Bias Reduction when Data are Rounded

Christopher S. Withers & Saralees Nadarajah

First version: 10 December 2013
Bias reduction when data are rounded

by

Christopher S. Withers
Applied Mathematics Group
Industrial Research Limited
Lower Hutt, NEW ZEALAND

Saralees Nadarajah
School of Mathematics
University of Manchester
Manchester M13 9PL, UK

Abstract: Analytical bias reduction methods are developed for univariate rounded data for the first time. Extensions are given to rounding of multivariate data, and to smooth functionals of several distributions. As a by product, we give for the first time the relation between rounded and unrounded multivariate cumulants.

Keywords: Bias reduction; Cumulants; Rounding.

1 Introduction

There are many reasons and many applied areas for rounding of measurements. Some recent examples are: “Measurements of continuous random variables are inevitably rounded to certain digits before or after the decimal point according to the precision of the measuring instruments or the recording/storage mechanism. For example, scientific measurements are often recorded to the nearest millimetres. The Consumer Price Index is commonly rounded to a single decimal place” (Li and Bai, 2011); “In general, measuring CD can be subjective, and such measurements are only estimates because observations are rounded to the nearest centimeter” (Hoh et al., 2012); “All observations are rounded to the nearest whole knot when recorded at the meteorological station, except that wind speeds below 1 kt are recorded as zero” (Thorarinsdottir and Johnson, 2012); “In practice, it is not uncommon that data are rounded. In the US, birth weights show ounces, for example 5 lb 3 oz, while in Europe, for example in Ukraine, intervals could be in 50 g or nearly 1.8 oz” (Wang and Wertelecki, 2012).

However, we are aware of no work in the literature considering bias reduction for rounded data. Bias reduction is a common method for providing accurate point as well as interval estimators. The aim of this paper to derive analytical bias reduction methods for univariate rounded data, multivariate rounded data and for multiple samples of rounded data.

Analytical bias reduction methods have several advantages over other nonparametric methods for bias reduction like bootstrapping and jackknifing. Two main advantages are: (i) they take only $O(n)$ calculations, while bootstrap estimators take $O(n^p)$ calculations, where $p$ denotes the order of bias reduction and $n$ denotes the sample size; (ii) they offer the advantage of analytic clarity as opposed to blind number crunching for a set of parameter...
choices that may miss some features.

Withers and Nadarajah (2008, 2012a) have developed analytic methods for nonparametric bias reduction that remove the need for computationally intensive methods like the bootstrap and the jackknife. They define an estimator as having \( p \)th order if its bias has magnitude \( n_0^{-p} \) as \( n_0 \to \infty \), where \( n_0 \) is the sample size or the minimum sample size if the estimator is a function of more than one sample. For general \( p \), Withers and Nadarajah (2008) have provided analytic \( p \)th order nonparametric estimators that require only \( O(N) \) calculations, where \( N \) is the total sample size. These estimators are given in terms of the von Mises derivatives of the functional being estimated, evaluated at the empirical distribution function. For \( p \leq 4 \), Withers and Nadarajah (2012a) have given explicit estimators in terms of the first \( 2p - 2 \) von Mises derivatives of the functional evaluated at the empirical distribution functions. These may be used to obtain unbiased estimators (UEs) when these exist and are of known form in terms of the sample sizes.

The analytic nonparametric bias reduction methods of Withers and Nadarajah (2008, 2012a) are for unrounded data. Here, we extend these methods for rounded data.

We have written programs in the R statistical software (R Development Core Team, 2013) to implement the general methods of this paper for rounded data: Theorems 2.1 to 2.5 for bias reduction for univariate rounded data; Theorem 3.1 for bias reduction for multivariate rounded data. The electronic versions of these programs can be obtained from the corresponding author, email: mbbsssn2@manchester.ac.uk We have chosen the R statistical software since it is freely available and since it can be implemented in any platform. These programs are believed to be the first programs in R for any analytical bias reduction. The programs can be used to yield ready and fast analytic bias reduction. The approach will be as accurate and far less time consuming than using bootstrapping, jackknifing and other computer intensive methods.

Let \( X_1, \ldots, X_n \) be a random sample of real univariate observations with empirical distribution function \( F_n \) from a distribution function \( F(x) \) with finite derivatives, mean \( \mu \), and density \( f(x) \). The problem we consider is to find an estimator of low bias for any smooth functional \( T(F) \) given a rounded sample from \( F \). By Withers and Nadarajah (2008, 2012a), for \( p \geq 1 \), an estimator of \( T(F) \) is \( S_{np}(F_n) \), where

\[
S_{np}(F,T) = S_{np}(F) = \sum_{i=0}^{p-1} S_i(F)/[n-1]_i, \quad [n-1]_i = (n-1) \cdots (n-i). \quad (1.1)
\]

In particular,

\[
S_0(F) = T(F), \quad S_1(F) = -T[2]/2, \quad S_2(F) = T[3]/3 + T[2^2]/8,
\]

\[
S_3(F) = -T[4]/4 + 3T[2^2]/8 - 3T[32]/6 - T[2^3]/48,
\]

\[
S_4(F) = -T[5]/5 - 2T[32]/3 - 3T[2^3]/16 + T[42]/8 + T[3^2]/18 + T[32^2]/24 + T[2^4]/384,
\]

where

\[
T[r] = \int \int T_F(x^r) dF(x), \quad T[r,s] = \int \int T_F(x^r, y^s) dF(x)dF(y),
\]

where \( T_F(x_1, \ldots, x_r) \) is the \( r \)th functional derivative of \( T(F) \), \( T_F(x^2) = T_F(x, x) \), \( T_F(x^3, y^2) = T_F(x, x, y, y) \), \( T[x^2] = T(xx) \), \( T[x^3] = T[xxx] \), \( T[xy^2] = T[xyy] \), and so on. \( S_{np}(F_n) \)
is an UE if $T(F)$ is a polynomial of order $p$ in $F$ such as the product of moments or cumulants of total order $p$. Tchouproff (1918), Fisher (1929) and James (1958) gave UEs of this form for $\kappa_p$ and $\mu_p$. We can expand (1.1) in the form

$$S_{np}(F) = T_{np}(F) + O \left( n^{-p} \right),$$

(1.3)

where

$$T_{np}(F) = \sum_{i=0}^{p-1} n^{-i} T_i(F), \quad p \geq 1.$$  

So, for $p \geq 1$, $T_{np}(F)$ also estimates $T(F)$ with bias $O(n^{-p})$.

Now suppose that only the rounded values of the sample are recorded, say $X_{h1}, \ldots, X_{hn}$, where $X_{hj}$ is $X_j$ rounded to the nearest integral multiple of a given constant $h > 0$. Let $F_{nh}$ be their empirical distribution function, and let $F_h(x)$ be the distribution function of $X_h = X_{h1}$.

Theorem 2.4 gives our main result for univariate data. We show that for any smooth functional $T(F)$, we have expansions of the form

$$T(F_h) = T(F) + \sum_{r=1}^{\infty} T_r(h)/r! = \sum_{k=0}^{\infty} H^k T_k(F) = T_h(F) \text{ say},$$

(1.4)

$$T(F) = T^h(F_h), \quad T^h(F) = T(F) + \sum_{r=1}^{\infty} T^h_r(F)/r! = \sum_{k=0}^{\infty} H^k T_k(h),$$

(1.5)

where $H = h^2/4$. (For example, if $T(F)$ is a smooth function of $(\mu, \mu_2)$, then $T(F_h) = T(F)$, that is, the expansions (1.4) and (1.5) for $T(F_h)$ and $T(F)$ only have one term.) So, $S_{np}^h(F_{nh}, T^h)$ is an estimator of $T(F)$ based on rounded data with bias $O(n^{-p})$. So,

$$\mathbb{E} \left[ \sum_{k=0}^{\infty} H^k S_{np}^h (F_{nh}, T^h_k) \right] = \mathbb{E} \left[ S_{np}^h (F_{nh}, T^h) \right] = T(F) + O(n^{-p}).$$

This and its multivariate and multisample extensions in Sections 3 and 4, are the key results of this paper.

If the series for $T^h$ is truncated at $I$ terms, an additional bias $O(H^I)$ is introduced. We can make this dimensionless by writing $O \left( \overline{H}^I \right)$ for $O(H^I)$, where $\overline{H} = H/\mu_2$. If $T(F)$ is a polynomial in moments or cumulants, so that it is a polynomial in $F$ of total degree $p$ say, then these expansions stop at $h^p$, as we now illustrate. Define the $p$th non-central and central moments of $X$ and $X_h$ by

$$m_p = m_p(F) = \mathbb{E} [X^p], \quad m_{ph} = m_p(F_h) = \mathbb{E} [X_h^p],$$

$$\mu_p = \mu_p(F) = \mathbb{E} [(X - \mu)^p], \quad \mu_{ph} = \mu_p(F_h),$$

and their $p$th cumulants by $\kappa_p = \kappa_p(F), \kappa_{ph} = \kappa_p(F_h), p \geq 1$. Then,

$$\kappa_{1h} = \kappa_1 = \mu, \quad \kappa_{ph} = \kappa_p + b_{ph},$$

(1.6)
where \( b_{ph} = B_p h^p / p \) for \( p \geq 2 \), and \( B_p \) is the \( p \)th Bernoulli number, defined by

\[
t / (e^t - 1) = \sum_{p=0}^{\infty} t^p B_p / p! = -t / 2 + \sum_{k=0}^{\infty} B_{2k} t^{2k} / (2k)!
\]

(This does not contradict (1.4) or (1.5) since \( B_p = 0 \) for \( p = 3, 5, 7, \ldots \). For \( B_p \) to \( p = 14 \), see Table 23.1 on page 809 of Abramowitz and Stegun (1964).) So, \( \kappa_{ph} = \kappa_p \) for \( p \) odd. Kendall (1938) gave a proof for bounded variables, and noted proofs of earlier authors. We give a proof in Example 2.2 below, and an extension to multivariate data in Example 3.1. UEs for cumulants and their products easily follow: see Examples 2.2, 3.1. Sheppard (1899, 1907) showed that

\[
m_{ph} = \sum_{k=0}^{[p/2]} H^k \left( \frac{p}{2k} \right) m_{p-2k} / (2k + 1)
\]  

(1.7)

for \( p \geq 1 \), where \([p/2]\) is the integral part of \( p/2 \). This is a special case of (1.4). For example,

\[
m_{1h} = m_1, \ m_{2h} = m_2 + H / 3, \ m_{3h} = m_3 + Hm_1, \ m_{4h} = m_4 + 2Hm_2 + H^2 / 5,
\]

\[
m_{5h} = m_5 + 10Hm_3 / 3 + H^2 m_1, \ m_{6h} = m_6 + 5Hm_4 + 3H^2 m_2 + H^3 / 7,
\]

\[
m_{7h} = m_7 + 7Hm_5 + 7H^2 m_3 + H^3 m_1,
\]

\[
m_{8h} = m_8 + 28Hm_6 / 3 + 14H^2 m_4 + 8H^3 m_2 + H^4 / 9,
\]

\[
m_{9h} = m_9 + 12Hm_7 + 12H^2 m_5 / 5 + 12H^3 m_3 + H^4.
\]

Kendall (1938) gave regularity conditions for (1.7) to hold, but only for bounded random variables. Corollary 2.1 extends (1.7) to show that for \( \text{any smooth function } g \),

\[
\mathbb{E} [g(X_h)] = \mathbb{E} [g_h(X)],
\]

where

\[
g_h(x) = \sum_{k=0}^{\infty} H^k g_{2k}(x) / (2k + 1)!
\]

Equation (37) of Sheppard (1907) inverted (1.7) to give an UE of \( m_p \) equivalent to that given in Example 2.1 below:

\[
m_p = \sum_{k=0}^{[p/2]} \binom{p}{2k} a_k H^k m_{p-2k,h} = m_p^h (F_h)
\]  

(1.8)

say, where \( p \geq 1 \),

\[
a_k = - \left( 2^{2k} - 2 \right) B_{2k} = 2^{2k} B_{2k}(1/2)
\]  

(1.9)

for \( k \geq 1 \). This is a special case of (1.5). For example,

\[
m_1 = m_{1h}, \ m_2 = m_{2h} - H / 3, \ m_3 = m_{3h} - Hm_{1h},
\]

\[
m_4 = m_{4h} - 2Hm_{2h} + 7H^2 / 15, \ m_5 = m_{5h} - 10Hm_{3h} / 3 + 7H^2 m_{1h} / 3,
\]

\[
m_6 = m_{6h} - 5Hm_{4h} + 7H^2 m_{2h} - 31H^3 / 21,
\]

\[
m_7 = m_{7h} - 7Hm_{5h} + 49H^2 m_{3h} / 3 - 31H^3 m_{1h} / 3,
\]

\[
m_8 = m_{8h} - 28Hm_{6h} / 3 + 98H^2 m_{4h} / 3 - 124H^3 m_{2h} / 3 + 127H^4 / 15,
\]

\[
m_9 = m_{9h} - 12Hm_{7h} + 294H^2 m_{5h} / 5 - 124H^3 m_{3h} + 381H^4 m_{1h} / 5.
\]
So, for $1 \leq p \leq n$, an UE of $m_p$ is

$$\hat{m}_p^h = m_p^h(F_nh),$$

where

$$m_p^h(F) = \sum_{k=0}^{[p/2]} \binom{p}{2k} a_k H^k m_{p-2k}.$$

Theorem 2.2 extends (1.8) to show that for any smooth function $g$,

$$\mathbb{E}[g(X)] = \mathbb{E}\left[g^h(X_h)\right],$$

where

$$g^h(x) = \sum_{k=0}^{\infty} H^k A_k g_{2k}(x), \quad A_k = a_k/(2k)!.$$ (1.10)

Example 2.2 gives an alternative proof of (1.6). Theorem 2.3 gives an UE of $\mathbb{E}[g(X)]$ for any smooth enough function $g$, based on the rounded sample. Theorem 2.4 extends this to nonlinear functionals of $F$. Equation (25) of Sheppard (1907) gave a result equivalent to

$$\mu_2 = \mu_2h - H/3, \quad \mu_3 = \mu_3h,$$

$$\mu_4 = \mu_4h - 2H \mu_2h + 7H^2/15, \quad \mu_5 = \mu_5h - 10H \mu_3h/3.$$

Example 2.3 proves that (1.7) and (1.8) hold with $\{m_i\}$ replaced by $\{\mu_i\}$, that is, for $p \geq 1$,

$$\mu_{ph} = \sum_{k=0}^{[p/2]} H^k \binom{p}{2k} \mu_{p-2k}/(2k + 1), \quad \mu_p = \sum_{k=0}^{[p/2]} \binom{p}{2k} a_k H^k \mu_{p-2k,h} = \mu_p^h(F_h).$$ (1.11)

These are also special cases of (1.4) and (1.5). For example, since $\mu_1 = 0$ this gives

$$\mu_6 = \mu_6h - 5H \mu_4h + 7H^2 \mu_2h - 31H^3/21,$$

$$\mu_7 = \mu_7h - 7H \mu_5h + 49H^2 \mu_3h/3,$$

$$\mu_8 = \mu_8h - 28H \mu_6h/3 + 98H^2 \mu_4h/3 - 124H^3 \mu_2h/3 + 127H^4/15,$$

$$\mu_9 = \mu_9h - 12H \mu_7h + 294H^2 \mu_5h/5 - 124H^3 \mu_3h.$$

The versions of the four expansions (1.7), (1.8), (1.11) given by the authors cited, are more cumbersome as they are in powers of $h$, not $H$. By (1.11), for $1 \leq p \leq n$, an UE of $\mu_p$ is

$$\hat{\mu}_p^h = \mu_{ph}^h(F_nh),$$

where

$$\mu_{ph}^h(F) = \sum_{k=0}^{[p/2]} \binom{p}{2k} a_k H^k \mu_{p-2k,n}(F),$$

and $\mu_{pn}(F_n)$ is the UE of $\mu_p$ given by James (1958) for $p \leq 6$ and by Withers and Nadarajah (2008) for $p \leq 12$ by a different method.
Section 3 generalizes Section 2 to $s$-dimensional rounding: we suppose that for $1 \leq j \leq s$ the $j$th component of the data is rounded to the nearest multiple of some constant $h_j > 0$. The multivariate relations between moments analogous to (1.8) and (1.11) are given in the appendix, using the method of Example 3.2.

Section 4 generalizes Section 3 to multisample data.

Removing the effect of rounding on parametric estimators. The results given here are nonparametric, but apply immediately to bias reduction of estimators of the form $\hat{\theta} = T(F_n)$.

Related results. Withers and Nadarajah (2011) have given UEs for moments for the linear regression model when observations are rounded. Withers and Nadarajah (2012b) have shown how to adjust Cornish-Fisher expansions and confidence intervals for the effect of roundoff. Bias reduction for parametric problems has also been well studied. Withers (1987) gave an estimator $t_{nk}(\hat{\theta})$ of bias $\sim n^{-k}$ for any smooth function $t(\theta) : \mathbb{R}^p \to \mathbb{R}$, when $F$ is determined by $\theta$.

Notation. Set
\[ [r]_k = r!/(r-k)! = r(r-1) \cdots (r-k+1), \]
\[ D = d/dx, \quad g_{k}(x) = D^k g(x) \text{ for } g : \mathbb{R} \to \mathbb{C}, \]
\[ \partial_i = \partial/\partial x_i, \quad g_{k_1, \ldots, k_r}(x) = \partial_1^{k_1} \cdots \partial_r^{k_r} g(x_1, \ldots, x_r) \text{ for } g : \mathbb{R}^r \to \mathbb{C}. \]
We use the notation $g$ to denote both the univariate and multivariate versions. The distinction should be clear from the context.

2 Proof of (1.4) and (1.5)

In this section, we give an UE of $E[g(X_1, \ldots, X_r)]$ and then prove (1.4) and (1.5). Let $g(x) : \mathbb{R} \to \mathbb{C}$ be any function with finite $k$th derivative $g_{k}(x)$, $k \geq 1$. Set
\[ \theta = E[g(X)], \quad \theta_h = E[g(X_h)] = \int g(x) dF_h(x), \]
\[ \hat{\theta}_h = \int g(x) dF_{nh}(x) = n^{-1} \sum_{j=1}^{n} g(X_{hj}), \quad (2.1) \]
so that $E[\hat{\theta}_h] = \theta_h$. Corollary 2.1 below shows that $\hat{\theta}_h$ estimates $\theta$ with bias
\[ \theta_h - \theta = \sum_{k=1}^{\infty} H^k E[g_{2k}(X)/(2k+1)!] = O(H) \]
(2.2)
as $H \to 0$. Since $F$ is continuous, $\hat{\theta}_h$ of (2.1) has mean $\theta_h$ given by
\[ E[g(X_h)] = \int g dF_h = \sum_{m=\infty}^{\infty} g(mh) \{ F(mh + h/2) - F(mh - h/2) \}. \quad (2.3) \]
We first expand this as a Taylor series in $h$, and then prove (2.2). Set
\[ G(x) = x^{-1} \sinh x = \sum_{k=0}^{\infty} x^{2k}/(2k+1)! \]
for \( x \in \mathbb{R} \). By equation (1.411.12) in Gradshteyn and Ryzhik (2000) or equation (23.1.21) in Abramowitz and Stegun (1964), for \(|x| < \pi\),

\[
G(x)^{-1} = x / \sinh x = \sum_{k=0}^{\infty} A_k x^{2k}
\]  

(2.4)

for \( A_k \) of (1.10). Set

\[
g_h(x) = G(hD/2)g(x) = \sum_{k=0}^{\infty} H^k g_{2k}(x)/(2k + 1)!,
\]

(2.5)

\[
g^h(x) = G(hD/2)^{-1}g(x) = \sum_{k=0}^{\infty} H^k A_k g_{2k}(x).
\]

(2.6)

**Theorem 2.1** Given \( g \) and \( F \),

\[
\mathbb{E}[g(X_h)] = \sum_{k=0}^{\infty} h^k L_k(g)
\]

when this converges, where

\[
L_k(g) = \begin{cases} 
2^{-k} (e_k(\infty) + e_k(-\infty))/k!, & k = 1, 3, 5, \ldots, \\
2^{-k} \int e_{k+1}(x)dx/(k+1)! + d_{k/2}(g), & k = 0, 2, 4, \ldots,
\end{cases}
\]

(2.7)

\[
e_k(x) = g(x)F_k(x), \quad d_0(g) = 0,
\]

\[
d_k(g) = \sum_{\ell=0}^{k-1} [B_{2k-2\ell}/(2k-2\ell)!] \alpha_{k-\ell}(e_{2\ell+1}) 2^{-2\ell}/(2\ell + 1)!, ~ k \geq 1,
\]

\[
\alpha_k(v) = \nu_{2k-1}(\infty) - \nu_{2k-1}(-\infty).
\]

So,

\[
e_k(x) \to 0 \text{ as } |x| \to \infty \text{ for } k \geq 1,
\]

\[
\mathbb{E}[g(X_h)] = \sum_{k=0}^{\infty} H^k \left[ \int e_{2k+1}(x)dx/(2k + 1)! + d_k(g) \right]
\]

(2.8)

when this converges.

**Proof:** Since

\[
F(x + h/2) - F(x - h/2) = 2 \sum_{\ell=0}^{\infty} (h/2)^{2\ell+1} F_{2\ell+1}(x)/(2\ell + 1)!,
\]

we have the right hand side of (2.3) equal to

\[
\sum_{m=-\infty}^{\infty} g(mh) \sum_{\ell=0}^{\infty} h^{2\ell+1} F_{2\ell+1}(mh) 2^{-2\ell}/(2\ell + 1)! = \sum_{\ell=0}^{\infty} h^{2\ell+1} g_{2\ell+1}(h, e_{2\ell+1})/(2\ell + 1)!, (2.9)
\]
where
\[
d(h, v) = \sum_{m=\infty}^{\infty} v(mh) = -v(0) + d(h, v) + \bar{d}(h, \bar{v}),
\]
\[
\bar{d}(h, v) = \sum_{m=0}^{\infty} v(mh) = h^{-1} \int_{-\infty}^{\infty} v(x) dx + \{v(0) + v(\infty)\} / 2 + \sum_{j=1}^{\infty} h^{2j-1} B_{2j} \bar{\pi}_j(v)/(2j)!,
\]
\[
\bar{\pi}(x) = v(-x), \quad \bar{\pi}_j(v) = v_{2j-1}(\infty) - v_{2j-1}(0)
\]
by the Euler-MacLaurin formula, equation (23.1.30), page 806 of Abramowitz and Stegun (1964) with \(a = 0, m = \infty\). So,
\[
d(h, v) = h^{-1} \int_{-\infty}^{\infty} v + \{v(\infty) + v(-\infty)\} / 2 + \sum_{j=1}^{\infty} h^{2j-1} B_{2j} \beta_j(v)/(2j)!, \tag{2.10}
\]
where
\[
\beta_j(v) = \bar{\pi}_j(v) + \bar{\pi}_j(\bar{v}) = v_{2j-1}(\infty) - 2v_{2j-1}(0) + v_{2j-1}(-\infty).
\]

Now apply (2.10) to (2.9) and collect terms to obtain \(E[g(X_h)]\) as being equal to the right hand side of (2.8) plus \(O_h\), where
\[
O_h = [C_h(\infty) + C_h(-\infty)] g(x),
\]
\[
C_h(x) = \sum_{l=0}^{\infty} (h/2)^{2l+1} F_{2l+1}(x)/(2l+1)! = \sinh(hD/2) F(x)
\]
\[
= (F(x + h/2) - F(x - h/2)) / 2 \to 0
\]
as \(x \to \pm\infty\), by Taylor’s expansion, so that \(O_h = 0\). \(\Box\)

**Corollary 2.1** Suppose that
\[
\sum_{k=1}^{\infty} h^{2k} B_{2k+1} \sum_{l=0}^{k-1} [e_{2l+1}(x) + e_{2l+1}(-x)] D^{2(k-l)-1} [e_{2l+1}(x) + e_{2l+1}(-x)] / (2k - 2l)! \to 0 \tag{2.11}
\]
as \(x \to \infty\), and
\[
g_i(x) f_j(x) \to 0 \tag{2.12}
\]
as \(x \to \pm\infty\) for \(i, j \geq 1\). Then, \(\theta_h\) of (2.1) is given in terms of \(g_h\), \(f_h\) of (2.5) by
\[
E[g(X_h)] = \int g(x) f_h(x) dx = \int f(x) g_h(x) dx = E[g_h(X)] = \sum_{k=0}^{\infty} H^k \mathbb{E}[g_{2k}(X)/(2k+1)!]. \tag{2.13}
\]
Proof: Under these conditions, (2.7) reduces to

\[ L_k(g) = I(k \text{ even})2^{-k} \int e_{k+1}(x)dx/(k+1)! , \]

so that

\[ \mathbb{E}[g(X_h)] = \int g(x) \sum_{k=0}^{\infty} H_k f_{2k}(x)dx/(2k+1)! = \int g(x)f_h(x)dx \]

if absolutely integrable.

Also, by integrating by parts, if \( g_i(x)f_{k-i}(x) \to 0 \) as \( x \to \pm \infty \) for \( 1 \leq i \leq k-1 \), then

\[ \int g(x)f_k(x)dx = -\int g_1(x)f_{k-1}(x)dx = (-1)^i \int g_i(x)f_{k-i}(x)dx = (-1)^k \int g_k(x)f(x)dx. \]

So, by (2.12),

\[ \int g f_h = \int g(x)G(hD/2)f(x)dx = \int f(x)G(hD/2)g(x)dx = \int fg_h. \]

The proof is complete. \( \square \)

The problem of finding an UE of \( \int gdF \) is essentially that of “inverting” this series. The solution is given in terms of \( g^h(x) \) of (2.6) and \( A_k \) of (1.10) by

**Theorem 2.2** If the derivatives of \( F \) and \( g \) tend to zero at \( \pm \infty \) as in (2.11) and (2.12), then for \( g^h \) of (2.6), \( g^h(X_h) \) is an UE of \( \mathbb{E}[g(X)] \):

\[ \mathbb{E}[g(X)] = \mathbb{E}[g^h(X_h)] = \sum_{k=0}^{\infty} H_k A_k \mathbb{E}[g_{2k}(X_h)]. \]  \hspace{1cm} (2.14)

**Proof:** Replace \( g(x) \) in (2.13) by \( g^h(x) \). The result follows from (2.4), (2.6). \( \square \)

So,

**Theorem 2.3** If the derivatives of \( F \) and \( g \) tend to zero at \( \pm \infty \) as in (2.11) and (2.12), then an UE of \( \mathbb{E}[g(X)] = \int gdF \) is \( T^h(F_{nh}) \), where

\[ T^h(F) = \mathbb{E}[g^h(X)] = \int g^h dF \]

for \( g^h \) of (2.6). So, for \( K \geq 1 \),

\[ T_{Kh}(F_{nh}) = \sum_{k=0}^{K-1} H_k A_k n^{-1} \sum_{i=1}^{n} g_{2k}(X_{hi}) \]

estimates \( \theta = \int gdF \) with bias \( O(\overline{h}^{2K}) \sim \overline{h}^{2K} \), where \( \overline{h} = h/\sigma, \sigma = \mu_2^{1/2} \).
Example 2.1 Note that (1.7) and (1.8) are just (2.13) and (2.14) with $g(x) = x^p$. Also $E[m_p^h(X_h)] = m_p$, where

$$m_p^h(x) = G(hD/2)^{-1}x^p = \sum_{k=0}^{[p/2]} A_k H^k D^{2k} x^p = \sum_{k=0}^{[p/2]} \binom{p}{2k} a_k H^k x^{p-2k}$$

(2.15)

since $D^{2k} x^p = [p]_{2k} x^{p-2k}$. Taking the mean of $m_p^h(X)$ gives (1.8).

That $m_p$ can be recovered from $\{m_{ph}\}$ even though there are infinitely many $F$ such that $\Pr((j-1)h/2 < X \leq jh/2)$ takes a given value $\Pr(X_h = jh)$, is a reminder that the moments need not determine a distribution.

Example 2.2 Take $g(x) = e^{xt}$, where $t \in \mathbb{C}$. Then,

$$g_n(x) = G(ht/2)g(x), \quad g^h(x) = G(ht/2)^{-1}g(x),$$

$$E[e^{X_{ht}}] / E[e^{X_{ht}}] = G(ht/2), \quad \ln E[e^{X_{ht}}] - \ln E[e^{X_{ht}}] = \ln G(ht/2).$$

But by equation (1.518.1), page 53 of Gradshteyn and Ryzhik (2000),

$$\ln G(t/2) = \sum_{p=1}^{\infty} t^{2p} k_{2p}/(2p)!,$$

(2.16)

where $k_{2p} = B_{2p}/(2p)$, so that

$$\kappa_{2p-1,h} = \kappa_{2p-1}, \quad \kappa_{2p,h} = \kappa_{2p} + h^{2p} B_{2p}/(2p),$$

proving Kendall’s (1.6). So, for $p \geq 2$, an UE of $\kappa_p$ based on rounded data is

$$\hat{\kappa}_{pn}^h = k_{pn} (F_{nh}) - b_{ph}$$

for $b_{ph}$ of (1.6), where $k_{pn} (F_n)$, Fisher’s $k_p$ statistic, an UE of $\kappa_p$ for unrounded data, see Section 12.6 of Stuart and Ord (1987).

So, for $T(F)$ a smooth function of the odd cumulants, $T(F_h) = T(F)$. Similarly, since

$$\kappa_p \kappa_q = (\kappa_p - b_{ph}) (\kappa_q - b_{qh}),$$

an UE of $\kappa_p \kappa_q$ is

$$\hat{\kappa}_{pq}^h = k_{pq} (F_{nh}) - \sum_{pq} b_{ph} k_{pq} (F_{nh}) + b_{ph} b_{qh},$$

where $k_{pq} (F_n)$, Fisher’s UE (polykay) of $\kappa_p \kappa_q$ for unrounded data, see Section 13.22 of Stuart and Ord (1987).

UEs of higher products of cumulants can be dealt with similarly.

Equation (2.13) fails if $E[g_{2k}(X)]$ is unbounded for some $k \geq 0$. Equation (2.14) fails if $E[g_{2k}(X_h)]$ is unbounded for some $k \geq 0$, for example, if $g_{2k}(0)$ is unbounded for some $k \geq 0$, as is the case for $g(x) = x^{1/2}$ or $\ln(x)$, and $\Pr(|X| < h/2) > 0$. So, (2.14) fails for these two cases of $g$ if $X$ has support (0, $\infty$), although it will work for $g(x) = (x + x_0)^{1/2}$ or $\ln(x + x_0)$ if $x_0 \geq h/2$.

So far we have been dealing with the functional $E[g(X)] = \int g dF$, using $g_h$, $g^h$ of (2.5) and (2.6). This is linear in $F$. Applying (2.13) and (2.14) $r$ times gives
Corollary 2.2 Given a smooth function \( g(x_1, \ldots, x_r) : \mathbb{R} \times \cdots \times \mathbb{R} \to \mathbb{C} \), and independent \( X_1, \ldots, X_r \) from a distribution function \( F \) on \( \mathbb{R} \), rounded as above to \( X_{1h}, \ldots, X_{rh} \) with distribution function \( F_h \) on \( \mathbb{R} \),

\[
\mathbb{E}[g(X_{1h}, \ldots, X_{rh})] = \mathbb{E}[g_h(X_1, \ldots, X_r)],
\]

(2.17)

\[
\mathbb{E}\left[g^h(X_{1h}, \ldots, X_{rh})\right] = \mathbb{E}[g(X_1, \ldots, X_r)],
\]

(2.18)

where

\[
g_h(x_1, \ldots, x_r) = G_h(\partial/2)g(x_1, \ldots, x_r),
\]

\[
g^h(x_1, \ldots, x_r) = G_h(\partial/2)^{-1}g(x_1, \ldots, x_r),
\]

\[
G_h(\partial/2) = \prod_{i=1}^{r} G_h(\partial_i/2).
\]

So,

\[
g_h(x_1, \ldots, x_r) = \sum_{k_1, \ldots, k_r = 1}^{\infty} H_1^{k_1 + \cdots + k_r} g_{2k_1, \ldots, 2k_r}(x_1, \ldots, x_r) / \prod_{i=1}^{r} (2k_i + 1)!,
\]

\[
g^h(x_1, \ldots, x_r) = \sum_{k_1, \ldots, k_r = 1}^{\infty} H_1^{k_1 + \cdots + k_r} A_{k_1} \cdots A_{k_r} g_{2k_1, \ldots, 2k_r}(x_1, \ldots, x_r).
\]

By (2.18), for \( 1 \leq r \leq n \), an UE of \( \mathbb{E}[g(X_1, \ldots, X_r)] \) is

\[
N_r^{-1} \sum_{r_c}^{N_r} g^h(Y_{1h}, \ldots, Y_{rh})
\]

summed over all \( N_r = \binom{n}{r} \) subsamples \( Y_{1h}, \ldots, Y_{rh} \) of \( X_{1h}, \ldots, X_{rh} \).

The next three examples give UEs of central moments. As noted, James (1958) and Example 6.3 of Withers and Nadarajah (2008) gave an UE for \( \mu_r \) of the form \( \mu_r(F_n) \) for \( r \leq 6 \) and \( r \leq 12 \), respectively.

Example 2.3 An UE of \( \mu_p \). Write

\[
\mu_p = \mathbb{E}[(X - \mu)^p] = \sum_{i=0}^{p} \binom{p}{i} (-\mu)^i m_{p-i} = \sum_{i=0}^{p} \binom{p}{i} (-1)^i J_{i+1},
\]

(2.19)

where

\[
J_{i+1} = \mathbb{E}[g_{i+1}(X_1, \ldots, X_{i+1})], \quad g_{i+1}(x_1, \ldots, x_{i+1}) = x_1 \cdots x_i x_{i+1}^{p-i},
\]

So, an UE of \( \mu_p \) is given by replacing \( J_{i+1} \) in (2.19) by

\[
\tilde{J}_{i+1}(X_{1h}, \ldots, X_{i+1,h}) = X_{1h} \cdots X_{ih} m_{p-i}(X_{h,i+1})
\]

for \( m_p^h(x) \) of Example 2.1. That is,

\[
\mu_p = \sum_{i=0}^{p} \binom{p}{i} (-1)^i \mu_h^i \mathbb{E}[m_{p-i}^h(X_h)],
\]
where \( \mathbb{E} [m_p^h (X_h)] \) is equal to the right hand side of (1.8). So,

\[
\mu_p = \sum_{k=0}^{[p/2]} a_k H^k M_{pkh},
\]

where

\[
M_{pkh} = \sum_{i=0}^{p} (-\mu_h)^i \left( \begin{array}{c} p \\ i \end{array} \right) m_{p-i-2k,h} = \left( \begin{array}{c} p \\ 2k \end{array} \right) \mu_{p-2k,h},
\]

proving (1.11). So, an UE of \( \mu_p \) is given by (1.12).

The next two examples illustrate this for \( p = 2, 3 \).

**Example 2.4** An UE of \( \mu_2 \) using unrounded data is \( \mu_{2n} (F_n) \), where

\[
\mu_{2n} (F) = (1 - n^{-1})^{-1} \mu_2 (F).
\]

So, an UE of \( \mu_2 = \mu_{2h} - H/3 \) using rounded data is \( \mu_{2n} (F_{nh}) - H/3 \).

**Example 2.5** An UE of \( \mu_3 \) using unrounded data is \( \mu_{3n} (F_n) \), where

\[
\mu_{3n} (F) = N_3^{-1} n^3 \mu_3 (F_{nh}), \quad N_3 = \binom{n}{3}.
\] (2.20)

So, an UE of \( \mu_3 = \mu_{3h} \) using rounded data is \( \mu_{3n} (F_{nh}) \).

Other polynomials in \( F \) of degree \( p \) say, can be estimated similarly by writing them as the sum of \( U \)-statistics:

\[
T(F) = \sum_{i=1}^{p} \int \cdots \int g_i (x_1, \ldots, x_i) dF (x_1) \cdots dF (x_i).
\]

In Section 3, we prove

**Theorem 2.4** Given a distribution function \( F(x) \) on \( \mathbb{R} \), let \( T(F) \) be a smooth functional with \( r \)th functional derivative \( T_F (x_1, \ldots, x_r) \), \( r \geq 1 \), as defined in Withers (1983). Then, (I):

\[
T (F_h) = T(F) + \sum_{r=1}^{\infty} T_{rh}(F)/r! = T_h(F)
\] (2.21)
say, where

\[
T_{rh}(F) = \mathbb{E} [T_F (X_{1h}, \ldots, X_{rh})] = \mathbb{E} [T_{Fh} (X_1, \ldots, X_r)],
\]

\[
T_{Fh} (x_1, \ldots, x_r) = G_h (\partial/2) T_F (x_1, \ldots, x_r) = \sum_{k=0}^{\infty} H^k T_{2kF}^r (x_1, \ldots, x_r),
\]

\[
T_{2kF}^r (x_1, \ldots, x_r) = \sum_{k_1 + \cdots + k_r = k} T_{F,2k_1,\ldots,2k_r} (x_1, \ldots, x_r)/\prod_{i=1}^{r} (2k_i + 1)!
\]
when (2.21) converges. Also by (2.17), for \( k \geq r \geq 1 \),

\[
T_{rh}(F) = \sum_{k=r}^{\infty} H^k T_{kr}^{r}(F) = O(\Pi^r), \quad T_{kr}^{r}(F) = \mathbb{E} \left[ T_{2kF}^{r} (X_1, \ldots, X_r) \right],
\]

\[
T_h(F) = \sum_{k=0}^{\infty} H^k T_k^{r}(F),
\]

where

\[
T_0^r(F) = T(F), \quad T_k^r(F) = \sum_{r=1}^{k} T_{kr}^{r}(F)/r!.
\]  (2.22)

Also (II):

\[
T(F) = T_h(F),
\]

where

\[
T^h(F) = T(F) + \sum_{r=1}^{\infty} T^h_r(F)/r!,
\]  (2.23)

\[
T^h_r(F) = \mathbb{E} \left[ T^h_r(F) (X_1, \ldots, X_r) \right],
\]

\[
T^h_r(x_1, \ldots, x_r) = G_h(\partial/2)^{-1} T^h_r(F) (x_1, \ldots, x_r) = \sum_{k=0}^{\infty} H^k T^h_{2kF} (x_1, \ldots, x_r),
\]

where

\[
T^h_{2kF} (x_1, \ldots, x_r) = \sum_{k_1 + \cdots + k_r = k} T_{F,2k_1,\ldots,2k_r}^h (x_1, \ldots, x_r) \prod_{i=1}^{r} A_{k_i}
\]

when (2.23) converges. So, for \( k \geq r \geq 1 \),

\[
T^h_r(F) = \sum_{k=r}^{\infty} H^k \mathbb{E} \left[ T^h_{2kF} (X_1, \ldots, X_r) \right] = O(\Pi^r),
\]

\[
T^h(F) = \sum_{k=0}^{\infty} H^k T^h_k(F),
\]

where

\[
T_0^h(F) = T(F), \quad T_k^h(F) = \sum_{r=1}^{k} T^h_{kr}(F)/r!, \quad T^h_{kr}(F) = \mathbb{E} \left[ T^h_{2kF} (X_1, \ldots, X_r) \right].
\]  (2.24)

For example,

\[
T_{k1}(F) = \mathbb{E} \left[ T_{F,2k} (X_1) / (2k + 1)! \right], \quad T_{k1}^h(F) = \mathbb{E} \left[ T_{F,2k} (X_1) A_k \right],
\]

\[
T_k^r(F) = \sum_{k_1=1}^{k-1} \left[ \mathbb{E} \left[ T_{F,2k_1,2k_2} (X_1, X_2) \right] / \prod_{i=1}^{2} (2k_i + 1) \right]_{k_2=k-k_1},
\]

\[
T_k^r(F) = \sum_{k_1=1}^{k-1} \left[ \mathbb{E} \left[ T_{F,2k_1,2k_2} (X_1, X_2) \right] \prod_{i=1}^{2} A_{k_i} \right]_{k_2=k-k_1}
\]
since $E[T_{F,0,2k_2}(X_1, X_2)] = 0$ for $k_2 \geq 1$.

For $T(F)$ a polynomial of degree $p$ in $F$, its $r$th derivatives are zero for $r > p$ so that $T_{rh}(F) = T_r^h(F) = 0$, and the expansions (2.21) and (2.23) have only $p$ terms.

**Example 2.6** Suppose that $T(F) = m_p$. Then,

$$T_F(x_1) = x_1^p - m_p,$$

$$T_k^l(F) = T_{k1}^l(F), \quad T_p^u(F) = T_{k1}^u(F),$$

$$(2k + 1)!T_k^l(F) = T_k^u(F)/A_k = E[T_{F,2k}(X_1)]/(2k + 1)! = \lceil p \rceil_{2k} m_{p-2k}$$

since $T_F(x_1, \ldots, x_r) = 0$ implies $T_{kr}^l(F) = T_{kr}^u(F) = 0$ for $k \geq r \geq 2$. This proves (1.7) and (1.8) again.

We now use this theorem to give another proof that the relations (1.7) and (1.8) between $m_p$ and $m_{ph}$ also hold between $\mu_p$ and $\mu_{ph}$, so that an UE of $\mu_p$ in terms of the rounded sample, is just a linear combination of the UEs of $\mu_p$ in terms of the unrounded sample, with $F$ and the unrounded sample replaced by $F_h$ and the rounded sample. To prove this, we need the following result, given in Example 5.6 of Withers and Nadarajah (2012a).

**Theorem 2.5** For $p, r$ in $\{1, 2 \cdots \}$, the $r$th functional derivative of $T(F) = \mu_p(F)$ is

$$T_{1,\ldots,r} = (-1)^r \left\{ [p]_r \mu_{p-r} - [p]_{r-1} \sum_{i=1}^{r} (h_i^{p-r} - \mu_{p-r+1} h_i^{r-1}) \right\} \prod_{i=1}^{r} h_i. \quad (2.25)$$

Each term in (2.25) is linear in $h_i$ except for the term with $h_i^{p-r+1}$.

**Example 2.7** Suppose that $T(F) = \mu_p$. Then, for $r \geq 2$, $k_1, \ldots, k_r \geq 1$,

$$T_{F,2k_1,\ldots,2k_r}(x_1, \ldots, x_r) = E[T_{F,2k_1,\ldots,2k_r}(X_1, \ldots, X_r)] = 0,$$

$$T_{rh}(F) = T_r^h(F) = 0.$$

Also for $k \geq 1$ and $A_k$ of (1.10),

$$T_{F,2k}(x_1) = \lceil p \rceil_{2k} h_1^{p-2k}$$

implies

$$T_k^l(F) = T_{k1}^l(F), \quad T_k^u(F) = T_{k1}^u(F),$$

$$(2k + 1)!T_k^l(F) = T_k^u(F)/A_k = E[T_{F,2k}(X)] = \lceil p \rceil_{2k} \mu_{p-2k}.$$

So,

$$T_{1h}(F) = \sum \left\{ \mu_{p-2k} \left( \frac{p}{2k} \right) H^k/(2k + 1) : 1 \leq k \leq p/2 \right\} = \mu_{ph} - \mu_p,$$

$$T_{1h}(F) = \sum \left\{ \left( \frac{p}{2k} \right) a_k H^k \mu_{p-2k} : 1 \leq k \leq p/2 \right\} = \mu_p^h - \mu_p,$$

and (1.11) holds. So, an UE of $\mu_p$ in terms of the rounded sample is given by (1.12).
A method for obtaining UEs using unrounded data, for products of moments, was given in Withers and Nadarajah (2012a).

**Example 2.8** By page 255 of Withers and Nadarajah (2012a), an UE of $\mu_2\mu_3$ is $\mu_{23n}(F_n)$, where

$$\mu_{23n}(F) = \left\{ \mu_2\mu_3 + (-\mu_5 - 2\mu_2\mu_3) n^{-1} + (\mu_5 + 5\mu_2\mu_3) n^{-2} \right\} / \prod_{i=1}^{4} (1 - i/n).$$

So, using rounded data, an UE of $\mu_2\mu_3 = (\mu_2h - H/3)\mu_3h$ is $\mu_{23n}(F_{nh})$, where

$$\mu_{23n}(F) = \mu_{23n}(F) - (H/3)\mu_{3n}(F)$$

for $\mu_{3n}(F)$ of (2.20).

Theorem 2.5 can be used to obtain estimators of low bias for any smooth function of moments. (In contrast, such a method does not exist for non-polynomial functions of cumulants since the derivatives of the general cumulant are not known.)

We now give two examples when $T(F)$ need not be a polynomial in $F$.

**Example 2.9** Estimating a function of the mean. Suppose that $T(F) = a(\mu)$ for some smooth function $a$. Since $m_1h = m_1$, we just need to substitute $F_{nh}$ for $F_n$ in the estimators $S_{np}(F_n)$, $T_{np}(F_n)$ of bias $O(n^{-p})$ given by (1.1) and (1.3). From Example 6.1 of Withers and Nadarajah (2008), $S_i(F)$ of (1.1) is given by

$$S_i(F) = \sum_{r=i+1}^{2i} S_{ir}a_r(\mu),$$

where

- $S_{12} = -\mu_2/2$, $S_{23} = \mu_3/3$, $S_{24} = \mu_2^2/8$,
- $S_{34} = -\mu_4/4 + 3\mu_2^2/8$, $S_{35} = -3\mu_3^2/2$, $S_{36} = \mu_2^3/48$,
- $S_{45} = \mu_5/5 - 2\mu_3\mu_2/3$, $S_{36} = -3\mu_3^2/16 + 4\mu_2\mu_3/3 - \mu_3^2/8$, $S_{47} = \mu_3\mu_2^2/24$,
- $S_{48} = \mu_2^4/24$,
- $S_{56} = -\mu_6/6 + 5\mu_4\mu_2/8 + 5\mu_3^2/18$, $S_{57} = -\mu_5\mu_2/10 - \mu_4\mu_3/12$,
- $S_{58} = 3\mu_2^4/64 - 4\mu_2\mu_3^2/32 - \mu_3^3\mu_3/36$, $S_{59} = -3\mu_3\mu_2^3/144$, $S_{510} = \mu_2^5/3840$.

So, $S_{np}(F_n)$ and $S_{np}(F_{nh})$ are estimators based on unrounded and rounded observations for $a(\mu)$ with bias $O(n^{-p})$ (and are UEs if $a(\mu)$ is a polynomial of degree less than or equal to 2p). For example, an estimator of bias $O((n^{-3})$ based on rounded observations is $S_{n3}(F_{nh})$, where

$$S_{n3}(F) = a(\mu) - (\mu_2/2) a_2(\mu)/(n - 1) + [(\mu_3/3) a_3(\mu) + (\mu_2^2/8) a_4(\mu)] / (n - 1)^2.$$

For $a(\mu) = \mu^q$, substitute $a_3(\mu) = [q]_q q^{q-i}$.
Example 2.10 Suppose that \( T(F) = a(\mu_2) \) for some smooth function \( a \). So, \( T(F) = T^h(F_n) \), where \( T^h(F) = a(\mu_2 - H/3) \). For \( T(F) = a(\mu_2) \), by Example 5.12 of Withers and Nadarajah (2012a), \( S_i(F) \), \( i = 1, 2, 3 \) needed for (1.1) are given by

\[
S_i(F) = \sum_{r=1}^{2i} S_{ir} a_i(\mu_2),
\]

where

\[
\begin{align*}
S_{11} &= \mu_2, \\
S_{12} &= -\left( \mu_4 - \mu_2^2 \right)/2, \\
S_{21} &= 0, \\
S_{22} &= -2\mu_4 + 7\mu_2^2/2, \\
S_{23} &= \mu_6/3 - \mu_3^2 - 3\mu_4\mu_2/2 + 7\mu_2^3/6, \\
S_{24} &= (\mu_4 - \mu_2^{2})^2/8, \\
S_{31} &= 0, \\
S_{32} &= -3\mu_4 + 9\mu_2^2/2, \\
S_{33} &= 3\mu_6 - 27\mu_4\mu_2/2 - 7\mu_3^2 + 13\mu_2^3, \\
S_{34} &= -\mu_8/4 + 4\mu_6\mu_2/3 + 2\mu_5\mu_3 + 11\mu_3^2/8 - 6\mu_4\mu_2^2 - 7\mu_3^2\mu_2 + 85\mu_2^4/24, \\
S_{35} &= (\mu_4 - \mu_2^2) (-4\mu_6 + 15\mu_4\mu_2 + 3\mu_3^2 - 11\mu_2^3)/24, \\
S_{36} &= - (\mu_4 - \mu_2^2)^3/48.
\end{align*}
\]

So, for \( 1 \leq p \leq 4 \), an estimator of \( a(\mu_2) \) with bias \( O(n^{-p}) \) based on unrounded observations is \( S_{np}(F_n, a) = S_{np}(F_n) \). But \( a(\mu_2) = a^h(\mu_2) \), where \( a^h(\mu_2) = a(\mu_2h - H/3) \). So, for \( 1 \leq p \leq 4 \), an estimator of \( a(\mu_2) \) with bias \( O(n^{-p}) \) based on rounded observations is \( S_{np}(F_{nh}, a^h) \). For example, an estimator of \( a(\mu_2) \) with bias \( O(n^{-2}) \) based on rounded observations is \( S_{n2}(F_{nh}, a^h) \), where

\[
S_{n2}(F, a^h) = a(\mu_2 - H/3) = \left[ \mu_2a_{1}(\mu_2 - H/3) - (\mu_4 - \mu_2^2)a_{2}(\mu_2 - H/3)/2 \right]/(n-1).
\]

For \( a(\mu_2) = \mu_2^q \), substitute \( a_{1}(\mu_2) = [q]_{0}^{q-1} \). For the standard deviation, \( \sigma = \mu_2^{1/2} \), \( q = 1/2 \).

Example 2.11 By Example 5.16 of Withers and Nadarajah (2012a), an estimator of \( T(F) = \mu/\sigma \) of bias \( O(n^{-p}) \) based on unrounded observations is given by \( S_{np}(F_n) \) of (1.1) with

\[
\begin{align*}
S_1(F) &= -\beta_3/2 - \beta(3\beta_4 + 1)/8, \\
S_2(F) &= (48\beta_5 - 15\beta_4\beta_3 - 23\beta_3)/64 \\
&\quad + (\mu/\sigma)(-80\beta_6 + 446\beta_4 - 327 + 105\beta_4^2 + 960\beta_3^2)/128,
\end{align*}
\]

where \( \beta_r = \mu_r\mu_2^{-r/2} \). Now replace each \( \mu_p \) by the right hand side of (1.11) and \( F_n \) by \( F_{nh} \) to obtain an estimator of bias \( O(n^{-p}) \), \( p = 1, 2, 3 \) based on rounded observations. For example, writing

\[
\tilde{\mu}_h = \mu(F_{nh}), \quad \tilde{\mu}_{rh} = \mu_r(F_{nh}), \quad (2.26)
\]

an estimator of \( \mu/\sigma \) with bias \( O(n^{-1}) \) based on rounded observations is \( \tilde{\mu}_h(\tilde{\mu}_{2h} - H/3)^{-1/2} \).

Example 2.12 By the appendix and Examples 5.22, 5.23 of Withers and Nadarajah (2012a), UEs of \( \mu_{11} \), the covariance of \( \tilde{X}_1 \) and \( \tilde{X}_2 \), and of \( \mu_{111} \) are \( \mu_{11}(F_{nh}) n/(n-1) \) and \( \mu_{111}(F_{nh}) n^2/[n-1]_2 \).
Example 2.13 The correlation of $\tilde{X}_1$ and $\tilde{X}_2$ is

$$\rho = T(F) = U_1 (U_2 U_3)^{-1/2} = g(U)$$

say, where

$$U_1 = \mu_{11} = \mu_{11h}, \ U_2 = \mu_{20} = \mu_{20h} - H_1/3, \ U_3 = \mu_{02} = \mu_{02h} - H_2/3.$$ 

So, an estimator of correlation with bias $O(n^{-1})$ using rounded observations is

$$\hat{\rho} = \hat{\mu}_{11h} (\hat{\mu}_{20h} - H_1/3)^{-1/2} (\hat{\mu}_{02h} - H_2/3)^{-1/2}.$$ 

An estimator of bias $O(n^{-2})$ using unrounded observations is given by $S_{n2}(F_n)$ of (1.1), where by equation (A.8) of Withers and Nadarajah (2012a),

$$T(1^2) = g_{ij} U_{ij} + g_i U_i(1^2),$$

where

$$U_{ij} = U_{ij}(1, 1) = \int U_i F(x) U_j F(x) dF(x),$$

where $g_i, g_{ij}$ are the first and second partial derivatives of $g$ at $\mu_2$, and the summation of $i, j$ over their range 1, 2, 3 is implicit. (Withers and Nadarajah (2012a) have used a different notation for multivariate moments.) Now substitute

$$g_1 = U_2^{-1/2} U_3^{-1/2}, \ g_2 = (-1/2) U_1 U_2^{-3/2} U_3^{-1/2}, \ g_3 = (-1/2) U_1 U_2^{-1/2} U_3^{-3/2},$$

$$g_{11} = 0, \ g_{12} = (-1/2) U_1 U_2^{-3/2} U_3^{-1/2}, \ g_{13} = (-1/2) U_1 U_2^{-1/2} U_3^{-3/2},$$

$$g_{22} = (3/4) U_1 U_2^{-5/2} U_3^{-1/2}, \ g_{23} = (1/4) U_1 U_2^{-3/2} U_3^{-1/2}, \ g_{33} = (3/4) U_1 U_2^{-1/2} U_3^{-5/2},$$

$$U_1(1^2) = -2 \mu_{11}, \ U_2(1^2) = -2 \mu_{20}, \ U_3(1^2) = -2 \mu_{02},$$

$$U_{11} = \mu_{22} - \mu_{11}^2, \ U_{12} = \mu_{31} - \mu_{11} \mu_{20}, \ U_{13} = \mu_{13} - \mu_{11} \mu_{02},$$

$$U_{22} = \mu_{40} - \mu_{20}^2, \ U_{23} = \mu_{22} - \mu_{20} \mu_{02}, \ U_{33} = \mu_{04} - \mu_{02}^2$$

to obtain $T(1^2)$ for $S_1(F)$ of (1.2) and an estimator $S_{n2}(F_{nh})$ of bias $O(n^{-2})$ using $S_{n2}(F)$ of (1.1).

Example 2.14 Example 6.6 of Withers and Nadarajah (2008) gives $T[2], T[3], T[2^2]$ needed for $S_i(F)$ of (1.1) to estimate the skewness, $T(F) = \mu_3/\mu_2^{3/2}$, to $O(n^{-p})$ for $p = 1, 2, 3$ using unrounded data. They are given in terms of the standardized moments, $\beta_r = \mu_r/\mu_2^{r/2}$. For rounded data, just replace each $\mu_p$ by the right hand side of (1.11) and $F_n$ by $F_{nh}$, as in Example 2.9. For example, in terms of $\hat{\mu}_{nh}, \hat{\mu}_{nh}$ of (2.26), an estimator of skewness with bias $O(n^{-1})$ using rounded observations is $\hat{\beta}_{3h}(\hat{\mu}_{2h} - H/3)^{-3/2}$.

Similarly, the bias reduced estimators of return times for unrounded data given in Examples 5.17-5.25 of Withers and Nadarajah (2012a), can be adapted to rounded data.

Example 2.15 By (1.6), for $r \geq 2$, an UE of $\kappa_r$ is $k_r(F_{nh}) - B_r h^r/r$, where $k_r = k_r(F_n)$ is Fisher’s $k$-statistic: see, for example, Stuart and Ord (1987).
3 Multivariate rounding

In this section, we suppose that $X$ is an $s$-vector with distribution function $F$ on $\mathbb{R}^s$, and that $g : \mathbb{R}^s \rightarrow \mathbb{C}$. Let $h = (h_1, \ldots, h_s)$ be given positive constants and suppose that $X_h$ is $X$ with its $i$th component rounded to the nearest multiple of $h_i$ for $1 \leq i \leq s$. Similarly, let $X_{1h}, \ldots, X_{nh}$ be a random sample $X_1, \ldots, X_n$ on $\mathbb{R}^s$ rounded in this way, and let $F_{nh}$ and $F_n$ be their empirical distribution functions.

What is $\mathbb{E} [g(X_h)]$? What is an UE of $\mathbb{E} [g(X)]$? For $x \in \mathbb{R}^s$ and $1 \leq j \leq s$, set $v_j = \text{var} \, [(X_1)_j]$,

$$\partial = \partial/\partial x, \quad \partial_j = \partial/\partial x_j, \quad \partial^k = \prod_{j=1}^s \partial_j^{k_j}, \quad g_{2k}(x) = \partial^{2k} g(x),$$

$$G_h(x) = \prod_{j=1}^s G(h_j x_j),$$

$$g_h(x) = G_h(\partial/2) g(x) = \sum_{k=0}^{\infty} H^k g_{2k}(x)/(2k + 1)!,$$

where

$$H^k = \prod_{j=1}^s H_j^{k_j}, \quad k! = \prod_{j=1}^s k_j!, \quad H_j = h_j^2/4, \quad H_0 = \max_{j=1}^s H_j/v_j, \quad (3.1)$$

$$g^h(x) = G_h(\partial/2)^{-1} g(x) = \sum_{k=0}^{\infty} A_k H^k g_{2k}(x), \quad A_k = \prod_{j=1}^s A_{k_j} \quad (3.2)$$

for $A_{k_j}$ of (1.9). So, $H_0$ is dimensionless. Applying (2.13) and (2.14) $s$ times gives

**Corollary 3.1** Given a smooth function $g : \mathbb{R}^s \rightarrow \mathbb{C}$,

$$\mathbb{E} [g(X_h)] = \mathbb{E} [g_h(X)],$$

$$\mathbb{E} [g^h(X_h)] = \mathbb{E} [g(X)].$$

Applying this $r$ times to a function of $r$ arguments gives

**Corollary 3.2** Given a smooth function $g(x_1, \ldots, x_r) : \mathbb{R}^s \times \cdots \times \mathbb{R}^s \rightarrow \mathbb{C}$, and independent $X_1, \ldots, X_r$ from a distribution function $F$ on $\mathbb{R}^s$, rounded as above to $X_{1h}, \ldots, X_{rh}$ with distribution function $F_h$ on $\mathbb{R}^s$,

$$\mathbb{E} [g(X_{1h}, \ldots, X_{rh})] = \mathbb{E} [g_h(X_1, \ldots, X_r)],$$

$$\mathbb{E} [g^h(X_{1h}, \ldots, X_{rh})] = \mathbb{E} [g(X_1, \ldots, X_r)], \quad (3.3)$$

where

$$g_h(x_1, \ldots, x_r) = G_h(\partial/2) g(x_1, \ldots, x_r),$$

$$g^h(x_1, \ldots, x_r) = G_h(\partial/2)^{-1} g(x_1, \ldots, x_r),$$

$$G_h(\partial/2) = \prod_{i=1}^r G_h(\partial_i/2), \quad G_h(\partial_i/2) = \prod_{j=1}^s G(h_j \partial_{ij}/2), \quad \partial_{ij} = \partial/\partial x_{ij}. $$

18
We now extend Example 2.2 to $F$ on $\mathbb{R}^s$. Set $\mathbb{N} = \{0,1,2,\ldots\}$. For $X \in \mathbb{R}^s$ and $p = (p_1,\ldots,p_s) \in \mathbb{N}^s$, the $p$th cumulant of $X$ is defined by

$$\ln \mathbb{E} \left[ e^{t^X} \right] = \sum_{|p| \geq 1} \kappa_p t^p / p!,$$

where

$$t^p / p! = \prod_{j=1}^{s} t_{j}^{p_j} / p!$$

for $t \in \mathbb{C}^s$.

**Example 3.1** Take $g(x) = e^{xt}$, where $t \in \mathbb{C}^s$. Then,

$$g_n(x) = G_h(t/2) g(x), \quad g^h(x) = G_h(t/2)^{-1} g(x),$$

$$\mathbb{E} \left[ e^{X^k_{ht}} \right] / \mathbb{E} \left[ e^{X^t} \right] = G_h(t/2),$$

$$\ln \mathbb{E} \left[ e^{X^k_{ht}} \right] - \ln \mathbb{E} \left[ e^{X^t} \right] = \ln G_h(t/2) = \sum_{j=1}^{s} \ln G(h_jt_j/2) = \sum_{p=1}^{\infty} \frac{k_{2p}}{(2p)!} \sum_{j=1}^{s} (h_jt_j)^{2p}$$

by (2.16). So, $\kappa_{ph} = \kappa_p$ for $p = (p_1,\ldots,p_s)$ except if $p_1 = \cdots = p_s = 2r$ for some positive integer $r$; in that case

$$\kappa_{ph} = \kappa_p + (h_1 \cdots h_s)^{2r} b_{2r,s}$$

for $r \geq 1$, where $b_{2r,s} = (2r!)^{s-1} B_{2r}/(2r)$. For example,

$$b_{2r,2} = (2r - 1)! B_{2r}, \quad b_{2r,3} = (2r - 1)! (2r)! B_{2r}.$$  

So, if $k_{ph}(F_n)$ is Fisher’s polykay, the UE of the multivariate cumulant $\kappa_p$ for unrounded data, then an UE of $\kappa_p$ for rounded data is $k_{np}(F_n)$ except if $p_1 = \cdots = p_s = 2r$ for some $r \geq 1$; in that case an UE of $\kappa_p$ is

$$\hat{\kappa}_{ph} = k_{ph}(F_{nh}) - b_{rs} (h_1 \cdots h_s)^{2r}$$

for the even marginals of $p$, which can be obtained from Example 2.2. Wold (1934), equation (5), page 254 considered the case $s = 2$.

**Theorem 3.1** Given a distribution function $F(x)$ on $\mathbb{R}^s$, Theorem 2.4 remains true if $0$, $k$, $k_i$ are interpreted as $s$-vectors, $H^k$, $k!$, $A_k$ are interpreted as in (3.1), (3.2), the expressions in (2.22), (2.24) for $T'_k(F)$, $T''_k(F)$ are replaced by

$$T'_k(F) = \sum_{1 \leq r \leq |k|} T'_{kr}(F) / r!, \quad T''_k(F) = \sum_{1 \leq r \leq |k|} T''_{kr}(F) / r!,$$

where $|k| = \sum_{j=1}^{s} k_j$, and $O\left( \Pi^r \right)$ is replaced by $O\left( \Pi^r_0 \right)$ for $H_0$ of (3.1).
Proof: By Withers (1983),

\[ T(G) = T(F) + \sum_{r=1}^{\infty} \int \cdots \int T_F(x_1, \ldots, x_r) \, dG(x_1) \cdots dG(x_r) / r!. \]

Putting \( G = F_h \) gives (2.21). Reversing \( F, F_h \) and applying (3.3) gives (2.23). That \( \mathbb{E}[T_{Fh}(X_1, \ldots, X_r)] = O(\overline{H}_0) \) follows from its expansion in powers of \( H \) by noting that

\[ H^{k_1 + \cdots + k_r} = O(\overline{H}_0) \]

if \( k_1 \geq 1, \ldots, k_r \geq 1 \). Furthermore, \( \mathbb{E}[T_{F}(X_1, x_2, \ldots, x_r)] = 0 \) implies \( \mathbb{E}[T_{F,0,k_2,\ldots,k_r}(X_1, x_2, \ldots, x_r)] = 0 \) if \( k_1 = 0 \) or \( k_2 = 0 \) or \( \cdots \) or \( k_r = 0 \). The proof is complete. \( \square \)

We need to distinguish between unrounded sample values, \( X_1, \ldots, X_n \), and the \( j \)th component of \( X = X_1 \), say \( \tilde{X}_j, 1 \leq j \leq s \). The multivariate moments are

\[
m_p = \mathbb{E} \left[ \prod_{j=1}^s \tilde{X}_j^{p_j} \right], \quad \bar{\mu}_j = \mathbb{E} \left[ \tilde{X}_j \right], \quad \mu_p = \mathbb{E} \left[ \prod_{j=1}^s \left( \tilde{X}_j - \bar{\mu}_j \right)^{p_j} \right]. \tag{3.4}
\]

Example 3.2 Take \( g(x) = x^p = \prod_{j=1}^s x_j^{p_j} \), \( T(F) = \mathbb{E}[g(X)] = m_p \). Then,

\[
g^h(x) = \prod_{j=1}^s G(h_j \partial_j/2)^{-1} x_j^{p_j}
\]

is the product of terms of the form (2.15). Taking the mean of \( m^h_p(X) \) gives

\[
m_p = \sum_{k=0}^{\infty} \binom{p}{2k} a_k H^k m_{p-2k}, \tag{3.5}
\]

where

\[
\binom{p}{2k} = \prod_{j=1}^s \binom{p_j}{2k_j}.
\]

This multiplication is easily done by writing (1.8) as

\[
m_p = P_{pH}(S)m_h,
\]

where

\[
P_{pH}(S) = \sum_{k=0}^{[p/2]} \binom{p}{2k} a_k H^k S^{2k}, \quad S^i m_h = m_{ih}.
\]

Its multivariate form is

\[
m_p = \prod_{j=1}^s P_{p_j, H_j}(S_j)m_h,
\]

20
where
\[ \prod_{j=1}^{s} S_j^{i_j} m_h = m_{i_1, \ldots, i_s}. \]

For example, for \( m_2, m_4 \) are given by (1.8) as

\[ P_{2H}(S) = S^2 - H/3, \quad P_{4H}(S) = S^4 - 2HS^2 + 7H^2/15. \]

So,

\[ P_{2H_2}(S_2) = S_2^2 - H_2/3, \quad P_{4H_1}(S_1) = S_1^4 - 2H_1S_1^2 + 7H_1^2/15. \]

So, setting \((rs) = m_{rsh}, m_{42} = P_{4H_1}(S_1) P_{2H_2}(S_2) m_h = (42) - 2H_1(22) + 7H_1^2(02)/15 \]

\[ - (H_2/3) ((40) - 2H_1(20) + 7H_1^2/15), \]

and an UE estimator of \( m_{42} \) based on rounded data is the right hand side of (3.6) with \( m_{rs}(F_h) = m_{rsh} \) replaced by \( m_{rs}(F_{nh}) \). In this way, we obtain the multivariate non-central moments of the appendix, and so UEs for them, by replacing \( F \) by \( F_{nh} \).

As in the univariate case, (3.5) also holds with moments replaced by central moments:

\[ \mu_p = \sum_{k=0}^{\infty} \binom{p}{2k} a_k H^k \mu_{p-2k,h}, \]

as illustrated in the appendix. An UE for \( \mu_p \) follows by replacing \( \mu_{p-2k} \) in the right hand side of (3.7) by an UE based on \( F_{nh} \). For our next example, we replace the notation for the central moments of (3.4) by

\[ \mu[i_1, \ldots, i_a] = \mathbb{E}[(X - \mu)_{i_1} \cdots (X - \mu)_{i_a}]. \]

For example, by Appendix A,

\[ \mu[1234] = \mathbb{E} \left( \prod_{j=1}^{4} (X - \mu)_j \right) = \mu_{1111} = \mu_{1111h}, \]

\[ \mu[1112] = \mathbb{E} \left( (X - \mu)_1^3 (X - \mu)_2 \right) = \mu_{31} = \mu_{31h} - H_1\mu_{11h}. \]

**Example 3.3** Estimating a function of a multivariate mean. Suppose that \( T(F) = a(\mu) \) for some smooth function \( a \) with partial derivatives \( a_{i_1, \ldots, i_r}(\mu) \) for \( i_1, \ldots, i_r \in \{1, 2, \ldots, s\} \). As in Example 2.9, we just need to substitute \( F_{nh} \) for \( F_n \) in the estimators \( S_{np}(F_n), T_{np}(F_n) \) of bias \( O(n^{-p}) \) given by (1.1) and (1.3). For \( a, b, \ldots \geq 1 \), define

\[ T[a] = a_{i_1, \ldots, i_a}(\mu) \mu[i_1, \ldots, i_a], \]

\[ T[ab] = a_{i_1, \ldots, i_a,j_1, \ldots, j_b}(\mu) \mu[i_1, \ldots, i_a] \mu[j_1, \ldots, j_b]. \]
and so on, and set \( T[a^2] = T[a a] \), and so on, with the tensor summation convention that repeated pairs of suffixes \( i_1, \ldots, i_a, j_1, \ldots, j_b, \ldots \) are implicitly summed over their range \( \{1, 2, \ldots, s\} \). For example,
\[
T[2^3] = a_{i_1, i_2, j_1, j_2, k_1, k_2}(\mu) \mu [i_1, i_2] \mu [j_1, j_2] \mu [k_1, k_2] = \sum_{i_1, \ldots, i_6 = 1}^s a_{i_1, \ldots, i_6}(\mu) \mu [i_1, i_2] \mu [i_3, i_4] \mu [i_5, i_6].
\]

So, for \( p \geq 1 \), an estimator of \( a(\mu) \) of bias \( O(n^{-p}) \) is \( S_{np}(F_{nh}) \) for \( S_{np}(F) \) of (1.1). \( S_i(F) \) has the form
\[
S_i(F) = \sum_{r = i + 1}^{2i} a_{j_1, \ldots, j_r}(\mu) S_{i, j_1, \ldots, j_r}.
\]

If \( a(\mu) \) is a polynomial of degree \( p_i \) in \( \mu_i \) for \( i = 1, \ldots, s \), then \( S_{np}(F_{nh}) \) is an UE of \( a(\mu) \) if \( p_1 + \cdots + p_s \leq p \).

### 4 The multisample case

Given \( k \) random samples with empirical distribution functions \( F_n = (F_{n_1}, \ldots, F_{n_k}) \) from distribution functions \( F = (F_1, \ldots, F_k) \) on \( \mathbb{R}^{s_1}, \ldots, \mathbb{R}^{s_k} \) and a smooth functional \( T(F) \), equation (3.10) of Withers and Nadarajah (2008) gives an estimator of \( T(F) \) with bias \( O(n^{-p}) \) of the form
\[
S_{np}(F_n) = \sum_{i=1}^{p-1} S_i^n(F_n)
\]
for \( p \leq 7 \), where \( n = \min(n_1, \ldots, n_k) \) is the minimum sample size and
\[
S_i^n(F) = T(F), \quad S_i^1(F) = -(1/2) \sum_{i=1}^k T(i^2) / (n_i - 1), \quad T(i^2) = \int T_F(i^2, x, x) dF_i(x),
\]
where \( T_F(i^2, x, y) \) is the second partial derivative of \( T(F) \) with respect to \( F_i, F_j \) as defined on page 194 of Withers and Nadarajah (2008), and so on: see page 199 of Withers and Nadarajah (2008). This was done using the expansion (2.21) with \( (F_h, F) \) replaced by \( (F, G) \) for \( G = (G_1, \ldots, G_k) \) arbitrary distribution functions on \( \mathbb{R}^{s_1}, \ldots, \mathbb{R}^{s_k} \), replacing \( \mathbb{E}[T_G(X_1, \ldots, X_r)] \) by an UE, (available for \( r \leq n \)), then replacing \( G \) by \( F_n \).

We illustrate how to adapt these results to rounded data with some examples. Let the rounding for the \( i \)th sample of dimension \( s_i \) have corresponding rounding vector \( h_i \in \mathbb{R}^{s_i} \). Let \( F_{in_i h_i} \) be the empirical distribution function of the \( i \)th sample after rounding. Set \( F_{nh} = (F_{in_1 h_1}, \ldots, F_{in_k h_k}) \), the set of empirical distribution functions. Let \( \mu_i \) be the mean of the \( i \)th distribution.

**Example 4.1** Suppose that \( T(F) = \sum_{i=1}^k U_i(F_i) \). Then, an estimator based on rounded data of bias \( O(n^{-p}) \) is
\[
\sum_{i=1}^k U_{ih_i}^{h_i}(F_{in_i h_i}).
\]
For example, if α_i are t × s_i constants, then an UE of \( \sum_{i=1}^{k} \alpha_i \mu_i \) based on rounded data is

\[
\sum_{i=1}^{k} \alpha_i \hat{\mu}_{ih_i}.
\]

For another example, suppose that s_i ≡ 1 and that α_i are given scalars. Denote the variance of \( F_i \) by \( \mu_2 \). Then, an UE of \( \sum_{i=1}^{k} \alpha_i \mu_2 \) based on rounded data is

\[
\sum_{i=1}^{k} \alpha_i (\hat{\mu}_{ih_i} - H_i/3)
\]

for \( H_i \) given by (3.1).

**Example 4.2** Suppose that \( k = 2 \), \( s_i \equiv 1 \) and \( T(F) = \mu_1 \mu_2 / (\mu_1 + \mu_2) \). Then, \( S_i^n(F) \) needed for (4.1) are given for \( 1 \leq i \leq 3 \) in Example 6.2.2, page 213 of Withers and Nadarajah (2008). Since \( \mu_i = \mu_{ih_i} \), an estimator of \( T(F) \) based on rounded data is \( S_{np} (F_{nh}) \) of (4.1).

**Example 4.3** For \( k \) univariate samples with \( T(F) = \alpha' \mu / \beta' \mu \), an estimator of bias \( O(n^{-p}) \) based on unrounded data was given in Example 5.5, page 247 of Withers and Nadarajah (2012a) of the form \( T_{np} \left( \hat{F} \right) \) of (1.3). For example,

\[
T_1(F) = (\beta' \mu)^{-2} \sum_{a=1}^{k} \left( n/n_a \right) \beta_a^2 \mu_2[a] (\alpha_a/\beta_a - T(F)).
\]

Since \( \mu_i = \mu_{ih_i} \), an estimator of \( T(F) \) based on rounded data is \( S_{np} (F_{nh}) \) of (4.1).

**Example 4.4** For two univariate samples with \( T(F) = \mu_1 \mu_2 / (\mu_1 + \mu_2) \), \( S_{n4}(F) \) of (4.1) is given by Example 6.2.2, page 213 of Withers and Nadarajah (2008). So, \( S_{n4} (F_{nh}) \) estimates \( T(F) \) with bias \( O(n^{-4}) \).

**Appendix A: Multivariate moments up to order seven**

By pages 252 and 253 of Wold (1934),

\[
\begin{align*}
m_{11} & = m_{11h}, \quad m_{21} = m_{21h} - H_1 m_{01h}/3, \\
m_{31} & = m_{31h} - H_1 m_{11h}, \quad m_{22} = m_{22h} - \sum_{i=1}^{2} H_1 m_{02h}/3 + H_1 H_2/9, \\
m_{41} & = m_{41h} - 2H_1 m_{21h} + 7H_1^2 m_{01h}/15, \\
m_{32} & = m_{32h} - (H_1/3) m_{12h} - H_2 (m_{30h} - (H_1/3) m_{10h}), \\
m_{51} & = m_{51h} - 10H_1 m_{31h} + 7H_1^2 m_{11h}/3, \\
m_{42} & = m_{42h} - 2H_1 m_{22h} + 7H_1^2 m_{02h}/15 - (H_2/3) (m_{40h} - 2H_1 m_{20h} + 7H_1^2/15), \\
m_{33} & = m_{33} - H_1 m_{13h} - H_2 (m_{31h} - H_1 m_{11h}).
\end{align*}
\]
Also, by the method explained in Example 3.2,

\[ m_{61} = m_{61h} - 5H_1 m_{41h} + 7H_1^2 m_{21h} - 31H_1^3 m_{01h}/21, \]

\[ m_{52} = m_{52h} - (10H_1/3) m_{32h} + (7H_1^2/3) m_{12h} \]
\[ - (H_2/3) \left[ m_{50h} - (10H_1/3) m_{30h} + (7H_1^2/3) m_{10h} \right], \]

\[ m_{43} = m_{43h} - H_2 m_{41h} - 2H_1 (m_{23h} - H_2 m_{21h}) + (7H_1^2/15) (m_{03h} - H_2 m_{01h}), \]

\[ m_{111} = m_{111h}, \quad m_{211} = m_{211h} - H_1 m_{011h}/3, \quad m_{311} = m_{311h} - H_1 m_{111h}/3, \]

\[ m_{221} = m_{221h} - H_1 m_{021h}/3 - H_2 m_{201h}/3 + H_1 H_2 m_{001h}/9, \]

\[ m_{411} = m_{411h} - 2H_1 m_{211h} + 7H_1^2 m_{021h}/15 \]
\[ - (H_2/3) (m_{401h}/3 - 2H_1 m_{201h} + 7H_1^2 m_{001h}/15), \]

\[ m_{331} = m_{331h} - H_1 m_{131h} - H_2 m_{311h} + H_1 H_2 m_{111h}, \]

\[ m_{1111} = m_{1111h}, \quad m_{2111} = m_{2111h} - H_1 m_{0111h}/3, \]

\[ m_{2211} = m_{2211h} - H_1 m_{0011h}/3 - H_2 m_{2011h}/3 + H_1 H_2 m_{001h}/9, \]

\[ m_{3111} = m_{3111h} - H_1 m_{1111h}/3, \]

\[ m_{2221} = m_{2221h} - H_1 m_{0221h}/3 - H_2 m_{2021h}/3 + H_1 H_2 m_{001h}/9, \]

\[ m_{4111} = m_{4111h} - 2H_1 m_{2111h}/3 + 7H_1^2 m_{0111h}/15, \]

\[ m_{3211} = m_{3211h} - H_1 m_{1211h}/3 - H_2 (m_{3011h}/3 - H_1 m_{1011h})/3, \]

\[ m_{214} = m_{214h} - H_1 m_{014h}/3, \quad m_{314} = m_{314h} - H_1 m_{14h}, \]

\[ m_{2213} = m_{2213h} - H_1 m_{0213h}/3 - H_2 m_{2013h}/3 + H_1 H_2 m_{013h}/9, \]

\[ m_{215} = m_{215h} - H_1 m_{015h}/3. \]
As before, we can replace moments by central moments, giving

\[ \mu_{11} = \mu_{11h}, \quad \mu_{21} = \mu_{21h}, \]
\[ \mu_{31} = \mu_{31h} - H_1\mu_{11h}, \quad \mu_{22} = \mu_{22h} - 2H_1\mu_{21h}, \quad \mu_{32} = \mu_{32h} - H_1\mu_{12h} - H_2\mu_{30h}, \]
\[ \mu_{41} = \mu_{41h} - 2H_1\mu_{21h}, \quad \mu_{33} = \mu_{33h} - H_1\mu_{13h} - H_2(\mu_{31h} - H_1\mu_{11h}), \]
\[ \mu_{51} = \mu_{51h} - 10H_1\mu_{31h}, \quad \mu_{52} = \mu_{52h} - 10H_1(\mu_{32h} + (7H_1^2/3)\mu_{12h} - (H_2/3)[\mu_{50h} - (10H_1/3)\mu_{30h}] - (H_2/3)\mu_{30h}), \]
\[ \mu_{42} = \mu_{42h} - 2H_1\mu_{22h} + 7H_1^2\mu_{02h}/15 - (H_2/3)(\mu_{40h} - 2H_1\mu_{20h} + 7H_1^2/15), \]
\[ \mu_{33} = \mu_{33h} - H_1\mu_{13h} - H_2(\mu_{31h} - H_1\mu_{11h}), \]
\[ \mu_{61} = \mu_{61h} - 5H_1\mu_{41h} + 7H_1^2\mu_{21h}, \]
\[ \mu_{52} = \mu_{52h} - 10H_1(\mu_{32h} + (7H_1^2/3)\mu_{12h} - (H_2/3)[\mu_{50h} - (10H_1/3)\mu_{30h}] - (H_2/3)\mu_{30h}), \]
\[ \mu_{43} = \mu_{43h} - H_2\mu_{41h} - 2H_1(\mu_{23h} - H_2\mu_{21h}) + 7H_1^2\mu_{03h}/15, \]
\[ \mu_{111} = \mu_{111h}, \quad \mu_{211} = \mu_{211h} - H_1\mu_{011h}/3, \quad \mu_{311} = \mu_{311h} - H_1\mu_{111h}/3, \]
\[ \mu_{221} = \mu_{221h} - H_1\mu_{211h}/3 - H_2\mu_{201h}/3, \]
\[ \mu_{411} = \mu_{411h} - 2H_1\mu_{211h}/3 + 7H_1^2\mu_{011h}/15, \]
\[ \mu_{321} = \mu_{321h} - H_1(\mu_{121h} - H_2\mu_{101h}/3), \]
\[ \mu_{222} = \mu_{222h} - 2H_1\mu_{022h}/3 + 3H_1H_2\mu_{002h}/9 - H_1H_2H_3/27, \]
\[ \mu_{511} = \mu_{511h} - 10H_1\mu_{311h}/3 + 7H_1^2\mu_{111h}/3, \]
\[ \mu_{421} = \mu_{421h} - 2H_1\mu_{221h} + 7H_1^2\mu_{021h}/15 - H_2\mu_{401h}/9 + 2H_1\mu_{201h}/3, \]
\[ \mu_{331} = \mu_{331h} - H_1\mu_{131h} - H_2\mu_{311h} + H_1H_2\mu_{111h}, \]
\[ \mu_{1111} = \mu_{1111h}, \quad \mu_{2111} = \mu_{2111h} - H_1\mu_{0111h}/3, \quad \mu_{3111} = \mu_{3111h} - H_1\mu_{1111h}/3, \]
\[ \mu_{2211} = \mu_{2211h} - H_1\mu_{0111h}/3 - H_2\mu_{2011h}/3, \]
\[ \mu_{4111} = \mu_{4111h} - 2H_1\mu_{2111h}/3 + 7H_1^2\mu_{0111h}/15, \]
\[ \mu_{3211} = \mu_{3211h} - H_1\mu_{1211h}/3 - H_2(\mu_{3011h}/3 - H_1\mu_{1011h})/3, \]
\[ \mu_{2221} = \mu_{2221h} - H_1\mu_{0221h}/3 - H_2\mu_{2021h}/3 + H_1H_2\mu_{0011h}/9, \]
\[ \mu_{2141} = \mu_{2141h} - H_1\mu_{0141h}/3, \quad \mu_{3141} = \mu_{3141h} - H_1\mu_{1141h}, \]
\[ \mu_{2133} = \mu_{2133h} - H_1\mu_{0213h}/3 - H_2\mu_{2013h}/3 + H_1H_2\mu_{0013h}/9, \]
\[ \mu_{2151} = \mu_{2151h} - H_1\mu_{0151h}/3. \]

References


