* + \delta_2) \frac{g(t)}{\theta_a t} \right) + o(ac_a e^{-\varepsilon t}) + o(\mathcal{F}(t)). \quad (114) $$

From (111) we know that $g(t)/\theta_{a t} = O(1)$ for $t \geq \delta t(a)$. Further, as one can easily see,

$$ ac_a e^{-\varepsilon t} \gg \mathcal{F}(t) \quad (115) $$

for $\delta t(a) \leq t \leq t(a) - C \ln(1/a)/\theta_a$ with $C$ sufficiently large. Hence,

$$ \mathbf{E}[G_{-c_a}(t - X^{(a)})] + LE[\hat{G}_{1-\varepsilon}(t - X^{(a)})] - G_{-c_a}(t) - L\hat{G}_{1-\varepsilon}(t) \leq 0 \quad (116) $$

for $\delta t(a) \leq t \leq t(a) - (C/\theta) \ln(1/a)$ and a small enough. If $t > t(a) - C \ln(1/a)/\theta_a$, (88) and (107) give

$$ \mathbf{E}[G_{-c_a}(t - X^{(a)})] + LE[\hat{G}_{1-\varepsilon}(t - X^{(a)})] - G_{-c_a}(t) - L\hat{G}_{1-\varepsilon}(t) $$

$$ \leq -ac_a e^{-\varepsilon t} + \frac{L\mathcal{F}(t)}{1 - \varepsilon} \left( -1 + (\gamma^* + \delta_2) \frac{g(t)}{\theta_a t} \right) + \frac{\mathcal{F}(t)}{1 - (\gamma^* + \delta_2) g(t)/\theta_a t} $$

$$ + o(ac_a e^{-\varepsilon t}) + o(\mathcal{F}(t)). \quad (117) $$

Since

$$ L = \lim_{a \to 0} \left( \frac{1}{1 - \gamma^* g(t)/\theta_a t} \right)^2, \quad (118) $$

one can easily see that for $\delta_2$ small enough,

$$ L \geq (1 - \varepsilon) \left( \frac{1}{1 - (\gamma^* + \delta_2) g(t)/\theta_a t} \right)^2. $$

Hence, we finally obtain

$$ \mathbf{E}[G_{-c_a}(t - X^{(a)})] + LE[\hat{G}_{1-\varepsilon}(t - X^{(a)})] - G_{-c_a}(t) - L\hat{G}_{1-\varepsilon}(t) \leq 0 \quad (119) $$

for a small enough in the case $t > t(a) - C \ln(1/a)/\theta_a$, too. Summing up all the above results this means that $Y_n^{(1)}$ is a nonnegative supermartingale.
Now, let us show that \( Y_n^{(2)} \) is a submartingale. Therefore it is sufficient to show that
\[
\mathbb{E} \tilde{G}_{c_n}(t - X^{(a)}) + \mathbb{E} \tilde{G}_{1+\varepsilon}(t - X^{(a)}) \geq \tilde{G}_{c_n}(t) + L \tilde{G}_{1+\varepsilon}(t)
\] (120)
for all \( t \geq \delta t(a) \) and
\[
\mathbb{E} \tilde{G}_{c_n}(t - X^{(a)}) \geq \tilde{G}_{c_n}(t)
\] (121)
for all \( 0 \leq t \leq \delta t(a) \).

Let us first examine \( \tilde{G}_{1+\varepsilon} \). Due to (91),
\[
\mathbb{E} \tilde{G}_{1+\varepsilon}(t - X) - \tilde{G}_{1+\varepsilon}(t) \geq \frac{\mathbb{F}(t)}{a(1+\varepsilon)} \left( - \frac{\mathbb{F}(t - \delta t(a))}{\mathbb{F}(t)} \mathbb{F}'(\delta t(a)) + \int_0^\nu \mathbb{F}(z) \frac{\mathbb{F}(t-z)}{\mathbb{F}(t)} dz \right) + \int_0^\nu \mathbb{F}(z) \frac{\mathbb{F}(t-z)}{\mathbb{F}(t)} dz - \int_{-\infty}^0 \mathbb{F}(z) \frac{\mathbb{F}(t-z)}{\mathbb{F}(t)} dz
\] (122)
for \( \nu \) such that \((t/g(t))^{1-\delta_1} \ll \nu \ll t/g(t)\) where \( \delta_1 \) is small. By using that \( \mathbb{E}|X^{(a)}|^{2+\varepsilon} < \infty \),
\[
\int_{-\infty}^{-\nu} \mathbb{F}(z) \frac{\mathbb{F}(t-z)}{\mathbb{F}(t)} dz \leq \int_{-\infty}^{-\nu} \mathbb{F}(z) dz \leq \nu^{-(1+\varepsilon)} \int_{-\infty}^{-\nu} |z|^{1+\varepsilon} \mathbb{F}(z) dz = o(\nu^{-(1+\varepsilon)}) = o(g(t)/t),
\] (123)
where we used that \( \nu \gg (t/g(t))^{1-\delta_1} \) with \( \delta_1 \) small enough in the last equality. Further, by (97),
\[
\frac{\mathbb{F}(t-\delta t(a))}{\mathbb{F}(t)} \mathbb{F}'(\delta t(a)) \leq \mathbb{F}'(\delta t(a)) = o(a).
\] (124)
To bound the other terms on the right hand side of (123), let us again consider two different types of distribution functions separately. First, regard \( g \) such that \( g(t) \ll t^{\varepsilon_1} \) for all \( \varepsilon_1 > 0 \). Since \( \mathbb{F}(t-z) \geq \mathbb{F}(t) \) for all \( z \geq 0 \) and \( \mathbb{F}(t-z) \leq \mathbb{F}(t) \) for all \( z \leq 0 \),
\[
\int_0^\nu \mathbb{F}(z) \frac{\mathbb{F}(t-z)}{\mathbb{F}(t)} dz + \int_0^\nu \mathbb{F}(z) \frac{\mathbb{F}(t-z)}{\mathbb{F}(t)} dz - \int_{-\nu}^0 \mathbb{F}(z) \frac{\mathbb{F}(t-z)}{\mathbb{F}(t)} dz \geq -a + o(a) + o(g(t)/t),
\] (125)
where we used (97) and (103).

Plugging the results from (124), (125) and (126) into (123), we obtain
\[
\mathbb{E} \tilde{G}_{1+\varepsilon}(t - X) - \tilde{G}_{1+\varepsilon}(t) \geq \frac{\mathbb{F}(t)}{a(1+\varepsilon)} \left( -a + o(a) + o\left(\frac{g(t)}{t}\right) \right) \geq \frac{\mathbb{F}(t)}{1+\varepsilon} \left( -1 + o(1) + o\left(\frac{g(t)}{\theta_n t}\right) \right) = -\frac{\mathbb{F}(t)}{1+\varepsilon} + o(\mathbb{F}(t)),
\] (126)
where we used (106) in the last equality. Now, consider \( g(t) \) such that \( g(t) \gg t^{\varepsilon_1} \) for some \( \varepsilon_1 \). In this case, using the definition (12) of \( \gamma^* \), the ratio \( g(t)/t^{\gamma^* - \delta_3} \) is increasing for all \( \delta_3 > 0 \) such that \( \gamma^* - \delta_3 > 0 \). One can easily show that this implies
\[
g(t) - g(t-w) \geq (\gamma^* - \delta_3)w \frac{g(t)}{t}, \quad w \ll t.
\] (127)
Hence, it follows analogously to (92) and (100) that
\[
\int_0^\nu F(z) \frac{F(t - z)}{F(t)} dz - \int_{-\nu}^0 F(z) \frac{F(t - z)}{F(t)} dz \\
\geq -a + \frac{(\gamma^* - \delta_3) a^2 g(t)}{2t} + o(a) + o \left( \frac{g(t)}{t} \right)
\]
and consequently by plugging this result, (124) and (125) into (123), we attain
\[
\mathbf{E} \hat{G}_{1+\varepsilon}(t - X) - \hat{G}_{1+\varepsilon}(t) \geq \frac{F(t)}{1 + \varepsilon} \left( -1 + \frac{\gamma^* - \delta_3}{\theta_a t} g(t) + o(1) + o \left( \frac{g(t)}{\theta_a t} \right) \right),
\]
where we used that \( \theta_a = 2a/\sigma^2 + o(a) \). Thus, by (106),
\[
\mathbf{E} \tilde{G}_{1+\varepsilon}(t - X) - \tilde{G}_{1+\varepsilon}(t) \geq \frac{F(t)}{1 + \varepsilon} \left( -1 + \frac{\gamma^* - \delta_3}{\theta_a t} g(t) + o(1) \right)
\]
for \( t \geq \delta t(a) \).

Now, let us examine \( \tilde{G}_{c_a} \). Put \( \lambda_a = \theta_a + c_a \), then
\[
\mathbf{E} \tilde{G}_{c_a}(t - X^{(a)})
\]
\[
= e^{-\lambda_a t} \left( \mathbf{E} \left[ e^{\lambda_a X^{(a)}} ; X^{(a)} \leq 1/a \right] + \mathbf{E} \left[ e^{\lambda_a X^{(a)}} ; X^{(a)} \in (1/a, t] \right] \right) + e^{\alpha t} F(t). \]

Using \( \mathbf{E}[X^{(a)}]^{2+\varepsilon} < \infty \), the bound \( e^x \geq 1 + x \) and the definition of \( \theta_a = 2a/\sigma^2 + o(a) \), we obtain
\[
\mathbf{E} \left[ e^{\lambda_a X^{(a)}} ; X^{(a)} \leq 1/a \right] \geq \mathbf{E} \left[ e^{\theta_a X^{(a)}} ; X^{(a)} \leq 1/a \right] + c_a \mathbf{E} \left[ X^{(a)} e^{\theta_a X^{(a)}} ; X^{(a)} \leq 1/a \right]
\]
\[
\geq 1 + c_a \mathbf{E} \left[ X^{(a)} ; X^{(a)} \leq 1/a \right] + \theta_a c_a \mathbf{E} \left[ (X^{(a)})^2 ; X^{(a)} \leq 1/a \right]
\]
\[
\geq 1 - ac_a + \theta_a c_a \sigma^2 + o(ac_a) = 1 + ac_a + o(ac_a).
\]

Plugging this result into (132) we attain
\[
\mathbf{E} \tilde{G}_{c_a}(t - X^{(a)}) - \tilde{G}_{c_a}(t)
\]
\[
\geq ac_a e^{-\lambda_a t} - c_a \mathbf{E} \left[ e^{\lambda_a X^{(a)}} ; X^{(a)} \in (1/a, t] \right] + e^{\alpha t} F(t) + o(ac_a e^{-\lambda_a t})
\]
for all \( t \geq 0 \). Hence, \( \mathbf{E} \tilde{G}_{c_a}(t - X^{(a)}) \geq \tilde{G}_{c_a}(t) \) (133) for \( 1/a \leq t \leq \delta t(a) \). In the case of \( 0 \leq t < 1/a \), the bound \( e^x \leq 1 + x + x^2 \), which is valid for \( x \leq 1 \), gives
\[
\mathbf{E} \left[ e^{\lambda_a X^{(a)}} ; X^{(a)} \in (t, 1/a] \right]
\]
\[
= \mathbf{E} \left[ e^{\theta_a X^{(a)}} ; X^{(a)} \in (t, 1/a] \right] + c_a \mathbf{E} \left[ X^{(a)} e^{\theta_a X^{(a)}} ; X^{(a)} \in (t, 1/a] \right] + O(c_a^2)
\]
\[
\leq F(t) + (\theta_a + c_a) \mathbf{E} \left[ X^{(a)} ; X^{(a)} > t \right] + (\theta_a^2 + \theta_a c_a) \mathbf{E} \left[ (X^{(a)})^2 ; X^{(a)} > t \right] + o(ac_a).
\]

Suppose that \( t \geq 0 \) is such that \( t = O(1) \) as \( a \to 0 \). Then, the latter inequality gives
\[
\mathbf{E} \left[ e^{\lambda_a X^{(a)}} ; X^{(a)} \in (t, 1/a] \right] \leq F(t) + o(1)
\]
and by plugging this into (133) we attain (134).
Now suppose $t \to \infty$ as $a \to 0$. Since the second moment is finite, integrating by parts gives
\[
\mathbb{E} \left[ (X^{(a)})^k ; X^{(a)} > t \right] = t^k \mathcal{F}(t) + k \int_t^\infty u^{k-1} \mathcal{F}(u)du
\] 
for $k \in \{1, 2\}$. Therefore, by (135)
\[
\mathbb{E} \left[ e^{\lambda_a X^{(a)}} ; X^{(a)} \in (t, 1/a) \right] \leq (1 + \theta_a t + \theta_a^2 t^2) \mathcal{F}(t) + (\theta_a + c_a) \mathcal{F}'(t) + 2(\theta_a^2 + \theta_a c_a) \int_t^\infty u \mathcal{F}(u)du + o(ac_a) + o(\mathcal{F}(t)),
\] 
where we used that $t \leq 1/a$ and that $c_a = o(a)$. Since $g$ is differentiable, one can easily see that
\[
\frac{\mathcal{F}'(t)}{\mathcal{F}(t)} \sim \frac{1}{g'(t)} \quad \text{and} \quad \frac{\int_t^\infty u \mathcal{F}(u)du}{\mathcal{F}(t)} \sim \frac{t}{g'(t)}
\] 
as $a \to 0$. On the other hand, we have $g(t) \geq (1 + o(1))(2 + \varepsilon) \ln t$, therefore $1/g'(t) = O(t)$ and consequently
\[
\mathbb{E} \left[ e^{\lambda_a X^{(a)}} ; X^{(a)} \in (t, 1/a) \right] \leq \left( 1 + \theta_a t + \theta_a^2 t^2 \frac{\theta_a}{g'(t)} + \frac{2\theta_a^2 t}{g'(t)} \right) \mathcal{F}(t) + o(ac_a) + o(\mathcal{F}(t)).
\] 
Now suppose that $t \ll 1/a$. Then,
\[
\theta_a t = o(1), \quad \frac{\theta_a}{g'(t)} = o(1) \quad \text{and} \quad \frac{\theta_a^2 t}{g'(t)} = o(1).
\] 
Hence,
\[
\mathbb{E} \left[ e^{\lambda_a X^{(a)}} ; X^{(a)} \in (t, 1/a) \right] \leq \mathcal{F}(t) + o(ac_a) + o(\mathcal{F}(t))
\] 
which immediately implies (134). If $t$ is such that $t \simeq 1/a$ with $t \leq 1/a$, then (44) gives $\mathcal{F}(t) = o(ac_a)$ and consequently by (140),
\[
\mathbb{E} \left[ e^{\lambda_a X^{(a)}} ; X^{(a)} \in (t, 1/a) \right] = o(ac_a),
\] 
where we again used that $1/g'(t) = O(t)$. That means (134) is also true in that case and consequently we have shown (122).

Now, let us consider $t$ such that $t > t(a) - C \ln(1/a)/\theta_a$ and without loss of generality assume that $t > C \ln(1/a)/\theta_a$ with $C$ defined like in the proof that $Y_n^{(1)}$ is a supermartingale. Integrating by parts, we obtain the following representation from (133):
\[
\mathbb{E} G_{c_a} (t - X^{(a)}) - \bar{G}_{c_a} (t) \\
\geq ac_a e^{-\lambda_a t} + \lambda_a e^{-\lambda_a t} \int_{t-C(1/\theta_a)\ln(1/a)}^t e^{\lambda_a u} \mathcal{F}(u)du + o(ac_a e^{-\lambda_a t}),
\]
To estimate the integral from the latter inequality let us first consider $g$ such that $g(t) \ll t^{\varepsilon_1}$ for all positive $\varepsilon_1$. Then, by the monotonicity of $g$,

$$\lambda_a e^{-\lambda_a t} \int_{t-C\ln(1/a)/\theta_a}^t e^{\lambda_a u} F(u) du = \lambda_a \int_0^{C\ln(1/a)/\theta_a} e^{-\lambda_a w-g(t-w)} dw$$

$$\geq \lambda_a \bar{F}(t) \int_0^{C\ln(1/a)/\theta_a} e^{-\lambda_a w} dw \geq \bar{F}(t)$$

(144)

for $t \geq C\ln(1/a)/\theta_a$ and hence

$$\mathbb{E} \tilde{G}_{c_a}(t-X(a)) - \tilde{G}_{c_a}(t) \geq ac_a e^{-\lambda_a t} + \bar{F}(t) \mathbf{1}\{t \geq t(a) - C\ln(1/a)/\theta_a\}.$$  

(145)

Combining this result with (127) we attain

$$\mathbb{E} \tilde{G}_{c_a}(t-X(a)) + \tilde{G}_{1+\varepsilon}(t-X(a)) - \tilde{G}_{c_a}(t) - \tilde{G}_{1+\varepsilon}(t)$$

$$\geq -\frac{\bar{F}(t)}{1+\varepsilon} + ac_a e^{-\lambda_a t} + \bar{F}(t) \mathbf{1}\{t \geq t(a) - C\ln(1/a)/\theta_a\} + o(\bar{F}(t))$$

(146)

Hence, (121) is true for $t \geq t(a) - C\ln(1/a)/\theta_a$. On the other hand, (116) implies that (121) is also true in the case $\delta t(a) \leq t(a) - C\ln(1/a)/\theta_a$.

Now, let $g$ be such that there exists a $\varepsilon_1 > 0$ such that $g(t) \gg t^{\varepsilon_1}$. Then, using (128) one can show very similar to (86) that for $t \geq t(a) - C\ln(1/a)/\theta_a$,

$$\lambda_a e^{-\lambda_a t} \int_{t-C\ln(1/a)}^t e^{\lambda_a u} F(u) du \geq \frac{\bar{F}(t)}{1 - (\gamma^* - \delta_3) g(t)/(\theta_a t)} + o(\bar{F}(t))$$

(147)

and by plugging this result into (144),

$$\mathbb{E} \tilde{G}_{c_a}(t-X(a)) - \tilde{G}_{c_a}(t)$$

$$\geq ac_a e^{-\lambda_a t} + \frac{\bar{F}(t)}{1+\varepsilon} \mathbf{1}\{t \geq t(a) - C\ln(1/a)/\theta_a\} + \frac{(\gamma^* - \delta_3) g(t)/(\theta_a t)}{1 - (\gamma^* - \delta_3) g(t)/(\theta_a t)} + ac_a e^{-\lambda_a t} + o(\bar{F}(t)).$$

(148)

Combining (131) and (149) we obtain

$$\mathbb{E} \tilde{G}_{c_a}(t-X(a)) + L \mathbb{E} \tilde{G}_{1+\varepsilon}(t-X(a)) - \tilde{G}_{c_a}(t) - L \tilde{G}_{1+\varepsilon}(t)$$

$$\geq ac_a e^{-\lambda_a t} + L \frac{\bar{F}(t)}{1 + \varepsilon} \left(-1 + \frac{(\gamma^* - \delta_3) g(t)}{\theta_a t}\right)$$

$$+ \frac{\bar{F}(t)}{1 - (\gamma^* - \delta_3) g(t)/(\theta_a t)} + o(ac_a e^{-\lambda_a t}) + o(\bar{F}(t)).$$

(149)

Combining the latter inequality with (116), we conclude that (121) is true for $t \leq t(a) - C\ln(1/a)/\theta_a$. On the other hand, one can easily see that for $\delta_3$ small enough,

$$L \leq (1 + \varepsilon) \left(\frac{1}{1 - (\gamma^* - \delta_3) g(t)/(\theta_a t)}\right)^2.$$

Hence, (134) is also true for $t > t(a) - C\ln(1/a)/\theta_a$ and therefore $Y_n^{(2)}$ is a non-negative submartingale.
2.3. Proof of Theorem 3. Fix some $\delta > 0$ such that

$$Y_n^{(1)} = G - c_a (x - S^{(a)}_{\mu}) + L\tilde{G}_1 (x - S_{\mu x - \delta x})$$

is a non-negative supermartingale for all $a$ small enough, which is possible due to Proposition 2. Then,

$$e^{-(\theta - c_a)x} + \frac{L}{a(1 - \varepsilon)} F'(x) = Y_0^{(1)} \leq \mathbb{E} Y_0^{(1)}$$

$$= \mathbb{E} \left[ G - c_a (x - S^{(a)}_{\mu}); \mu x < \infty \right] + \mathbb{L} \mathbb{E} \left[ \tilde{G}_1 (x - S_{\mu x - \delta x}); \mu x - \delta x < \infty \right]$$

Further, by the definition of $G$ and $\tilde{G}$,

$$\mathbb{E} [ G - c_a (x - S^{(a)}_{\mu}); \mu x < \infty ] = \mathbb{P} (\mu x < \infty) = \mathbb{P} (M^{(a)} > x).$$

Plugging these results into (151) we attain

$$\mathbb{P} (M^{(a)} > x) \leq (1 + o(1)) \left( e^{-(\theta - c_a)x} + \frac{L}{a(1 - \varepsilon)} F'(x) \right)$$

It is $(1/a)F'(x) \gg e^{-\theta x}$ for all $x \gg x(a)$ and, by the definition of $c_a$, $c_a x \to 0$ for all $x$ such that $x/x(a) = O(1)$. Hence, we conclude by letting $\varepsilon \to 0$ that

$$\mathbb{P} (M^{(a)} > x) \leq (1 + o(1)) \left( e^{-\theta x} + \frac{L}{a} F'(x) \right).$$

On the other hand, there exists some $\delta > 0$ such that $Y_n^{(2)}$ is a non-negative submartingale. Hence,

$$e^{-(\theta + c_a)x} + \frac{L}{a(1 + \varepsilon)} F'(x) = Y_0^{(2)} \geq \mathbb{E} Y_0^{(2)}$$

$$= \mathbb{E} \left[ \tilde{G}_c (x - S^{(a)}_{\mu}); \mu x < \infty \right] + \mathbb{L} \mathbb{E} \left[ \tilde{G}_1 (x - S_{\mu x - \delta x}); \mu x - \delta x < \infty \right]$$

$$= \mathbb{P} (M^{(a)} > x)$$

and consequently by again letting $\varepsilon \to 0$ and since $c_a x \to 0$ for $x$ such that $x/x(a) = O(1)$ and $(1/a)F'(x) \gg e^{-\theta x}$ for all $x \gg x(a)$,

$$\mathbb{P} (M^{(a)} > x) \geq (1 + o(1)) \left( e^{-\theta x} + \frac{L}{a} F'(x) \right).$$

**References**


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