Perpetual Game Options in Incomplete Markets

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A game option is a financial contract of the American type which allows the holder to exercise it and the writer to cancel it at any time for a predetermined price. If the writer terminates the contract before the holder exercises it then in addition to the value of the contract at the time of cancellation the writer must pay a penalty to the holder. In this work we solve a nonzero-sum game of optimal stopping which is associated with the exercise and cancellation time of perpetual game capped call option in incomplete markets. In particular, by considering geometric Brownian motion to model the price of the underlying and exponential utility functions for both the holder and the writer of the option we shall determine the time when the holder will exercise and the writer will cancel the option in such a way as to maximize the expected utility of their terminal wealth.

1. Introduction

A game option, also known as Israeli option, is a contract of the American type which also allows the writer to cancel it at any time provided that in addition to the value of the option at the cancellation time a penalty fee is given to the holder. The main advantage of such options is that they can be sold cheaper on the market than their American counterpart. Game options were introduced by Kifer [19]. The author showed, through hedging arguments, that in complete markets the price of a game option can be recovered by solving a zero-sum game of optimal stopping. Moreover the author derived a saddle point of optimal stopping times which corresponds to the optimal exercise time and optimal cancellation time of the holder and writer respectively. Since the seminal paper of Kifer, various authors have contributed in determining the price of specific game contingent claims in complete markets. Kyprianou [25] derived the price of two perpetual game options, a game put option and a Russian option whereas in [3] Baurdoux and Kyprianou derived the price of \( \delta \)-penalty integral options. Kühn, Kyprianou and Van Schaik [23] derived dual pricing formula for general Israeli options, which for contracts with more complex payoffs, can be utilised to obtain approximate prices via Monte Carlo simulation.

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Carlo simulation. Ekström [6], determined explicitly, via the link between excessive functions and concave functions, the price of two perpetual game options; a put option and a capped call option. A similar approach was used by Ekström and Villeneuve [7], to determine the price of two perpetual call options. Gapeev [8] considered a perpetual spread game option and obtained explicitly the no-arbitrage price provided that the parameters of the model satisfy certain conditions. Emmerling [5] and Yam, Yung and Zhou [34] derived the price of perpetual game call options in the presence of dividends. For other literature on the pricing of game options in complete markets one can refer to [9], [33], [22], [24] and [34].

In general financial markets may be incomplete and so it is impossible to find a portfolio that completely mimics the payoff of the derivative at the time of exercise. Thus the risk generated from this random payoff will not be completely eliminated through the investment. Financial markets might be incomplete due to transaction costs or trading restrictions. For example in executive stock options, due to insider trading reasons, the company’s shares cannot be typically traded. Game options in incomplete markets have been initially studied by Kühn [21]. By assuming that there is no liquid market for the game option the author explains the contract as a generalized nonzero-sum Dynkin game from which utility indifference prices can be derived. From a buyer’s point of view the utility indifference price \( p^b \) is the price at which the buyer of the derivative is indifferent between not engaging in the derivative (and thus faces a classical Merton optimal investment problem) and on the other hand taking into account the cost \( p^b \) of buying the derivative at time of inscription of the contract and the contract proceeds at the time of exercise. From a writer’s point of view the utility indifference price is the price \( p^w \) at which the writer is indifferent between not engaging in the derivative and thus invests optimally (in the classical Merton sense) a given initial wealth and on the other hand taking into account the proceeds \( p^w \) of selling the derivative at time of inscription of the contract and the liabilities at the time the holder exercises the contract. Utility indifference pricing has been studied extensively by several authors in the valuation of European and American options. One can refer to [31], [4], [10],[11],[12],[13],[26],[27],[28],[29] and [32] for further details. Kallsen and Kühn in [17] show, by considering a market made up of the underlying and game options, that the set of no-arbitrage prices coincide with the value of a zero-sum optimal stopping game under a suitably chosen equivalent martingale measure whereas in [18] the authors derive the so-called neutral derivative prices. Neutral derivative prices are obtained by maximizing the expected utility and by assuming that derivative supply and demand are balanced.

Following Kühn’s [21] interpretation of a game contingent claim, the purpose of the present work is to explicitly solve a nonzero-sum game of optimal stopping related to perpetual game capped call options written on a nontraded asset. For simplicity of exposition we shall not consider the possibility of investing in a correlated asset. Our focus here is not to price the option but rather to derive explicitly the exercise and cancellation times of the holder and writer respectively which maximize their expected utility. Exponential utility functions for both the holder and the writer of the option will be considered. One way to recover a price for such options in this case might be via the so-called certainty-equivalent price, which is obtained by inverting the optimal expected utility of the terminal payoff. A more general approach, however, would be to consider utility indifference prices, however the inclusion of a financial market would require solving explicitly a nonzero-sum game of optimal stopping and control and we leave this for future research. A similar approach to ours has been considered by Kadam,
Lakner and Srinivasan ([15],[16]) for standard American call options.

The structure of this paper is as follows: In section 2 we give the underlying setup and define the optimal stopping game whereas in sections 3 to 5 we solve the problem for different values of the parameters involved.

2. Problem Formulation

Let $X$ be the price of a nontraded risky asset which is assumed to be modelled by a geometric Brownian motion defined by

$$
\begin{align*}
\frac{dX_t}{X_t} &= \mu dt + \nu dB_t \\
X_0 &= x \quad (2.1)
\end{align*}
$$

for $x > 0$, where $\mu \in \mathbb{R}$ is the drift, $\nu > 0$ the volatility and $B$ a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. It is well known that the process $(X_t^x)_{t \geq 0}$ where

$$
X_t^x = xe^{(\mu - \nu^2/2)t + \nu B_t} \quad (2.2)
$$

for each $t \geq 0$, is the unique strong solution of (2.1)-(2.2). We shall denote by $P_x$ the law of the process $X^x$ on the canonical space so that under $P_x$, the coordinate process $X$ starts at $x$. It is well known that $X$ is a strong Markov process under $P_x$.

Suppose that a game capped call option with strike $K$ and cap $L > K$ is written on this asset. So if the holder of the option selects an exercise time $\tau$ and the writer selects a cancellation time $\sigma$, then the writer pledges to pay the holder the amount

$$
R(\tau, \sigma) = (X_\tau \wedge L - K)^+ I(\tau \leq \sigma) + (X_\sigma \wedge L - K)^+ + \delta) I(\sigma < \tau) \quad (2.4)
$$

at time $\tau \wedge \sigma$, where $\delta$ is a penalty paid by the writer for cancelling the option prior to the holder’s exercise time and $\tau$ and $\sigma$ are stopping times of $X$. Stopping times of $X$ are relative to the natural filtration of $X$.

Let $U_1, U_2 : \mathbb{R} \to \mathbb{R}$ be non-decreasing concave functions representing the utility function of the holder and the writer of this option respectively. We aim at solving the nonzero-sum game of optimal stopping

$$
\begin{align*}
V_\sigma^1(x) &= \sup_{\tau} M_\tau^1(x, \tau, \sigma) = \sup_{\tau} E_x R(\tau, \sigma) \\
V_\tau^2(x) &= \sup_{\sigma} M_\sigma^2(x, \tau, \sigma) = \sup_{\sigma} E_x R(\tau, \sigma) \quad (2.5)
\end{align*}
$$

where $E_x$ is the expectation with respect to the probability measure $P_x$. Problem (2.5)-(2.6) can equivalently be written as

$$
\begin{align*}
V_\sigma^1(x) &= \sup_{\tau} E_x [G_1(X_\tau) I(\tau \leq \sigma) + H_1(X_\sigma) I(\sigma < \tau)] \\
V_\tau^2(x) &= \sup_{\sigma} E_x [G_2(X_\sigma) I(\sigma < \tau) + H_2(X_\tau) I(\tau \leq \sigma)] \quad (2.6)
\end{align*}
$$
where the payoff functions are given by

\[ G_1(x) = U_1((x \wedge L - K)^+) \]
\[ H_1(x) = U_1((x \wedge L - K)^+ + \delta) \]
\[ G_2(x) = U_2(-(x \wedge L - K)^+ + \delta)) \]
\[ H_2(x) = U_2(-(x \wedge L - K)^+) \]

We shall assume that the stopping times of \( X \) may take the value infinity (Markov times) and if \( \rho \) is any given stopping time then for any given measurable function \( f : (0, \infty) \to \mathbb{R} \) we shall set

\[ f(X_{\rho(\omega)})(\omega) = \limsup_{t \to \infty} f(X_t(\omega)) \]

on the set \( \{\omega \in \Omega : \rho(\omega) = \infty\} \). To model the agents’ preferences we will consider the family of exponential utility functions, so the payoff functions are explicitly given by

\[ G_1(x) = -e^{-\gamma_1(x \wedge L - K)^+}, \quad H_1(x) = -e^{-\gamma_1((x \wedge L - K)^+ + \delta)} \]
\[ G_2(x) = -e^{\gamma_2((x \wedge L - K)^+ + \delta)}, \quad H_2(x) = -e^{\gamma_2(x \wedge L - K)^+} \]

where \( \gamma_1 > 0 \) and \( \gamma_2 > 0 \) are the risk aversion parameters of the holder and the writer of the option respectively. We will look for stopping times \( \tau_* \) and \( \sigma_* \) that are in Nash equilibrium (a saddle point of optimal stopping times), that is

\[ M_1^x(\tau, \sigma_*) \leq M_2^x(\tau_*, \sigma_*) \quad \text{and} \quad M_2^x(\tau, \sigma) \leq M_2^x(\tau_*, \sigma_*) \]

for all stopping times \( \tau \) and \( \sigma \) and for all \( x > 0 \). To solve the optimal stopping game (2.7)-(2.8) we shall make use of a scale function of \( X \), given by

\[ S : (0, \infty) \to \mathbb{R} \]
\[ S(x) = \begin{cases} \frac{x^p}{p} & \text{for } p \neq 0 \\ \ln x & \text{otherwise} \end{cases} \]

where \( p = 1 - \frac{2\mu}{L^2} \) (see e.g. [14]).

3. Solution to the problem when \( \delta \geq L - K \)

In this section we solve the optimal stopping game (2.7)-(2.8) for \( \delta \geq L - K \). Let us consider first the case \( \mu \geq \frac{L^2}{2} \) and suppose that the holder of the option selects the stopping time \( \tau_* = \inf\{t \geq 0 : X_t \geq L\} \). It is well known that \( \tau_* < \infty \) \( P_x \)-a.s. and so \( E_x H_2(X_{\tau_*}) = H_2(L) \). Thus for any stopping time \( \sigma \) we have that

\[ M_2^x(\tau_*, \sigma) = E_x[(G_2(X_\sigma)I(\sigma < \tau_*) + H_2(X_{\tau_*})I(\tau_* \leq \sigma))I(\sigma < \infty) + H_2(X_{\tau_*})I(\sigma = \infty)] \]
\[ \leq E_x[H_2(X_{\tau_*})I(\sigma < \infty) + H_2(X_{\tau_*})I(\sigma = \infty)] \]
\[ = H_2(L) \]
\[ = M_2^x(\tau_*, \infty) \]
where the inequality follows from the fact that \( \sup_{x>0} G_2(x) \leq \inf_{x>0} H_2(x) \) whenever \( \delta \geq L - K \). From this it is clear that the writer has no incentive of cancelling the option at all. On the other hand we see that

\[
\sup_{\tau} M_1^1(\tau, \infty) = \sup_{\tau} E_x [G_1(X_\tau)] \\
= \sup_{\tau} E_x \left[ G_1(X_\tau) I(\tau < \infty) + \lim_{t \to \infty} G_1(X_t) I(\tau = \infty) \right] \\
\leq G_1(L) = M_1^1(\tau_*, \infty)
\]

(3.1)

for all stopping times \( \tau \). The inequality follows from the fact that \( \lim_{t \to 0} X_t = \infty \) \( P_x \)-a.s. whenever \( \mu \geq \frac{\nu^2}{2} \). From the above we can conclude that the pair \((\tau_*, \infty)\) is a Nash equilibrium point (see figure 1).

![Figure 1: A graphical representation of the payoff functions and the value functions in the case \( \mu \geq \frac{\nu^2}{2} \) and \( \delta \geq L - K \).](image)

Now suppose that \( \mu < \frac{\nu^2}{2} \). It is well known that in this case \( \lim_{t \to \infty} X_t = 0 \) \( P_x \)-a.s. If the writer will never cancel the option then his expected utility is given by

\[
M_2^2(\tau, \infty) = E_x H_2(X_\tau).
\]

(3.2)

On the other hand, for any stopping time \( \sigma \) we have that

\[
M_2^2(\tau, \sigma) \leq E_x [(H_2(X_\tau)I(\sigma < \tau) + H_2(X_\tau)I(\tau \leq \sigma))I(\tau < \infty) \\
+ \lim_{t \to \infty} (H_2(X_t)I(\sigma < \infty) + \lim_{t \to \infty} H_2(X_t)I(\sigma = \infty))I(\tau = \infty)] \\
= M_2^2(\tau, \infty)
\]

(3.3)

where the inequality follows again from the fact that \( \inf_{x>0} H_2(x) \geq \sup_{y>0} G_2(y) \). From this observation it is clear that an optimal strategy for the writer is to never cancel the option irrespective of the strategy of the holder of the option. Now if the writer does not cancel the option then the holder needs to solve the optimal stopping problem

\[
V_1^1(\tau) = \sup_{\tau} E_x G_1(X_\tau).
\]

(3.4)
It is well known (see for example [30] page 145) that problem (3.4) is equivalent to
\[
V_1^\infty(x) = \sup_{\rho} E_x \tilde{G}_1(\tilde{B}_\rho)
\]
where \( S(x) = \frac{x^p}{p} \) is the scale function of \( X \) (see (2.10)), \( \tilde{G}_1 = G_1 \circ S^{-1} \) and \( \tilde{B} \) is a standard Brownian motion. The supremum in (3.5) is taken over all stopping times of \( \tilde{B} \). To solve (3.5) we let
\[
W_1(y) = \sup_{\rho} E_y \tilde{G}_1(\tilde{B}_\rho)
\]
for \( y \in (0, \infty) \). From the general theory of optimal stopping (see [30]) we know that \( W_1 \) is the smallest concave function majorising \( \tilde{G}_1 \) in \( (0, \infty) \) (it is well known that for Brownian motion, superharmonic functions are concave functions) and the stopping time \( \beta_* \) is \( \inf \{ t : B_t \in D \} \) where \( D = \{ y \in (0, \infty) : W_1(y) = \tilde{G}_1(y) \} \) is optimal. By noting that \( \tilde{G}_1''(y) < 0 \) for all \( y \in (\frac{K^p}{p}, \frac{L^p}{p}) \) and \( (\frac{K^p}{p}, \tilde{G}_1(\frac{L^p}{p})) \) it is clear that if \( 1 - p > \gamma_1 L \) then the smallest concave majorant of \( \tilde{G}_1 \) in \( (0, \infty) \) is given by
\[
W_1(y) = \begin{cases} 
\frac{l(y)}{\tilde{G}_1(y)} & \text{if } 0 < y \leq \frac{L^p}{p} \\
\tilde{G}_1(y) & \text{if } \frac{L^p}{p} < y
\end{cases}
\]
where \( l \) is the line segment joining the points \( (0, \tilde{G}_1(0)) \) and \( (\frac{L^p}{p}, \tilde{G}_1(\frac{L^p}{p})) \). So the value function of player one is \( V_1^\infty(x) = W_1(S(x)) \) and the corresponding optimal stopping time is \( \tau_* = \inf \{ t \geq 0 : X_t \geq L \} \). If on the other hand \( 1 - p > \gamma_2 \), to determine the smallest concave majorant of \( \tilde{G}_1 \) we will study the existence of point \( A_* \in (\frac{K^p}{p}, \frac{L^p}{p}) \) such that
\[
\frac{\tilde{G}_1(A_*) - \tilde{G}_1(0+)}{A_*} = \tilde{G}_1'(A_*-)
\]
(smooth fit principle)
For this we introduce the function
\[
\Psi : \left( \frac{K^p}{p}, \frac{L^p}{p} \right) \rightarrow \mathbb{R}
\]
\[
\Psi(y) = \tilde{G}_1(0+) - \tilde{G}_1(y) + \tilde{G}_1'(y)y
\]
Suppose first that \( 1 - p > \gamma_2 \). It is easy to see that \( \Psi'(\frac{K^p}{p+\gamma_1}) = 0 \) that \( \Psi'(y) > 0 \) for all \( y < \frac{1}{p}(\frac{L^p}{p})^p \) and that \( \Psi'(y) < 0 \) for all \( y > \frac{1}{p}(\frac{L^p}{p})^p \). Since \( \Psi(\frac{K^p}{p+}) > 0 \), then if \( \Psi(\frac{L^p}{p-}) < 0 \) we get, by continuity of \( \Psi \), the existence of a unique point \( A_* \in (\frac{K^p}{p}, \frac{L^p}{p}) \) which solves (3.8) and so we have
\[
W_1(y) = \begin{cases} 
\frac{l(y)}{\tilde{G}_1(y)} & \text{if } 0 < y \leq A_* \\
\tilde{G}_1(y) & \text{if } A_* < y
\end{cases}
\]
where \( l \) is the line segment joining the points \( (0, \tilde{G}_1(0)) \) and \( (A_*, \tilde{G}_1(A_*)) \). If on the other hand \( \Psi(\frac{L^p}{p-}) = 0 \) we can set \( A_* = \frac{L^p}{p} \) in (3.8). So the value function of player one is given by \( V_1^\infty(x) = W_1(S(x)) \) and the optimal stopping time is \( \tau_* = \inf \{ t \geq 0 : X_t \geq A_* \} \) with
\(a_s = S^{-1}(A_s) = (pA_s)^{\frac{1}{p}}\). If \(\Psi\left(\frac{a}{\mu} - \frac{\delta}{\nu} - \frac{\gamma}{K}\right) > 0\) we have that \(\Psi(A) > 0\) for all \(A \in \left(\frac{K_a^p}{p}, \frac{L_a^p}{p}\right)\). So in this case the smooth fit breaks down (since \(\hat{C}_1\) is not smooth at \(\frac{L_a^p}{p}\)) and the function \(W_1\) is given by (3.7). The case \(1 - p \leq \gamma_2 K\) can be dealt with in a similar way by noting that in this special case \(\Psi\) is decreasing. A typical representation of the value functions in the case \(\mu < \frac{\nu^2}{2}\) and \(\delta \geq L - K\) is given in figure 2.

Figure 2: A graphical representation of the payoff functions and the value functions in the case \(\mu < \frac{\nu^2}{2}\) and \(\delta \geq L - K\).

4. Solution to the problem when \(\delta < L - K\) and \(\mu \geq 0\)

In this section we solve the optimal stopping game (2.7)-(2.8) for \(\delta < L - K\) and \(\mu \geq 0\). Consider first the case \(\mu \geq \frac{\nu^2}{2}\) and suppose that the process starts at \(x > L\). Then the holder of the option will want to exercise the option immediately as by doing so he will avoid the possibility that the price falls in future and his utility decreases. If on the other hand \(x < K\), then since \(X\) will reach any level greater than \(x\) in finite time, it seems reasonable for the writer to cancel the option immediately as by doing so he avoids that the possibility that the price increases in future and thus his utility decreases (note that \(H_2(L) < G_2(K)\) whenever \(\delta < L - K\)). Now suppose that \(x \in (L,K)\). Then the holder might want to exercise the option when the price is sufficiently close to \(L\) as his expected utility might be higher than that obtained if he had to wait until the process actually reaches \(L\). Similarly the writer might want to cancel the option whenever the price is close to \(K\). Motivated by these arguments we will look for optimal stopping times of the form

\[
\tau_s = \inf\{t \geq 0 : X_t \geq a_s\}
\]

\[
\sigma_s = \inf\{t \geq 0 : X_t \leq b_s\}
\]

for some optimal stopping boundaries \(K \leq b_s < a_s \leq L\) to be determined.

Let us now suppose that \(0 \leq \mu < \frac{\nu^2}{2}\). As discussed above if \(x > L\) the holder of the option will want to exercise the option immediately whereas if \(x \in (L,K)\) he might want to exercise the option when the price is sufficiently close to \(L\). With this in mind we guess that the optimal stopping time of the holder is of the form (4.1) for some optimal stopping
boundary $K < a_* \leq L$ to be determined. However in this case there are sample paths of positive measure that will not reach any given level greater than $x$ in finite time. So if $x < K$ the writer will consider not cancelling the option at all so as to allow the process to decrease in future, and hence increase his expected utility (we note, from the form of the payoff function $M_2(\tau, \sigma)$, that on the set where both players will not exercise/cancel the option the utility of the writer is $\limsup_{t \geq 0} H_2(X_t) = H_2(0+)$. On the other hand if $x \in (L, K)$, then the writer might either not cancel the option at all (if his expected utility is sufficiently large) or he might want to exercise the option when the price is sufficiently close to $L$. So in certain cases (to be determined) it will never optimal for the writer to cancel the option and the holder will solve a standard optimal stopping problem as in the case $\delta \geq L - K$ whereas in other cases the form of the optimal stopping time of the writer is given by

$$
\sigma_* = \inf\{t \geq 0 : X_t \in [b_+^1, b_+^2]\}
$$

for some optimal stopping boundaries $K \leq b_+^1 \leq b_+^2 < a_* \leq L$ to be determined.

We shall now make use of the results on the double partial superharmonic characterisation of the value functions (see [1] and [2]) to determine the optimal stopping boundaries introduced above and hence solve the optimal stopping game (2.7)-(2.8). This characterisation allow us to solve the game by reformulating the problem in terms of standard Brownian motion $\tilde{B}$ (see section 6 in [2] for further details). As shall be seen in section (4.0.1) the latter problem will be easier to deal with geometrically as one can make use of results from convex analysis to get to the solution. To this end let us consider the nonzero-sum game

$$
W_1^1(y) = \sup_{\eta} \tilde{M}_1^1(\eta, \beta_*)
$$

$$
W_2^1(y) = \sup_{\beta} \tilde{M}_2^1(\eta_*, \beta)
$$

where $y \in \mathcal{R}(S)$ with $\mathcal{R}(S)$ denoting the range of the scale function $S$ of $X$, and

$$
\tilde{M}_1^1(\eta, \beta) = \mathbb{E}_y \left[ \tilde{G}_1(\tilde{B}_\eta)I(\eta \leq \beta) + \tilde{H}_1(\tilde{B}_\beta)I(\beta < \eta) \right]
$$

$$
\tilde{M}_2^1(\eta, \beta) = \mathbb{E}_y \left[ \tilde{G}_2(\tilde{B}_\beta)I(\beta < \eta) + \tilde{H}_2(\tilde{B}_\eta)I(\eta \leq \beta) \right]
$$

with $\tilde{G}_i = G_i \circ S^{-1}$ and $\tilde{H}_i = H_i \circ S^{-1}$ for $i = 1, 2$. We note that the above suprema are taken over all stopping times $\eta$ and $\beta$ of $\tilde{B}$. We guess that the optimal stopping time of the holder of the option for the transformed problem is of the form

$$
\eta_* = \inf\{t \geq 0 : \tilde{B}_t \geq A_*\}
$$

for some optimal stopping boundary $S(K) < A_* < S(L)$ to be determined. If $\mu \geq \nu^2$ then the optimal stopping time of the writer that we are after will be of the form

$$
\beta_* = \inf\{t \geq 0 : \tilde{B}_t \leq B_*\}
$$

for some optimal stopping boundary $S(K) < B_* < A_*$ to be determined. If on the other hand $0 \leq \mu < \frac{\nu^2}{2}$ then we guess that the optimal strategy of the writer will either be not to stop the process at all or else it is given by the first entry time

$$
\beta_* = \inf\{t \geq 0 : \tilde{B}_t \geq [B_1^1, B_2^1]\}
$$
Consider the transformed problem (4.4)-(4.5) and let us first suppose that

\[ 4.0.1. \text{Free-boundary problem} \]

unknown points \( S(K) \leq B_1^1 \leq B_1^2 < A_\ast \) to be determined. As shall be
seen in the verification theorems (see section (4.0.3)), once the optimal stopping boundaries for
the transformed problem have been determined, the optimal exercise and cancellation times
can be recovered by setting \( a_\ast = S^{-1}(A_\ast) \), \( b_1^\ast = S^{-1}(B_1^1) \) and \( b_1^2 = S^{-1}(B_1^2) \).

**4.0.1. Free-boundary problem**

Consider the transformed problem (4.4)-(4.5) and let us first suppose that \( \mu > \frac{\nu^2}{2} \). The results
on the double partial superharmonic characterisation of the value functions (see [1] and [2])
suggest to formulate the following free-boundary problem for unknown functions \( \tilde{u} \) and \( \tilde{v} \) and
unknown points \( \frac{K_p}{p} \leq B_\ast < A_\ast \leq \frac{L_p}{p} : \)

\[
\begin{align*}
(4.9) & \quad \mathcal{L}_B \tilde{u} = 0 \text{ and } \mathcal{L}_B \tilde{v} = 0 \text{ for } y \in (B_\ast, A_\ast) \\
(4.10) & \quad \tilde{u}(A_\ast) = \tilde{G}_1(A_\ast) \text{ and } \tilde{v}(B_\ast) = \tilde{G}_2(B_\ast) \\
(4.11) & \quad \tilde{u}(B_\ast) = \tilde{H}_1(B_\ast) \text{ and } \tilde{v}(A_\ast) = \tilde{H}_2(A_\ast) \\
(4.12) & \quad \tilde{u}(y) = \tilde{G}_1(y) \text{ for } y \in (A_\ast, \infty) \\
(4.13) & \quad \tilde{v}(y) = \tilde{G}_2(y) \text{ for } y \in (-\infty, B_\ast) \\
(4.14) & \quad \tilde{u}(y) > \tilde{G}_1(y) \text{ and } \tilde{v}(y) > \tilde{G}_2(y) \text{ for } y \in (B_\ast, A_\ast) \\
(4.15) & \quad \tilde{u}(y) = \tilde{H}_1(y) \text{ for } y \in (-\infty, B_\ast) \\
(4.16) & \quad \tilde{v}(y) = \tilde{H}_2(y) \text{ for } y \in (A_\ast, 0)
\end{align*}
\]

where \( \mathcal{L}_B \) is the Laplace operator acting on twice differentiable functions. To solve the free-
boundary problem we note from (4.9) that

\[
\begin{align*}
(4.17) & \quad \tilde{u}(y) = c_1 y + d_1 \\
(4.18) & \quad \tilde{v}(y) = c_2 y + d_2
\end{align*}
\]

for \( y \in (B_\ast, A_\ast) \) where \( c_i \) and \( d_i \) for \( i = 1, 2 \) are undetermined constants. From conditions
(4.10)-(4.11) we further get that

\[
\begin{align*}
(4.19) & \quad c_1 = \frac{\tilde{G}_1(A_\ast) - \tilde{H}_1(B_\ast)}{A_\ast - B_\ast}, \quad d_1 = \tilde{H}_1(B_\ast) - c_1 B_\ast \\
(4.20) & \quad c_2 = \frac{\tilde{G}_2(B_\ast) - \tilde{H}_2(A_\ast)}{B_\ast - A_\ast}, \quad d_2 = \tilde{H}_2(A_\ast) - c_2 A_\ast
\end{align*}
\]

Moreover, in certain cases (which shall be specified below) the double smooth fit principle (see [1])

\[
\begin{align*}
(4.21) & \quad \tilde{u}'(A_\ast-) = \tilde{G}'_1(A_\ast-) \\
(4.22) & \quad \tilde{v}'(B_\ast+) = \tilde{G}'_2(B_\ast+)
\end{align*}
\]

will also hold. Equations (4.21) – (4.22) together with (4.19) – (4.20) imply that the optimal
stopping boundaries \( A_\ast \) and \( B_\ast \) must satisfy the system of nonlinear equations

\[
\begin{align*}
(4.23) & \quad \tilde{G}_1(A_\ast) - \tilde{H}_1(B_\ast) - \tilde{G}'_1(A_\ast-)(A_\ast - B_\ast) = 0 \\
(4.24) & \quad \tilde{G}_2(B_\ast) - \tilde{H}_2(A_\ast) - \tilde{G}'_2(B_\ast+)(B_\ast - A_\ast) = 0
\end{align*}
\]

9
To study existence of a solution to (4.23)-(4.24) we define the functions \( \Theta : [\frac{K^p}{p}, \frac{L^p}{p}] \times [\frac{K^p}{p}, \frac{L^p}{p}] \to \mathbb{R} \) and \( \Gamma : [\frac{K^p}{p}, \frac{L^p}{p}] \times [\frac{K^p}{p}, \frac{L^p}{p}] \to \mathbb{R} \) as follows:

\[
\Theta (A, B) = \begin{cases} 
\tilde{G}_1 (A) - \tilde{H}_1 (B) - \tilde{G}_1' (A) (A - B) & \text{if } \frac{K^p}{p} < A < \frac{L^p}{p} \\
\tilde{G}_1 (A) - \tilde{H}_1 (B) - \tilde{G}_1' (A+) (A - B) & \text{if } A = \frac{K^p}{p} \\
\tilde{G}_1 (A) - \tilde{H}_1 (B) - \tilde{G}_1' (A-) (A - B) & \text{if } A = \frac{L^p}{p} \\
\tilde{G}_2 (B) - \tilde{H}_2 (A) - \tilde{G}_2' (B) (B - A) & \text{if } \frac{K^p}{p} > B < \frac{L^p}{p} \\
\tilde{G}_2 (B) - \tilde{H}_2 (A) - \tilde{G}_2' (B+) (B - A) & \text{if } B = \frac{K^p}{p} \\
\tilde{G}_2 (B) - \tilde{H}_2 (A) - \tilde{G}_2' (B-) (B - A) & \text{if } B = \frac{L^p}{p} 
\end{cases}
\]

(4.26)

\[
\Gamma (A, B) = \begin{cases} 
\tilde{H}_2 (A) - \tilde{G}_2 (B) & \text{if } \frac{K^p}{p} < B < \frac{L^p}{p} \\
\tilde{H}_2 (A) - \tilde{G}_2 (B+) & \text{if } B = \frac{K^p}{p} \\
\tilde{H}_2 (A) - \tilde{G}_2 (B-) & \text{if } B = \frac{L^p}{p} 
\end{cases}
\]

From the theory of convex analysis it is well known that if a differentiable function \( f \) on the interval \([a, b]\) is strictly convex, then

\[
\frac{f(c) - f(d)}{c - d} < f'(c)
\]

for any given points \( c, d \in [a, b] \) such that \( c > d \). If \( 1 - p \geq \gamma_1 L \) then \( \tilde{G}_1 \) is strictly convex in \([\frac{K^p}{p}, \frac{L^p}{p}]\). So from (4.27) and the fact that \( \tilde{H}_2 > \tilde{G}_2 \) it follows that \( \Theta (A, B) < 0 \) for all \( A > B \) and so there exists no \( A_* > B_* \) satisfying (4.23). In this case we set \( A_* = \frac{L^p}{p} \) in the free-boundary problem (4.9)-(4.16). To see whether there exists \( B_* \) for which \( \Gamma (\frac{L^p}{p}, B_*) = 0 \) we first note that the mapping \( B \mapsto \Gamma (\frac{L^p}{p}, B) \) is decreasing (since \( \tilde{G}_2 \) is strictly concave in \([\frac{K^p}{p}, \frac{L^p}{p}]\) ) and that \( \Gamma (\frac{L^p}{p}, \frac{L^p}{p}) < 0 \). So if \( \Gamma (\frac{L^p}{p}, \frac{K^p}{p}) < 0 \) then there exists no such \( B_* \) and we will also set \( B_* = \frac{K^p}{p} \) in the free-boundary problem (4.9)-(4.16). If on the other hand \( \Gamma (\frac{L^p}{p}, \frac{K^p}{p}) \geq 0 \) we get the existence of a unique point \( B_* \in [\frac{K^p}{p}, \frac{L^p}{p}] \) solving (4.24) with \( A_* = \frac{L^p}{p} \).

Let us now suppose that \( 1 - p < \gamma_1 L \) so that \( \tilde{G}_1 \) is convex in \([\frac{K^p}{p}, \frac{L^p}{p}]\) and concave in \([\frac{1}{p+1} \frac{(L^p)}{1 - p - 1}, \frac{L^p}{p}]\). We consider first the case \( \Gamma (\frac{L^p}{p}, \frac{K^p}{p}) < 0 \). As above we can see that \( \Gamma (\frac{L^p}{p}, B) < 0 \) for all \( B \in [\frac{K^p}{p}, \frac{L^p}{p}] \) and since \( \tilde{H}_2 \) is concave in \([\frac{K^p}{p}, \frac{L^p}{p}]\), then for any given \( B \in (\frac{K^p}{p}, \frac{L^p}{p}) \) we have that \( A \mapsto \Gamma (A, B) \) is convex for all \( A \in (\frac{K^p}{p}, \frac{L^p}{p}) \). Now since \( \Gamma (B, B) > 0 \) and \( \Gamma (\frac{L^p}{p}, B) < 0 \) for any given \( B \in [\frac{K^p}{p}, \frac{L^p}{p}] \) we get that \( \Gamma (A, B) < 0 \) for all \( A \in (\frac{L^p}{p}, \frac{K^p}{p}) \). So in this case we see that there exist no solution to (4.24) and we shall set \( A_* = \frac{L^p}{p} \) in the free-boundary problem (4.9)-(4.16). If in addition \( \Theta (\frac{L^p}{p}, \frac{K^p}{p}) \geq 0 \), then there exists a unique point \( A_* \in (\frac{1}{p+1} \frac{(L^p)}{1 - p - 1}, \frac{L^p}{p}] \) solving (4.23). This follows from the fact that the mapping \( A \mapsto \Theta (A, \frac{K^p}{p}) \) is increasing in \([\frac{L^p}{p}, \frac{(L^p)}{1 - p - 1}]\) (by concavity of \( \tilde{G}_1 \) in \([\frac{1}{p+1} \frac{(L^p)}{1 - p - 1}, \frac{K^p}{p}]\) ) and that \( \Theta (\frac{1+1}{p+1} \frac{(L^p)}{1 - p - 1}, \frac{K^p}{p}) < 0 \). If on the other hand \( \Theta (\frac{L^p}{p}, \frac{K^p}{p}) < 0 \), then by using again the monotonicity properties of the mapping \( A \mapsto \Theta (A, \frac{K^p}{p}) \) we get that \( \Theta (A, \frac{K^p}{p}) < 0 \) for all \( A \in [\frac{L^p}{p}, \frac{(L^p)}{1 - p - 1}]\). From this we conclude that there exist no solution to the system of equations (4.23)-(4.24) and we set \( A_* = \frac{L^p}{p} \) in the free-boundary problem (4.9)-(4.16). Let us now consider the case \( \Gamma (\frac{L^p}{p}, \frac{K^p}{p}) \geq 0 \) and denote by \( B_*^\gamma \in [\frac{K^p}{p}, \frac{L^p}{p}] \) the unique root of \( \Gamma (\frac{L^p}{p}, B) = 0 \). Suppose first that \( 1 - p > \gamma_1 K \). If \( \Theta (\frac{L^p}{p}, B_*^\gamma) < 0 \) then we will set \( A_* = \frac{L^p}{p} \) and \( B_* = B_*^\gamma \).
in the free-boundary problem (4.9)-(4.16). Now suppose that \( \Theta(L_p^p, B_s^{L_p^p}) \geq 0 \). By strict convexity of \( \tilde{G}_1 \) in \( [K_p^p, \frac{1}{p}((p-1)^p) B(p-1)^p] \) and the fact that \( \tilde{H}_1 > \tilde{G}_1 \) we see, from (4.27), that \( \Theta(\frac{1}{p}(p-1)^p, B) < 0 \) for all \( B < \frac{1}{p}(p-1)^p \), in particular for \( B = K_p^p \). If \( \Theta(L_p^p, K_p^p) \geq 0 \) we get the existence of a unique point \( A_s^{K_p^p} \in (\frac{1}{p}(p-1)^p, L_p^p] \) such that \( \Theta(A_s^{K_p^p}, K_p^p) = 0 \). Now let us denote by \( A_s^{R_p^R} \) the unique root of \( \Gamma(A, K_p^p) = 0 \) (note that since we are considering the case \( \Gamma(K_p^p, L_p^p) \geq 0 \) then uniqueness of \( A_s^{R_p^R} \) follows from the fact that \( \Gamma(K_p^p, K_p^p) < 0 \) and that the mapping \( A \mapsto (A, K_p^p) \) is convex). If \( A_s^{K_p^p} < A_s^{R_p^R} \) then we will set \( B_s = K_p^p \) and \( A_s = A_s^{K_p^p} \) in the free-boundary problem (4.9)-(4.16). Now suppose that \( A_s^{K_p^p} \geq A_s^{R_p^R} \). If \( B_s^{L_p^p} = K_p^p \) then by uniqueness of \( A_s^{R_p^R} \) we must have that \( A_s^{R_p^R} = L_p^p \) and so (4.23)-(4.24) are satisfied with \( A_s = A_s^{K_p^p} = L_p^p \) and \( B_s = K_p^p \). Now suppose that \( B_s^{L_p^p} > K_p^p \) (so that \( A_s^{R_p^R} < L_p^p \)). If \( \Theta(L_p^p, B_s^{L_p^p}) = 0 \) then (4.23)-(4.24) are satisfied with \( A_s = L_p^p \) and \( B_s = B_s^{L_p^p} \). If on the other hand \( \Theta(L_p^p, B_s^{L_p^p}) > 0 \) then by the implicit function theorem (note that \( \Theta(A, B) \neq 0 \) and \( \Gamma_B(A, B) \neq 0 \) for all \( A > B > \frac{1}{p}(p-1)^p \)) there exists a unique \( C^1 \) function \( \phi: (B_1^{1, \Theta, L_p^p}, B_2^{1, \Theta, L_p^p}) \rightarrow \mathbb{R}, \) for some points \( K_p^p \leq B_1^{1, \Theta, L_p^p} < B_s < B_2^{1, \Theta, L_p^p} \), \( \Theta(L_p^p, B_s^{1, \Theta, L_p^p}) = \Theta(L_p^p, B_2^{1, \Theta, L_p^p}) = 0 \), and a unique \( C^1 \) function \( \psi: [A_s^{R_p^R}, L_p^p] \rightarrow \mathbb{R} \) such that \( \Theta(\phi(B), B) = 0 \) for all \( B \in [B_1^{1, \Theta, L_p^p}, B_2^{1, \Theta, L_p^p}] \) and \( \Gamma(A, \psi(A)) = 0 \) for all \( A \in [A_s^{R_p^R}, L_p^p] \). This clearly implies the existence of points \( A_s \) and \( B_s \) satisfying (4.23)-(4.24). If \( \Theta(L_p^p, K_p^p) < 0 \) we similarly get the existence of \( L_p^p > A_s > B_s > K_p^p \) satisfying (4.23)-(4.24) (note that in this case \( B_s^{1, \Theta, L_p^p} > K_p^p \)).

**Remark 4.1** The case \( \mu < \frac{\nu^2}{2} \) can be treated in the same way as the case \( \mu > \frac{\nu^2}{2} \) by noting that the scale function is given by \( S(x) = \ln(x) \) and so \( \mathcal{R}(S) = (-\infty, \infty) \).

Let us now consider the case \( 0 \leq \mu < \frac{\nu^2}{2} \). Define the function

\[
\Xi : \left[ \frac{K_p^p}{p}, \frac{L_p^p}{p} \right] \rightarrow \mathbb{R}
\]

\[
\Xi(B) = \begin{cases} 
\tilde{H}_2(0+) - \tilde{G}_2(B) - \tilde{G}_2(B) B & \text{if } \frac{K_p^p}{p} < B < \frac{L_p^p}{p} \\
\tilde{H}_2(0+) - \tilde{G}_2(B) - \tilde{G}_2(B+1) B & \text{if } B = \frac{K_p^p}{p} \\
\tilde{H}_2(0+) - \tilde{G}_2(B) - \tilde{G}_2(B-1) B & \text{if } B = \frac{L_p^p}{p}
\end{cases}
\]

(4.28)

If \( \Xi(\frac{K_p^p}{p}) > 0 \) and \( \Xi(\frac{L_p^p}{p}) < 0 \) we shall set \( l \) to be the line passing through the point \((0, \tilde{H}_2(0+))\) and tangent to \( \tilde{G}_2(B) \) for some \( B \in (\frac{K_p^p}{p}, \frac{L_p^p}{p}) \) (note that since \( \tilde{G}_2 \) is concave we have that \( \Xi \) is decreasing and so \( \tilde{B} \) is unique) whereas if \( \Xi(\frac{L_p^p}{p}) \geq 0 \) we let \( l \) be the line joining the points \((0, \tilde{H}_2(0+))\) and \((\frac{L_p^p}{p}, \tilde{G}_2(\frac{L_p^p}{p}))\), and in this case we set \( \tilde{B} = \frac{L_p^p}{p} \). If on the
other hand \( \Xi(\overline{K}_p^p) \leq 0 \) we will identify \( l \) with the line joining the points \((0, \tilde{H}_2(0+))\) and \((\overline{K}_p^p, \tilde{G}_2(\overline{K}_p^p))\), and in this case we set \( \overline{B} = \overline{K}_p^p \).

Consider the optimal stopping problem (3.6) and denote by \( A^*_0 \in [\overline{K}_p^p, \frac{L_p^p}{1-p}] \) the optimal stopping boundary. Since \( \tilde{G}_2 \) is concave in \((0, \frac{L_p^p}{1-p}]\) then \( l \) supports the hypograph of \( \tilde{G}_2 \) in \((0, \frac{L_p^p}{1-p}]\). So if \( \tilde{H}_2(A^*_0) > 1(A^*_0) \) then the line connecting the points \((0, \tilde{H}_2(0+))\) and \((A^*_0, \tilde{H}_2(A^*_0))\) majorises \( \tilde{G}_2 \). We will show in the next section that in this case it is not optimal for the writer to cancel the option. Now suppose that \( \tilde{H}_2(A^*_0) \leq 1(A^*_0) \). In this case we shall formulate the following free-boundary problem for unknown functions \( \tilde{u} \) and \( \tilde{v} \), and unknown points \( \overline{K}_p^p \leq B^*_1 \leq B^*_2 < A^*_0 \leq \frac{L_p^p}{1-p} \):

\[
\begin{align*}
(4.29) & \quad \mathbb{L}_B \tilde{u} = 0 \text{ and } \mathbb{L}_B \tilde{v} = 0 \text{ for } y \in (0, B^*_1) \\
(4.30) & \quad \lim_{y \downarrow 0} \tilde{u}(y) = \lim_{y \downarrow 0} \tilde{G}_1(y) \text{ and } \lim_{y \downarrow 0} \tilde{v}(y) = \lim_{y \downarrow 0} \tilde{H}_2(y) \\
(4.31) & \quad \tilde{u}(y) = \tilde{H}_1(y) \text{ for } y \in [B^*_1, B^*_2] \\
(4.32) & \quad \tilde{v}(y) = \tilde{G}_2(y) \text{ for } y \in [B^*_1, B^*_2] \\
(4.33) & \quad \mathbb{L}_B \tilde{u} = 0 \text{ and } \mathbb{L}_B \tilde{v} = 0 \text{ for } y \in (B^*_2, A^*_0) \\
(4.34) & \quad \tilde{u}(A^*_0) = \tilde{G}_1(A^*_0) \text{ and } \tilde{v}(B^*_2) = \tilde{G}_2(B^*_2) \\
(4.35) & \quad \tilde{u}(B^*_2) = \tilde{H}_1(B^*_2) \text{ and } \tilde{v}(A^*_0) = \tilde{H}_2(A^*_0) \\
(4.36) & \quad \tilde{u}(y) = \tilde{G}_1(y) \text{ for } y \in (A^*_0, \infty) \text{ and } \tilde{v}(y) = \tilde{H}_2(y) \text{ for } y \in (A^*_0, \infty) \\
(4.37) & \quad \tilde{u}(y) > \tilde{G}_1(y) \text{ and } \tilde{v}(y) > \tilde{G}_2(y) \text{ for } y \in (0, B^*_1) \cup (B^*_2, A^*_0)
\end{align*}
\]

Moreover, the smooth fit principle

\[
(4.38) \quad \tilde{v}'(B^*_1-) = \tilde{G}_2'(B^*_1-)
\]

and the double smooth fit principle

\[
\begin{align*}
(4.39) & \quad \tilde{u}'(A^*_0-) = \tilde{G}_1'(A^*_0-)

(4.40) & \quad \tilde{v}'(B^*_2+) = \tilde{G}_2'(B^*_2+)
\end{align*}
\]

will also hold in certain cases (to be specified below). Prior to studying this free-boundary problem together with (4.38)-(4.40) we note the following:

**Remark 4.2**

i. If \( \hat{l} \) is the line segment joining the points \((0, \tilde{G}_1(0+))\) and \((B, \tilde{H}_1(B))\) then \( \hat{l} \) will majorise \( \tilde{G}_1 \) in \((0, \overline{B})\). For this it is sufficient to show that \( \hat{l} \) majorises \( \tilde{G}_1 \) in \((\overline{K}_p^p, \overline{B})\). Suppose, for contradiction, that \( \hat{l} \) crosses \( \tilde{G}_1 \) at some point \( A \in (\overline{K}_p^p, \overline{B}) \) (note that this can only happen if \( \tilde{G}_1 \) is concave in the region \((y, \overline{B})\) where \( y > \overline{K}_p^p \)).

But then this would imply that \( A^*_0 < \overline{B} \) and so by concavity of \( \tilde{H}_2 \) we must have that \( \tilde{H}_2(A^*_0) > 1(A^*_0) \). Thus in the case \( \tilde{H}_2(A^*_0) \leq 1(A^*_0) \), the line \( \hat{l} \) will majorise \( \tilde{G}_1 \) in \((0, \overline{B})\).

ii. By concavity of \( \tilde{H}_2 \) in \((0, \frac{L_p^p}{1-p}]\) we see that whenever \( \tilde{H}_2(A^*_0) \leq 1(A^*_0) \) we must have \( \Xi(\overline{K}_p^p) < 0 \).

We next state the following result from convex analysis.
Proposition 4.3 Let $h$ be a concave function in an interval $[c,d]$ and $l$ any straight line cutting $h$ at two points $a < b$. Then $h < l$ in $(b,d]$.

To study existence of solution to the free-boundary problem (4.29)-(4.37) together with (4.38) and (4.39)-(4.40) we proceed analogously to the case $\mu \geq \frac{v^2}{2}$. Indeed, if $1-p \geq \gamma_1 L$ then (4.39) fails to hold and in this case we can set $A_* = \frac{L^p}{p}$ in the free-boundary problem (4.29)-(4.37). Moreover, if $\Gamma(\frac{L^p}{p}, \frac{K^p}{p}) < 0$, then (4.40) will also fail to hold and we set $B_1^* = B_2^* = \bar{B} = \frac{K^p}{p}$ in the free-boundary problem (4.29)-(4.37). If on the other hand $\Gamma(\frac{L^p}{p}, \frac{K^p}{p}) \geq 0$ then (4.40) holds and so we set $B_2^* = B_*^{\text{LP}}$ where we recall that $B_*^{\text{LP}}$ is the unique root of the equation $\Gamma(\frac{L^p}{p}, B) = 0$. If $\Xi(\frac{K^p}{p}) \geq 0$ then $B_1^* = \bar{B}$ and (4.38) will also hold whereas if $\Xi(\frac{K^p}{p}) < 0$ we set $B_1^* = \bar{B} = \frac{K^p}{p}$ and (4.38) breaks down.

Remark 4.4 Note that since $\bar{H}_2$ is concave in $(0, \frac{L^p}{p} ]$, then from proposition (4.3) together with the fact that $A_* \leq \frac{L^p}{p}$ we get that $\bar{H}_2(\frac{L^p}{p}, K^p) \leq l(\frac{L^p}{p})$.

Now suppose that $1-p < \gamma_1 L$. If $\Gamma(\frac{L^p}{p}, \frac{K^p}{p}) < 0$ and $\Theta(\frac{L^p}{p}, \frac{K^p}{p}) < 0$ the double smooth fit principle (4.39)-(4.40) breaks down. Moreover the smooth fit condition (4.38) also fails to hold and so in this case we set $A_* = \frac{L^p}{p}$ and $B_1^* = B_2^* = \bar{B} = \frac{K^p}{p}$ in the free-boundary problem (4.29)-(4.37) (see remark (4.4)). If on the other hand $\Gamma(\frac{L^p}{p}, \frac{K^p}{p}) < 0$ and $\Theta(\frac{L^p}{p}, \frac{K^p}{p}) \geq 0$ then (4.38) and (4.40) will break down and so we set $B_1^* = B_2^* = \frac{K^p}{p}$. On the other hand we see that (4.39) holds with $A_* = \frac{K^p}{p}$ where $A_*^{\text{LP}}$ is the unique root of the equation $\Theta(A^*, \frac{K^p}{p}) = 0$. In order to guarantee that a solution to the game exists with the optimal stopping times of the form (4.1)-(4.3) we make the following assumption:

Assumption 4.5 $\bar{H}_2(A_*^{\text{LP}}, \frac{K^p}{p}) \leq l(A_*^{\text{LP}})$

Now suppose that $\Gamma(\frac{L^p}{p}, \frac{K^p}{p}) \geq 0$. As above if $\Xi(\frac{K^p}{p}) \geq 0$ then $B_1^* = \bar{B}$ and (4.38) will also hold whereas if $\Xi(\frac{K^p}{p}) < 0$ then we set $B_1^* = \bar{B} = \frac{K^p}{p}$ and (4.38) will break down. To determine the point $B_2^*$ we can proceed analogously to the case $\mu \geq \frac{v^2}{2}$, however when there exists $A_* > B_*$ such that $\Theta(A_*, B_2^*) = \Gamma(A_*, B_2^*) = 0$ we need to assume that the following assertion holds:

Assumption 4.6 If $\Theta(A_*, B_2^*) = \Gamma(A_*, B_2^*) = 0$ for some point $A_* < \frac{L^p}{p}$ then $\bar{H}_2(A_*) \leq l(A_*)$.

4.0.2. The case $\gamma_1 \leq \gamma_2$

We will now show that in the special case $\gamma_1 \leq \gamma_2$ and $\Gamma(\frac{L^p}{p}, \frac{K^p}{p}) \geq 0$ there exists no solution to (4.23)-(4.24). We shall only prove the result for $\mu \geq \frac{v^2}{2}$. The case $0 \leq \mu < \frac{v^2}{2}$ can be dealt with analogously. For this we note that if there exist $\frac{L^p}{p} \geq A_* > B_* \geq \frac{K^p}{p}$ such that $\Gamma(A_*, B_*) = 0$ then we must have that

\[(4.41) \quad (pA_*)^{\frac{1}{p}} - (pB_*)^{\frac{1}{p}} - \delta > 0\]
This follows from the fact that $\tilde{G}_2' < 0$ in $[K^p_p, L^p_p]$. Let us assume first that $\gamma_1 < \gamma_2$ and suppose, for contradiction, that there exist $\frac{L^p_p}{p} \geq A_* > B_* \geq \frac{K^p_p}{p}$ such that

(4.42) \quad \Gamma(A_*, B_*) = 0
(4.43) \quad \Theta(A_*, B_*) \geq 0.

Multiplying (4.42) by $-e^{-\gamma_2((pA_*)_1^\frac{1}{p} - (pB_*)_1^\frac{1}{p} - K_{1+\delta})}$ and (4.43) by $-e^{-\gamma_1((pA_*)_1^\frac{1}{p} - K)}$ we obtain

(4.44) \quad 1 - e^{-\gamma_2((pA_*)_1^\frac{1}{p} - (pB_*)_1^\frac{1}{p} - \delta)} + \gamma_2(pB_*)_1^\frac{1}{p} - 1(A_* - B_*) = 0
(4.45) \quad 1 - e^{-\gamma_1((pA_*)_1^\frac{1}{p} - (pB_*)_1^\frac{1}{p} - \delta)} + \gamma_1(pA_*)_1^\frac{1}{p} - 1(A_* - B_*) \leq 0

Subtracting (4.45) from (4.44) we get

(4.46) \quad e^{-\gamma_1((pA_*)_1^\frac{1}{p} - (pB_*)_1^\frac{1}{p} - \delta)} - e^{-\gamma_2((pA_*)_1^\frac{1}{p} - (pB_*)_1^\frac{1}{p} - \delta)} + (A_* - B_*)(\gamma_2((pB_*)_1^\frac{1}{p} - 1) - \gamma_1((pA_*)_1^\frac{1}{p} - 1) \geq 0

Rearranging the terms in (4.46) (note that since $\gamma_1 < \gamma_2$ and (4.41) must hold, then for (4.46) to be satisfied we must have that $\gamma_2((pB_*)_1^\frac{1}{p} - 1 - \gamma_1((pA_*)_1^\frac{1}{p} - 1 > 0$) and using (4.42) we obtain

(4.47) \quad \frac{e^{-\gamma_2((pA_*)_1^\frac{1}{p} - (pB_*)_1^\frac{1}{p} - \delta)} - e^{-\gamma_2((pB_*)_1^\frac{1}{p} - (pB_*)_1^\frac{1}{p} - \delta)} + (A_* - B_*)(\gamma_2((pB_*)_1^\frac{1}{p} - 1) - \gamma_1((pA_*)_1^\frac{1}{p} - 1)}{\gamma_2((pB_*)_1^\frac{1}{p} - 1) - \gamma_1((pA_*)_1^\frac{1}{p} - 1)} = A_* - B_*

from which it follows that

(4.48) \quad (e^{-\gamma_2((pA_*)_1^\frac{1}{p} - (pB_*)_1^\frac{1}{p} - \delta)} - e^{-\gamma_2((pB_*)_1^\frac{1}{p} - (pB_*)_1^\frac{1}{p} - \delta)})(\gamma_2((pB_*)_1^\frac{1}{p} - 1) - \gamma_1((pA_*)_1^\frac{1}{p} - 1))

\geq \gamma_2((pB_*)_1^\frac{1}{p} - 1) - e^{-\gamma_2((pB_*)_1^\frac{1}{p} - (pB_*)_1^\frac{1}{p} - \delta)} - e^{-\gamma_2((pA_*)_1^\frac{1}{p} - (pB_*)_1^\frac{1}{p} - \delta)}).

Simplifying the expressions in (4.48) we get

(4.49) \quad \gamma_1((pA_*)_1^\frac{1}{p} - 1(1 - e^{-\gamma_2((pA_*)_1^\frac{1}{p} - (pB_*)_1^\frac{1}{p} - \delta)} - \gamma_2((pB_*)_1^\frac{1}{p} - 1(1 - e^{-\gamma_1((pA_*)_1^\frac{1}{p} - (pB_*)_1^\frac{1}{p} - \delta)} \geq 0.

Now let us set $\Phi(A, B) = \gamma_1((pA)_1^\frac{1}{p} - 1(1 - e^{-\gamma_2((pA)_1^\frac{1}{p} - (pB)_1^\frac{1}{p} - \delta)} - \gamma_2((pB)_1^\frac{1}{p} - 1(1 - e^{-\gamma_1((pA)_1^\frac{1}{p} - (pB)_1^\frac{1}{p} - \delta)}$ for $\frac{K^p_p}{p} < B < A < \frac{L^p_p}{p}$ and note that $\Phi(\frac{(pB)_1^\frac{1}{p} + \delta}{p}, B) = 0$. Moreover, for each given $B$ we have that

$$
\Phi_A(A, B) = \gamma_1((1 - p)(pA)_1^\frac{1}{p} - 2(1 - e^{-\gamma_2((pA)_1^\frac{1}{p} - (pB)_1^\frac{1}{p} - \delta)}) \\
- \gamma_1\gamma_2((pA)_1^\frac{1}{p} - 2(1 - e^{-\gamma_2((pA)_1^\frac{1}{p} - (pB)_1^\frac{1}{p} - \delta)}]) \\
+ \gamma_1\gamma_2((pB)_1^\frac{1}{p} - 2(1 - e^{-\gamma_2((pA)_1^\frac{1}{p} - (pB)_1^\frac{1}{p} - \delta)})) \\
< \gamma_1\gamma_2((pA)_1^\frac{1}{p} - 2((pB)_1^\frac{1}{p} - 1) - (pA)_1^\frac{1}{p} - 2((pB)_1^\frac{1}{p} - (pB)_1^\frac{1}{p} - \delta)) \\
= \gamma_1\gamma_2((pB)_1^\frac{1}{p} - 2((pB)_1^\frac{1}{p} - 1 - e^{-\gamma_2((pA)_1^\frac{1}{p} - (pB)_1^\frac{1}{p} - \delta)} - (pA)_1^\frac{1}{p} - 2((pB)_1^\frac{1}{p} - (pB)_1^\frac{1}{p} - \delta))$$

14
\[ < \gamma_1 \gamma_2 (pA)^{\frac{1}{p} - 1} (pB)^{\frac{1}{p} - 1} [e^{\gamma_1 ((pA)^{\frac{1}{p}} - (pB)^{\frac{1}{p}} - \delta)} - e^{\gamma_2 ((pA)^{\frac{1}{p}} - (pB)^{\frac{1}{p}} - \delta)}] \]

\[ \leq \gamma(pB_s) \leq e^{-\gamma((pB_s)^{\frac{1}{p} - 1} - (pA_s)^{\frac{1}{p} - 1} - \delta)} \]

for all \( A > B \) such that \((pA)^{\frac{1}{p} - 1} - (pB)^{\frac{1}{p} - 1} - \delta > 0\). The first inequality follows from the fact that \( p < 1 \) whereas for the penultimate inequality we have also used the fact that \( \gamma_1 < \gamma_2 \).

From this we have that the mapping \( A \mapsto \Phi(A,B) \) is decreasing and so for each given \( B \) we can conclude that \( \Phi(A,B) < 0 \) for all \( A > B \) such that \((pA)^{\frac{1}{p} - 1} - (pB)^{\frac{1}{p} - 1} - \delta > 0\). But since (4.41) must hold for \( \Gamma(A_s, B_s) = 0 \) to be satisfied, then we get a contradiction. From this we conclude that if there exists \( \frac{Lp}{p} > A_s > B_s > \frac{Kp}{p} \) such that \( \Gamma(A_s, B_s) = 0 \), then \( \Theta(A_s, B_s) < 0 \) and so there exists no solution to the system of nonlinear equations (4.23)-(4.24). Now suppose that \( \gamma_1 = \gamma_2 = \gamma \) and again assume, for contradiction, that (4.42)-(4.43) are satisfied. Rearranging terms in (4.42)-(4.43) we get that

\[ A_s - B_s = \frac{e^{\gamma((pA_s)^{\frac{1}{p}} - K)} - e^{\gamma((pB_s)^{\frac{1}{p}} - K + \delta)}}{\gamma((pB_s)^{\frac{1}{p}} - K + \delta)} \leq \frac{e^{-\gamma((pB_s)^{\frac{1}{p}} - K + \delta)} - e^{-\gamma((pA_s)^{\frac{1}{p}} - K)}}{\gamma((pA_s)^{\frac{1}{p}} - K)} \]

from which it follows that

\[ (pA_s)^{\frac{1}{p} - 1} (1 - e^{\gamma((pB_s)^{\frac{1}{p}} - (pA_s)^{\frac{1}{p}} + \delta)}) \leq (pB_s)^{\frac{1}{p} - 1} (1 - e^{\gamma((pB_s)^{\frac{1}{p}} - (pA_s)^{\frac{1}{p}} + \delta)}). \]

Since (4.41) has to be satisfied for \( \Gamma(A_s, B_s) = 0 \) to hold, then (4.52) simplifies to \((pA_s)^{\frac{1}{p} - 1} \leq (pB_s)^{\frac{1}{p} - 1} \). But this contradicts the fact that \( 0 > A_s > B_s \) (recall that \( p < 0 \)).

4.0.3. Verification Theorems

We are now in a position to provide verification results which link the solutions of the free-boundary problems formulated in the previous section, with the value functions (2.7)-(2.8) and thus establish Nash equilibrium.

**Theorem 4.7** Suppose that \( \mu \geq \frac{v^2}{2} \) and denote the right hand side of expressions (4.17) and (4.18) by \( \hat{u}(y; B_s, A_s) \) and \( \hat{v}(y; B_s, A_s) \) for given points \( S(K) \leq B_s < A_s \leq S(L) \) and \( y \in (B_s, A_s) \).

(A) If \( \Theta(S(L), S(K)) \leq 0 \) and \( \Gamma(S(L), S(K)) \leq 0 \) then the value functions (2.7)-(2.8) take the form

\[ V^1_{A_s}(x) = \begin{cases} H_1(x) & \text{if } 0 < x \leq K \\ \hat{u}(S(x); S(K), S(L)) & \text{if } K < x < L \\ G_1(x) & \text{if } L \leq x \end{cases} \]

and

\[ V^2_{A_s}(x) = \begin{cases} G_2(x) & \text{if } 0 < x \leq K \\ \hat{v}(S(x); S(K), S(L)) & \text{if } K < x < L \\ H_2(x) & \text{if } L \leq x \end{cases} \]

and the optimal stopping times \( \tau_* \) and \( \sigma_* \) are of the form (4.1) and (4.2) respectively with \( a_* = L \) and \( b_* = K \) (see figure 3)
(B) If \( \Theta(S(L), S(K)) > 0 \) and \( \Gamma(S(L), S(K)) \leq 0 \) then the value functions (2.7)-(2.8) take the form

\[
V_{\sigma_*}^1(x) = \begin{cases} 
H_1(x) & \text{if } 0 < x \leq K \\
\tilde{u}(S(x); S(K), A_*) & \text{if } K < x < a_* \\
G_1(x) & \text{if } a_* \leq x 
\end{cases}
\]

and

\[
V_{\tau_*}^2(x) = \begin{cases} 
G_2(x) & \text{if } 0 < x \leq K \\
\tilde{v}(S(x); S(K), A_*) & \text{if } K < x < a_* \\
H_2(x) & \text{if } a_* \leq x 
\end{cases}
\]

where \( A_* < S(L) \) is the unique root of the equation \( \Theta(A, S(K)) = 0 \) and \( a_* = S^{-1}(A_*) \).

The optimal stopping times \( \tau_* \) and \( \sigma_* \) are of the form (4.1) and (4.2) respectively where \( b_* = K \) (see figure 4).

(C) Suppose that \( \Gamma(S(L), S(K)) > 0 \) so that there exists \( B_*^{S(L)} > S(K) \) solving \( \Gamma(S(L), B) = 0 \). If \( \Theta(S(L), B_*^{S(L)}) \leq 0 \), then the value functions (2.7)-(2.8) take the form

\[
V_{\sigma_*}^1(x) = \begin{cases} 
G_1(x) & \text{if } 0 < x \leq b_* \\
\tilde{u}(S(x); B_*, S(L)) & \text{if } b_* < x < L \\
H_1(x) & \text{if } L \leq x 
\end{cases}
\]

and

\[
V_{\tau_*}^2(x) = \begin{cases} 
H_2(x) & \text{if } 0 < x \leq b_* \\
\tilde{v}(S(x); B_*, S(L)) & \text{if } b_* < x < L \\
G_2(x) & \text{if } L \leq x 
\end{cases}
\]

where \( B_* = B_*^{S(L)} \) and \( b_* = S^{-1}(B_*) \). The optimal stopping times \( \tau_* \) and \( \sigma_* \) are of the form (4.1) and (4.2) respectively where \( a_* = L \) (see figure 5).

(D) Suppose that \( \Gamma(S(L), S(K)) > 0 \) so that there exists \( B_*^{S(L)} > S(K) \) solving \( \Gamma(S(L), B) = 0 \). Suppose also that \( \Theta(S(L), B_*^{S(L)}) > 0 \). If \( \Theta(S(L), S(K)) < 0 \) then the value functions (2.7)-(2.8) take the form

\[
V_{\sigma_*}^1(x) = \begin{cases} 
G_1(x) & \text{if } 0 < x \leq b_* \\
\tilde{u}(S(x); B_*, A_*) & \text{if } b_* < x < a_* \\
H_1(x) & \text{if } a_* \leq x 
\end{cases}
\]

and

\[
V_{\tau_*}^2(x) = \begin{cases} 
H_2(x) & \text{if } 0 < x \leq b_* \\
\tilde{v}(S(x); B_*, A_*) & \text{if } b_* < x < a_* \\
G_2(x) & \text{if } a_* \leq x 
\end{cases}
\]

where \( S(L) > A_* > B_* > S(K) \) are the roots of the equations \( \Theta(A, B) = \Gamma(A, B) = 0 \), \( a_* = S^{-1}(A_*) \) and \( B_* = S^{-1}(B_*) \). The optimal stopping times \( \tau_* \) and \( \sigma_* \) are of the
form (4.1) and (4.2) respectively (see figure 6). If \( \Theta(S(L),S(K)) \geq 0 \) let \( A^{S(K)}_{s} \leq S(L) \) be the unique root of \( \Theta(A,S(K)) = 0 \). Denote by \( A^{\Gamma,S(K)}_{s} < S(L) \) the unique root of \( \Gamma(A,S(K)) = 0 \) and let \( B^{S(L)}_{s} > S(K) \) be the unique root of the equation \( \Gamma(S(L),B) = 0 \). If \( A^{S(K)}_{s} < A^{\Gamma,S(K)}_{s} \) then the value functions (2.7)-(2.8) and the optimal stopping times \( \tau_{s} \) and \( \sigma_{s} \) take the form presented in part (B) above. If on the other hand \( A^{S(K)}_{s} \geq A^{\Gamma,S(K)}_{s} \) then the value functions are given by (4.59)-(4.60) above with \( S(L) \geq A_{s} > B_{s} > S(K) \) being the roots of the equations \( \Theta(A,B) = \Gamma(A,B) = 0 \), \( a_{s} = S^{-1}(A_{s}) \) and \( B_{s} = S^{-1}(B_{s}) \). The optimal stopping times \( \tau_{s} \) and \( \sigma_{s} \) are of the form (4.1) and (4.2) respectively.

**Proof.** We shall only prove the above theorem for part (D) and when there exist \( A_{s} > B_{s} \) such that \( \Theta(A_{s},B_{s}) = \Gamma(A_{s},B_{s}) = 0 \) as this represents the most general case. The other assertions can be proved using similar arguments. To this end let us denote the right hand side of the expression in (4.59) by \( u \). We first show that \( V_{\sigma_{s}}^{1}(x) \leq u(x) \) for all \( x > 0 \). Since \( u \) is absolutely continuous and \( u' \) (which exists a.e.) is of bounded variation then \( u' \) can be written as the difference of two convex functions. So we can apply Itô-Tanaka formula to \( u(X_{t}) \) to get

\[
    u(X_{t}) = u(x) + \int_{0}^{t} u'(X_{s})dX_{s} + \frac{1}{2} \int_{0}^{\infty} l_{t}^{x}du'(x)
\]

\[
= u(x) + \int_{0}^{t} u'(X_{s})dX_{s} + \frac{1}{2} \int_{0}^{\infty} l_{t}^{x}du''(x)I(x \neq K, x \neq b_{s}, x \neq a_{s}, x \neq L)dx
\]

\[
+ \frac{1}{2} \int_{0}^{\infty} l_{t}^{x}I(x = K)du'(x) + \frac{1}{2} \int_{0}^{\infty} l_{t}^{x}I(x = b_{s})du'(x)
\]

\[
+ \frac{1}{2} \int_{0}^{\infty} l_{t}^{x}I(x = a_{s})du'(x) + \frac{1}{2} \int_{0}^{\infty} l_{t}^{x}I(x = L)du'(x)
\]

\[
= u(x) + M_{t} + \frac{1}{2} \int_{0}^{t} \mathbb{L}_{X}u(X_{s})I(X_{s} \neq K, X_{s} \neq b_{s}, X_{s} \neq a_{s}, X_{s} \neq L)ds
\]

\[
+ \frac{1}{2} l_{t}^{K}(u'_{+}(K) - u'_{-}(K)) + \frac{1}{2} l_{t}^{b_{s}}(u'_{+}(b_{s}) - u'_{-}(b_{s}))
\]

\[
+ \frac{1}{2} l_{t}^{a_{s}}(u'_{+}(a_{s}) - u'_{-}(a_{s})) + \frac{1}{2} l_{t}^{L}(u'_{+}(L) - u'_{-}(L))
\]

\[
\leq u(x) + M_{t} + \frac{1}{2} \int_{0}^{t} \mathbb{L}_{X}u(X_{s})I(X_{s} \neq K, X_{s} \neq b_{s}, X_{s} \neq a_{s}, X_{s} \neq L)ds
\]

\[
+ \frac{1}{2} l_{t}^{K}(u'_{+}(K) - u'_{-}(K)) + \frac{1}{2} l_{t}^{b_{s}}(u'_{+}(b_{s}) - u'_{-}(b_{s}))
\]

\[
(4.61)
\]

where \( l_{y}^{x} \) is the local time of \( X \) at a given point \( y \in (0, \infty) \), defined by

\[
(4.62)
\]

\[
l_{t}^{x} = P_{x} - \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{0}^{t} I(y < X_{s} < y + \varepsilon)ds
\]

and \( (M_{t})_{t \geq 0} \) is the continuous local martingale

\[
(4.63)
\]

\[
\int_{0}^{t} \sigma X_{s}u'_{-}(X_{s})dB_{s}.
\]
The third equality follows from the occupation time space formula whereas the inequality follows from the fact that that \((u'(L) - u'(L)) \leq 0\) and that \(u\) is smooth at \(a_+\). The latter assertion follows from (4.21) and the fact that \(a_+ = S^{-1}(A_x)\). Now for each stopping time \(\tau\) we have

\[
G_1(X_\tau)I(\tau \leq \sigma_x) + H_1(X_{\sigma_x})I(\sigma_x < \tau)
= (G_1(X_\tau)I(\tau \leq \sigma_x) + H_1(X_{\sigma_x})I(\sigma_x < \tau))I(\sigma_x < \infty)
\]

\[
+ (G_1(X_\tau)I(\tau \leq \sigma_x) + H_1(X_{\sigma_x})I(\sigma_x < \tau))I(\sigma_x = \infty)
\]

\[
\leq (u(X_\tau)I(\tau \leq \sigma_x) + u(X_{\sigma_x})I(\sigma_x < \tau))I(\sigma_x < \infty)
\]

\[
+ (G_1(X_\tau)I(\sigma_x = \infty)
\]

\[
= (u(X_{\tau \wedge \sigma_x})I(\sigma_x < \infty) + (G_1(X_\tau)I(\tau < \infty)
\]

\[
+ \limsup G_1(X_\tau)I(\tau = \infty))I(\sigma_x = \infty)
\]

(4.64)

\[
\leq u(X_{\tau \wedge \sigma_x})
\]

Note that if \(\mu = \frac{\nu^2}{2}\) then \(\sigma_x < \infty\) \(P_x\)-a.s. The inequalities follow from the fact that \(u \leq G_1\).

The latter can be deduced from the fact that \(\tilde{u}(y; B_\ast, A_\ast)\) majorises \(\tilde{G}_1\) in \((B_\ast, A_\ast)\). Taking the \(P_x\)-expectation in (4.64) we conclude that

(4.65)

\[
E_x[G_1(X_\tau)I(\tau \leq \sigma_x) + H_1(X_{\sigma_x})I(\sigma_x < \tau)] \leq E_x u(X_{\tau \wedge \sigma_x})
\]

for all stopping times \(\tau\). Now let \((\tau_n)_{n=1}^\infty\) be a localising sequence of stopping times for the process \(M = (M_t)_{t \geq 0}\). Then by (4.61) together with Doob’s optional sampling theorem and the structure of the stopping time \(\sigma_x\), we get that

(4.66)

\[
E_x u(X_{\tau \land \tau_n \land \sigma_x}) \leq u(x)
\]

\[
+ \frac{1}{2} E_x \int_0^{\tau \land \tau_n \land \sigma_x} L_x u(X_s)I(X_s \neq K, X_s \neq b_x, X_s \neq a_x, X_s \neq L) ds
\]

for all \(\tau\). The second inequality follows from the fact that \(A_\ast > \frac{1}{p}(\frac{1-p}{1+\frac{1}{p}})\) in the case \(\mu > \frac{\nu^2}{2}\), whereas \(A_\ast > \ln(\frac{1}{1+\frac{1}{p}})\) when \(\mu = \frac{\nu^2}{2}\) (note that \(A_\ast \leq S(L)\) is in the region where \(\tilde{G}_1\) is concave). So \(\tilde{G}_1' \leq 0\) in \([A_\ast, \infty)\) which implies that \(L_x G_1 \leq 0\) in \([a_+ , \infty)\). Taking the limit as \(n \to \infty\) in (4.65) we get, by Fatou’s Lemma, that

(4.67)

\[
E_x u(X_{\tau \land \sigma_x}) \leq u(x)
\]

From (4.65) and (4.67) it follows that

(4.68)

\[
E_x[G_1(X_\tau)I(\tau \leq \sigma_x) + H_1(X_{\sigma_x})I(\sigma_x < \tau)] \leq u(x)
\]

for all stopping times \(\tau\). Taking the supremum over all \(\tau\) we get that \(V_1(x) \leq u(x)\). We next show that (4.68) holds with equality if \(\tau\) is replaced by \(\tau_x\). Indeed, from (4.61) and the structure of the stopping times \(\tau_x\) and \(\sigma_x\), we have that

(4.69)

\[u(X_{\tau_x \land \tau_n \land \sigma_x}) = u(x) + M_{\tau_x \land \tau_n \land \sigma_x}\]

Taking the \(P_x\)-expectation on both sides of (4.69) and letting \(n \to \infty\) it follows, by Doob’s optional sampling theorem and Lebesgue dominated convergence theorem, that \(E_x u(X_{\tau_x \land \sigma_x}) =\)
Since \( E_x u(X_{\tau_*,\sigma_*}) = M^1_x(\tau_*,\sigma_*) \) (note that \( \tau_* < \infty \) a.s. whenever \( \mu \geq \frac{\nu^2}{2} \)) the result follows.

In a similar way it can be shown that \( V^{2,\tau}_x(x) \) coincides with the right hand side of the expression in (4.54).

**Figure 3:** Case (A): A graphical representation of the payoff functions and the value functions when \( \mu \geq \frac{\nu^2}{2} \) and \( \delta < L - K \)

**Figure 4:** Case (B): A graphical representation of the payoff functions and the value functions when \( \mu \geq \frac{\nu^2}{2} \) and \( \delta < L - K \)
To state the verification result in the case $0 \leq \mu < \frac{v^2}{2}$ we recall from section 3 that the first entry time in $[a_*, \infty)$ where $a_* \in (K, L]$, is an optimal stopping time for the optimal stopping problem $\sup_{\tau} \mathbb{E}_x[G_1(X_\tau)]$. To avoid confusion in notation, in this section we set $a_0^* := a_*$.

**Theorem 4.8** Suppose that $0 \leq \mu < \frac{v^2}{2}$. Let $f = l \circ S$ where $l$ is the line defined in the text following (4.28) and suppose that $H_2(a^*_0) > f(a^*_0)$. Then the value functions in (2.7)-(2.8)

![Graphical representation of Case (C)](image1)

**Figure 5:** Case (C): A graphical representation of the payoff functions and the value functions when $\mu \geq \frac{v^2}{2}$ and $\delta < L - K$

![Graphical representation of Case (D)](image2)

**Figure 6:** Case (D): A graphical representation of the payoff functions and the value functions when $\mu \geq \frac{v^2}{2}$ and $\delta < L - K$
take the form

\begin{align*}
V^1_{\tau_*} (x) &= \begin{cases} 
\frac{G_1(a_0^* - G_1(0^+))}{a_0^*} S(x) + G_1(0^+) & \text{if } 0 < x < a_0^*, \\
\frac{G_1(a_0^*)}{G_1(x)} & \text{if } a_0^* \leq x
\end{cases} \\
\text{and}

V^2_{\tau_*} (x) &= \begin{cases} 
\frac{H_2(a_0^* - H_2(0^+))}{a_0^*} S(x) + H_2(0^+) & \text{if } 0 < x < a_0^*, \\
\frac{H_2(a_0^*)}{H_2(x)} & \text{if } a_0^* \leq x
\end{cases}
\end{align*}

The optimal stopping time \( \tau_* \) is of the form \((4.1)\) with \( a_* = a_0^* \) whereas \( \sigma_* = \infty \).

**Proof.** This theorem can be proved in the same way as the proof of theorem \((4.7)\) upon using \((2.9)\) and by noting that

i. if \( H_2(a_0^*) > f(a_0^*) \) then the expression on the right hand side of \((4.71)\) majorises \( G_2 \),

ii. in the case \( \mu < \frac{v^2}{2} \),

\begin{equation}
P_x(\tau_* < \infty) = \begin{cases} 
(x/\delta)^{1-p} & \text{if } 0 < x < a_0^* \\
1 & \text{if } a_0^* \geq x
\end{cases}
\end{equation}

so that \( E_x H_2(X_{\tau_*}) = f(x) \) for \( x < a_0^* \).

\[\blacksquare\]

**Theorem 4.9** Suppose that \( 0 \leq \mu < \frac{v^2}{2} \). Let \( f = l \circ S \) and \( \hat{f} = \hat{l} \circ S \), where \( l \) is the line defined in the text following \((4.28)\), and \( \hat{l} \) the line defined in remark \((4.2 \ i.)\). Assume that \( H_2(a_0^*) \leq f(a_0^*) \).

(A) If \( \Theta(S(L), S(K)) \leq 0 \) and \( \Gamma(S(L), S(K)) \leq 0 \) then the value functions \((2.7)-(2.8)\) take the form

\begin{equation}
V^1_{\sigma_*} (x) = \begin{cases} 
\frac{H_1(K) - G_1(0^+)}{H_1(x)} S(x) + G_1(0^+) & \text{if } 0 < x < K \\
\tilde{u}(S(x); S(K), S(L)) & \text{if } K < x < L \\
G_1(x) & \text{if } L \leq x
\end{cases}
\end{equation}

and

\begin{equation}
V^2_{\tau_*} (x) = \begin{cases} 
\frac{G_2(K) - H_2(0^+)}{G_2(x)} S(x) + H_2(0^+) & \text{if } 0 < x < K \\
\tilde{v}(S(x); S(K), S(L)) & \text{if } K < x < L \\
H_2(x) & \text{if } L \leq x
\end{cases}
\end{equation}

and the optimal stopping times \( \tau_* \) and \( \sigma_* \) are of the form \((4.1)\) and \((4.3)\) respectively with \( a_* = L \) and \( b_*^1 = b_*^2 = K \) (see figure 7)
(B) Suppose that $\Theta(S(L), S(K)) > 0$ so that there exists $A^S(K) < S(L)$ such that $\Theta(A^S(K), S(K)) = 0$ and let $\Gamma(S(L), S(K)) \leq 0$. If assumption (4.5) holds the value functions (2.7)-(2.8) take the form

$$V^1_{\sigma*}(x) = \begin{cases} \frac{H_1(b^1)}{H_1(x)} G_1(0+) + S(x) \Theta(x; S(K), A^S) & \text{if } 0 < x < K \\ \Theta(x; S(K), A^S) & \text{if } K < x < a^S \\ b^S(x) & \text{if } a^S \leq x \end{cases}$$

and

$$V^2_{\tau*}(x) = \begin{cases} \frac{G_2(b^2)}{G_2(x)} H_2(0+) + S(x) \Theta(x; S(K), A^S) & \text{if } 0 < x < K \\ \Theta(x; S(K), A^S) & \text{if } K < x < a^S \\ b^S(x) & \text{if } a^S \leq x \end{cases}$$

where $A^S = A^S(K)$ and $a^S = S^{-1}(a^S)$. The optimal stopping times $\tau^*$ and $\sigma^*$ are of the form (4.1) and (4.3) respectively where $b^1 = b^2 = K$ (see figure 8).

(C) Suppose that $\Gamma(S(L), S(K)) > 0$ so that there exists $B^S(L) > S(K)$ such that $\Gamma(S(L), B^S(L)) = 0$ and let $\Theta(S(L), B^S(L)) \leq 0$. Then the value functions (2.7)-(2.8) take the form

$$V^1_{\sigma*}(x) = \begin{cases} \frac{H_1(b^1)}{H_1(x)} G_1(0+) + S(x) \Theta(x; B^2, S(L)) & \text{if } 0 < x < b^1 \\ \Theta(x; B^2, S(L)) & \text{if } b^1 \leq x \leq b^2 \\ \Theta(x; B^2, S(L)) & \text{if } b^2 < x < L \end{cases}$$

and

$$V^2_{\tau*}(x) = \begin{cases} \frac{G_2(b^2)}{G_2(x)} H_2(0+) + S(x) \Theta(x; B^2, S(L)) & \text{if } 0 < x < b^1 \\ \Theta(x; B^2, S(L)) & \text{if } b^1 \leq x \leq b^2 \\ \Theta(x; B^2, S(L)) & \text{if } b^2 < x < L \end{cases}$$

where $b^1 = S^{-1}(B)$ and $b^2 = S^{-1}(B^S(L))$. The optimal stopping times $\tau^*$ and $\sigma^*$ are of the form (4.1) and (4.3) respectively (see figure 9).

(D) Suppose that $\Gamma(S(L), S(K)) > 0$ so that there exists $B^S(L) > S(K)$ such that $\Gamma(S(L), B^S(L)) = 0$. Suppose first that $\Theta(S(L), S(K)) < 0$ so that there exist $S(L) > A^S > B^2 > S(K)$ such that $\Theta(A^S, B^2) = \Gamma(A^S, B^2) = 0$. If assumption (4.6) holds then the value functions (2.7)-(2.8) take the form

$$V^1_{\sigma*}(x) = \begin{cases} \frac{H_1(b^1)}{H_1(x)} G_1(0+) + S(x) \Theta(x; B^2, A^S) & \text{if } 0 < x < b^1 \\ \Theta(x; B^2, A^S) & \text{if } b^1 \leq x \leq b^2 \\ \Theta(x; B^2, A^S) & \text{if } b^2 < x < a^S \end{cases}$$
and

\[ V^2_{\tau^*} (x) = \begin{cases} \frac{G_2 (b_1^*) - H_2 (0+)}{G_2 (x)} S(x) + H_2 (0+) & \text{if } 0 < x < b_1^* \\ \tilde{v} (S(x); B_2^*, A_*) & \text{if } b_1^* \leq x \leq b_2^* \\ \frac{H_2 (0+)}{H_2 (x)} & \text{if } b_2^* < x < a_* \\ \tilde{\tilde{v}} (S(x); B_2^*, A_*) & \text{if } a_* \leq x \end{cases} \]

where \( a_* = S^{-1} (A_*), \ b_2^* = S^{-1} (B_2^*) \) and \( b_1^* = S^{-1} (\bar{B}) \). The optimal stopping times \( \tau^* \) and \( \sigma^* \) are of the form (4.1) and (4.3) respectively (see figure 10). Now suppose that \( \Theta (S(L), S(K)) \geq 0 \). Let \( A_{s}^{S(K)} < S(L) \) be the unique root of \( \Theta (A, S(K)) = 0 \) and \( A_{s}^{\Gamma, S(K)} \) the unique root of \( \Gamma (S(L), B) = 0 \). If \( A_{s}^{S(K)} < A_{s}^{\Gamma, S(K)} \) then the value functions (2.7)-(2.8) take the form presented in part (B) above provided that assumption (4.5) holds. If on the other hand \( A_{s}^{S(K)} \geq A_{s}^{\Gamma, S(K)} \) then the value functions (2.7)-(2.8) are given by (4.79)-(4.80) provided that assumption (4.6) holds.

**Proof.** The proof can be carried out in the same way as the proof of theorem (4.7). If we consider case (D) for example, and denote by \( u \) and \( v \) the expressions on the right hand side of (4.79) and (4.80) then the theorem can be proved by using (2.9) and by noting

i. that assumption (4.6) is equivalent to \( H_2 (a_*) \leq f (a_*) \), which justifies the existence of \( b_2^* \geq b_1^* \),

ii. that \( u \) and \( v \) majorise \( \tilde{G}_1 \) and \( \tilde{G}_2 \) respectively. The fact that \( u \) majorises \( G_1 \) in \((0, b_1^*)\) follows from Remark (4.2 i.). Indeed the latter implies that \( \tilde{f} \geq G_1 \) in \((0, b_1^*)\).

The fact that \( v \) majorises \( G_2 \) in \((0, b_1^*)\) can be seen by noting that \( l \) supports the hypograph of \( \tilde{G}_2 \) in \((0, \bar{B})\) (this follows by concavity of \( \tilde{G}_2 \)). From this we get that \( f \geq G_2 \) in \((0, b_1^*)\).

iii. \( \lim_{t \to \infty} X_t = 0 \) \( \mathbb{P}_x \)-a.s. whenever \( \mu < \frac{\nu^2}{2} \).

\( \blacksquare \)
Figure 7: Case A: A graphical representation of the payoff functions and the value functions when $0 \leq \mu \leq \frac{\nu^2}{2}$ and $\delta < L - K$

Figure 8: Case B: A graphical representation of the payoff functions and the value functions when $0 \leq \mu \leq \frac{\nu^2}{2}$ and $\delta < L - K$
Figure 9: Case C: A graphical representation of the payoff functions and the value functions when $0 \leq \mu \leq \frac{\nu^2}{2}$ and $\delta < L - K$

Figure 10: Case D: A graphical representation of the payoff functions and the value functions when $0 \leq \mu \leq \frac{\nu^2}{2}$ and $\delta < L - K$

5. The case $\delta < L - K$ and $\mu < 0$

We shall conclude this study by solving the optimal stopping game (2.7)-(2.8) for $\mu < 0$. This case will yield additional complexities on the structure of the optimal stopping times and for this we shall solve the game for certain cases only. Recall that when $0 \leq \mu < \frac{\nu^2}{2}$ the writer will not cancel the option whenever $x < K$ so as to allow the process to decrease in future,
and hence increase his expected utility. From this we concluded that the optimal stopping time of the writer is either not to cancel the option at all or to choose an optimal stopping time of the form (4.3). Similarly in the case \( \mu < 0 \) the writer may consider not cancelling the option at all if his expected utility is sufficiently large (given the strategy of the holder). In other cases the writer might have an incentive to cancel the option, however since the process will tend to move more to the left (one can see this by noting that the scale function is convex) his optimal strategy might not necessarily be of the form (4.3). This is due to the fact that the writer might consider not cancelling the option whenever \( K < x < L \) and sufficiently close to \( K \). With this, the optimal stopping time of the writer that we are after will either be

\[
(5.1) \quad \sigma_* = \inf\{ t \geq 0 : X_t = K \}
\]

or of the form

\[
(5.2) \quad \sigma_* = \inf\{ t \geq 0 : X_t \in K \cup [b_1^*, b_2^*] \}
\]

for some optimal stopping boundary \( K \leq b_1^* \leq b_2^* < a_* \leq L \) to be determined. On the other hand, the optimal stopping of the holder that we shall be looking for will be of the form (4.1).

**Remark 5.1** If \( H_1(b_2^*) \) is sufficiently small, this might create an incentive for the holder to exercise the option when the process is close to \( K \), however to simplify matters we shall rule out such cases.

As in the case \( \mu \geq 0 \), to be able to use results from convex analysis we shall consider the transformed problem (4.4)-(4.5). In this case \( \tilde{G}_1, \tilde{H}_1 \) are concave in \( \left( \frac{K^p}{p}, \frac{L^p}{p} \right) \) whereas \( \tilde{G}_2 \) and \( \tilde{H}_2 \) are convex in \( \left( \frac{K^p}{p}, \frac{1}{p}(\frac{p-1}{p})^p \right) \) and concave in \( \left( \frac{1}{p}(\frac{p-1}{p})^p, \frac{L^p}{p} \right) \). Prior to formulating the corresponding free-boundary problems we shall state and prove the following preliminary lemmas. To this end let us define the functions \( \Theta \) and \( \Gamma \) by (4.25) and (4.26) respectively.

**Lemma 5.2** Suppose that there exist points \( \frac{L^p}{p} > A_1 > B_1 > \frac{K^p}{p} \) and \( \frac{L^p}{p} > A_2 > B_2 > \frac{K^p}{p} \) such that \( \Theta(A_1, B_1) = 0 \) and \( \Theta(A_2, B_2) = 0 \). If \( B_2 > B_1 \) then \( A_2 > A_1 \).

**Proof.** Suppose, for contradiction, that \( A_2 \leq A_1 \). Since \( \tilde{G}_1 \) is concave in \( \left( \frac{K^p}{p}, \frac{L^p}{p} \right) \), then for any given \( B \in \left( \frac{K^p}{p}, \frac{L^p}{p} \right) \), the mapping \( A \rightarrow \Theta(A, B) \) is increasing in \( \left[ B, \frac{L^p}{p} \right) \). From this we get that \( 0 = \Theta(A_1, B_1) \geq \Theta(A_2, B_1) \). By concavity of \( \tilde{H}_1 \) in \( \left( \frac{K^p}{p}, \frac{L^p}{p} \right) \), we get that \( B \rightarrow \Theta(A_2, B) \) is convex. Now since \( \Theta(A_2, B_2) = 0 \geq \Theta(A_2, B_1) \) we get that \( \Theta(A_2, A_2) > 0 \), which contradicts the fact that \( \Theta(A_2, A_2) < 0 \). □

**Lemma 5.3** Suppose that there exist points \( \frac{L^p}{p} > A_1 > B_1 > \frac{1}{p}(\frac{p-1}{p})^p \) and \( \frac{L^p}{p} > A_2 > B_2 > \frac{1}{p}(\frac{p-1}{p})^p \) such that \( \Gamma(A_1, B_1) = 0 \) and \( \Gamma(A_2, B_2) = 0 \). If \( A_2 > A_1 \) then \( B_2 > B_1 \).

**Proof.** The proof can be carried out in the same way as the proof of lemma 5.2 by recalling that \( \tilde{G}_2 \) and \( \tilde{H}_2 \) are concave in \( \left[ \frac{1}{p}(\frac{p-1}{p})^p, \frac{L^p}{p} \right] \). □

Prior to stating the final preliminary lemma we shall introduce the function

\[
\Lambda : \left[ \frac{K^p}{p}, \frac{L^p}{p} \right] \rightarrow \mathbb{R}
\]
\[\Lambda(B) = \begin{cases} 
\tilde{G}_2\left(\frac{K_p}{p}\right) - \tilde{G}_2(B) - \tilde{G}_2'(B)\left(\frac{K_p}{p} - B\right) & \text{if } \frac{K_p}{p} < B < \frac{L_p}{p} \\
\tilde{G}_2\left(\frac{K_p}{p}\right) - \tilde{G}_2(B) - \tilde{G}_2'(B+)\left(\frac{K_p}{p} - B\right) & \text{if } B = \frac{K_p}{p} \\
\tilde{G}_2\left(\frac{K_p}{p}\right) - \tilde{G}_2(B) - \tilde{G}_2'(B-)\left(\frac{K_p}{p} - B\right) & \text{if } B = \frac{L_p}{p} 
\end{cases}\]

Upon computing \(\Lambda'\) we see that \(\Lambda\) is decreasing in \([\frac{K_p}{p}, \frac{1}{p} - \frac{1}{p^{27}}]\) and increasing in \([\frac{1}{p} - \frac{1}{p^{27}}, \frac{L_p}{p}]\). Moreover we have that \(\Lambda\left(\frac{K_p}{p}\right) = 0\). If \(\Lambda\left(\frac{L_p}{p}\right) > 0\) then we shall set \(\tilde{l}\) to be the line passing through the point \((\frac{K_p}{p}, \tilde{G}_2\left(\frac{K_p}{p}\right))\) and tangent to \(\tilde{G}_2(\tilde{B})\) where \(\tilde{B} \in \left(\frac{1}{p} - \frac{1}{p^{27}}, \frac{L_p}{p}\right)\) is the unique point solving \(\Lambda(B) = 0\). If on the other hand \(\Lambda\left(\frac{L_p}{p}\right) \leq 0\) we let \(\tilde{l}\) be the line joining the points \((\frac{K_p}{p}, \tilde{G}_2\left(\frac{K_p}{p}\right))\) and \((\frac{L_p}{p}, \tilde{G}_2\left(\frac{L_p}{p}\right))\) and in this case we set \(\tilde{B} = \frac{L_p}{p}\).

**Lemma 5.4** Let \(A^p, \frac{K_p}{p}\) be the unique root of \(\Theta(A, \frac{K_p}{p}) = 0\) when \(\Theta\left(\frac{L_p}{p}, \frac{K_p}{p}\right) > 0\) and let us set \(A^p, \frac{K_p}{p}\) when \(\Theta\left(\frac{L_p}{p}, \frac{K_p}{p}\right) \leq 0\). Suppose that \(\tilde{H}_2(A^p, \frac{K_p}{p}) \leq \tilde{l}(A^p, \frac{K_p}{p})\) and that \(A^p, \frac{K_p}{p} > \tilde{B}\). Then the sequences \(A_n^p, \frac{K_p}{p}\) and \(B_n^p, \frac{K_p}{p}\) defined recursively as follows:

\[B_1 = \frac{K_p}{p} \text{ and for } n \geq 1,\]

\[\text{if } \Theta\left(\frac{L_p}{p}, B_n\right) > 0 \text{ then } A_n \text{ is the root of the equation } \Theta(A,B_n) = 0,\]

\[\text{if } \Theta\left(\frac{L_p}{p}, B_n\right) \leq 0 \text{ then } A_n = \frac{L_p}{p},\]

\[B_{n+1} \text{ is the root of the equation } \Gamma(A_n,B) = 0,\]

with \(A_n > B_n\) for each \(n \geq 1\), are decreasing with limits \(A_* \in (\tilde{B}, \frac{L_p}{p})\) and \(B_* \in [\tilde{B}, \frac{L_p}{p}]\). Moreover \(\Gamma(A_*,B_*) = 0\) and \(\Theta(A_*,B_*) \leq 0\).

**Proof.** Consider first the case \(\Theta\left(\frac{L_p}{p}, \frac{K_p}{p}\right) \leq 0\) so that \(A_1 = A_*^p, \frac{K_p}{p} = \frac{L_p}{p}\). Using the hypotheses \(\tilde{H}_2(A_*^p, \frac{K_p}{p}) \leq \tilde{l}(A_*^p, \frac{K_p}{p})\) and \(A_*^p > \tilde{B}\) we get that \(\Gamma(A_1,\tilde{B}) \geq 0\). Since the mapping \(B \mapsto \Gamma(A_1, B)\) is decreasing in \((\tilde{B}, A_1)\) and \(\Gamma(A_1, A_1) < 0\) we get the existence of a unique point \(B_2 \in [\tilde{B}, A_1]\) such that \(\Gamma(A_1, B_2) = 0\). Moreover, since \(\tilde{H}_1\) is concave in \([\frac{K_p}{p}, \frac{L_p}{p}]\) then for any given \(A \in [\frac{K_p}{p}, \frac{L_p}{p}]\) it follows that the mapping \(B \mapsto \Theta(A,B)\) is convex for all \(B \in [\frac{K_p}{p}, A]\). This together with the fact that \(\Theta\left(\frac{L_p}{p}, \frac{K_p}{p}\right) \leq 0\) implies that \(\Theta\left(\frac{L_p}{p}, B_2\right) < 0\) and so \(A_2 = A_1\). Hence \(B_3 = B_2\). Continuing in this manner we get that \(A_* = \lim_{n \to \infty} A_n = A_1\) and \(B_* = \lim_{n \to \infty} B_n = B_2\) and so \(\Gamma(A_*, B_*) = 0\) and \(\Theta(A_*, B_*) < 0\) as claimed. Now suppose that \(\Theta\left(\frac{L_p}{p}, \frac{K_p}{p}\right) > 0\) so that \(A_1 = A_*^p, \frac{K_p}{p} < \frac{L_p}{p}\). Again by using the hypotheses \(\tilde{H}_2(A_*^p, \frac{K_p}{p}) \leq \tilde{l}(A_*^p, \frac{K_p}{p})\) and \(A_*^p > \tilde{B}\) we get the existence of a unique \(B_2 \in [\tilde{B}, A_1]\) such that \(\Gamma(A_1, B_2) = 0\). If \(\Theta\left(\frac{L_p}{p}, B_2\right) > 0\) then there exists a unique point \(A_2\) such that \(\Theta(A_2, B_2) = 0\). Moreover, by Lemma 5.2 we have that \(A_2 > A_1\) (recall that \(B_2 > B_1\)) and so by concavity of \(\tilde{H}_2\) it follows that \(\tilde{H}_2(A_2) < \tilde{l}(A_2)\) (see proposition (4.3)). This implies the existence of a unique point \(B_3 \in (\tilde{B}, A_2)\) such that \(\Gamma(A_2, B_3) = 0\) and by Lemma 5.3 we have that \(B_3 > B_2\). If on the other hand \(\Theta\left(\frac{L_p}{p}, B_2\right) \leq 0\) then \(A_2 = \frac{L_p}{p} > A_1\) and
so \( B_3 = B_2 \), which implies that \( \Theta(L^p_p, B_3) \leq 0 \). Continuing in this manner we either get that \( A_\ast = \lim_{n \to \infty} A_n \) and \( B_\ast = \lim_{n \to \infty} A_n \) with \( \Theta(A_\ast, B_\ast) = \lim_{n \to \infty} \Theta(A_n, B_n) = 0 \) and \( \Gamma(A_\ast, B_\ast) = \lim_{n \to \infty} \Gamma(A_n, B_n) = 0 \), where \( \tilde{B} < B_\ast < A_\ast \leq \frac{L_p}{p} \), or that there exists \( m \geq 2 \) such that \( \Theta(L^p_p, B_m) \leq 0 \), which implies that \( A_n = A_m = \frac{L_p}{p} \) and \( B_{n+2} = B_{m+1} \) for all \( n \geq m \). In this case we conclude that \( \Theta(A_\ast, B_\ast) = \Theta(L^p_p, B_{m+1}) < 0 \) and \( \Gamma(A_\ast, B_\ast) = \Gamma(L^p_p, B_{m+1}) = 0 \).

To study solutions for \( \mu < 0 \) it is sufficient to assume that \( p - 1 > \gamma_2 K \). Indeed, the case \( p - 1 \leq \gamma_2 K \) can be dealt with in the same way as the case \( 0 \leq \mu < \frac{\nu^2}{2} \) (recall that when \( \mu < 0 \), \( \bar{G}_i, \tilde{H}_i \) for \( i = 1, 2 \), are concave in \( [\frac{Kp}{p}, \frac{Lp}{p}] \)). Moreover, under assumption (4.5) and by convexity of \( \tilde{H}_i \) one can see, by Lemma 5.4, that assumption (4.6) is always satisfied. We note, however, that the result obtained in section (4.0.2) might not be true in this case. Similar conclusions, nevertheless, can be drawn in the case \( \gamma_1 \geq \gamma_2 \) (see remark (5.7)). To this end consider first the optimal stopping problem (3.6) and let \( A^0_\ast \) be the optimal stopping boundary. Set \( \tilde{l} \) to be the line joining the points \((0, \bar{H}_2(0+))\) and \((\frac{Kp}{p}, \tilde{G}_2(\frac{Kp}{p})) \). We shall see in the next section that if one of the following statements hold;

i. \( \tilde{H}_2(A^0_\ast) > \max(\bar{l}(A^0_\ast), \tilde{l}(A^0_\ast)) \)

ii. \( \bar{l}(A^0_\ast) \geq H_2(A^0_\ast) > \tilde{l}(A^0_\ast) \) and \( A^0_\ast < \tilde{B} \)

then it will not be optimal for the writer to cancel the option. Now suppose that (i.) and (ii.) do not hold. To guarantee existence of a solution we shall assume that:

**Assumption 5.5**

\[
\tilde{H}_2(A^0_\ast) \leq \tilde{l}(A^0_\ast).
\]

Moreover, to guarantee that the optimal stopping time of the holder is of the form given in (4.1) and that of the writer is of the form given in (5.1) or (5.2) we shall consider only the case \( \tilde{l} < \bar{l} \) in \( [\frac{Kp}{p}, \infty) \). To this end let us assume first that \( A^0_\ast \leq \tilde{B} \), where \( A^0_\ast \) is the point defined in Lemma 5.4. Since \( \tilde{H}_2 > \tilde{G}_2 \), then we see that the line \( l_1 \) joining the points \((\frac{Kp}{p}, \tilde{G}_2(\frac{Kp}{p})) \) and \((\frac{Kp}{p}, \tilde{H}_2(A^0_\ast)) \) majorises the line \( l_2 \) joining the points \((\frac{Kp}{p}, \tilde{G}_2(\frac{Kp}{p})) \) and \((\frac{Kp}{p}, \tilde{G}_2(A^0_\ast)) \) in \( [\frac{Kp}{p}, \infty) \). If \( A^0_\ast \leq \frac{1}{p} (\frac{p-1}{72})^p \) then by convexity of \( \tilde{G}_2 \) in \( (\frac{Kp}{p}, \frac{1}{p} (\frac{p-1}{72})^p) \) we see that \( l_2 \) majorises \( \tilde{G}_2 \) in \( (\frac{Kp}{p}, \frac{Kp}{p}) \). If on the other hand \( A^0_\ast > \frac{1}{p} (\frac{p-1}{72})^p \) then by concavity of \( \tilde{G}_2 \) in \( (\frac{1}{p} (\frac{p-1}{72})^p, \frac{Kp}{p}) \) we see that \( l_2 \) majorises \( \tilde{G}_2 \) in \( (\frac{1}{p} (\frac{p-1}{72})^p, A^0_\ast) \) and \( l_2 \) majorises the line \( l_3 \) joining the points \((\frac{Kp}{p}, \tilde{G}_2(\frac{Kp}{p})) \) and \((\frac{1}{p} (\frac{p-1}{72})^p, \tilde{G}_2(\frac{1}{p} (\frac{p-1}{72})^p)) \) (see proposition (4.3)). Thus by convexity of \( \tilde{G}_2 \) in \( (\frac{Kp}{p}, \frac{1}{p} (\frac{p-1}{72})^p) \) we get that \( l_3 \) majorises \( \tilde{G}_1 \) in \( (\frac{Kp}{p}, \frac{1}{p} (\frac{p-1}{72})^p) \) and hence we can conclude that \( l_1 \geq \tilde{G}_1 \) in \( (\frac{Kp}{p}, A^0_\ast) \). From this it follows that there exist continuous functions \( \tilde{u} \) and \( \tilde{v} \) and point \( A_\ast = A^0_\ast \) solving the free-boundary problem

\[
L_B \tilde{u} = 0 \text{ and } L_B \tilde{v} = 0 \text{ for } y \in (0, \frac{Kp}{p})
\]
for all \( y \)

\[
(5.17) \quad \lim_{y \downarrow 0} \tilde{u}(y) = \lim_{y \downarrow 0} \tilde{G}_1(y) \quad \text{and} \quad \lim_{y \downarrow 0} \tilde{v}(y) = \lim_{y \downarrow 0} \tilde{H}_2(y)
\]

\[
(5.18) \quad \tilde{u}(y) = \tilde{G}_1(y) \quad \text{and} \quad \tilde{v}(y) = \tilde{H}_2(y) \quad \text{for} \quad y = \frac{K_p}{p}
\]

\[
(5.19) \quad \mathbb{L}_B \tilde{u} = 0 \quad \text{and} \quad \mathbb{L}_B \tilde{v} = 0 \quad \text{for} \quad y \in \left( \frac{K_p}{p}, A_* \right)
\]

\[
(5.20) \quad \tilde{u}(A_*) = \tilde{G}_1(A_*) \quad \text{and} \quad \tilde{v}(A_*) = \tilde{H}_2(A_*)
\]

\[
(5.21) \quad \tilde{u}(y) > \tilde{G}_1(y) \quad \text{and} \quad \tilde{v}(y) > \tilde{G}_2(y) \quad \text{for} \quad y \in (0, \frac{K_p}{p}) \cup \left( \frac{K_p}{p}, A_* \right)
\]

Together with the smooth fit principle

\[
(5.22) \quad \tilde{u}''(A_*) = \tilde{G}'_1(A_*)
\]

whenever \( \Theta\left( \frac{L_p}{p}, \frac{K_p}{p} \right) \geq 0 \). If on the other hand \( \Theta\left( \frac{L_p}{p}, \frac{K_p}{p} \right) < 0 \) then \( A_* = \frac{L_p}{p} \) in (5.5)-(5.11) and the smooth fit principle (5.22) breaks down.

Now let us suppose that \( A_* > B \). To guarantee that the optimal stopping time of the holder of the option is of the form (4.1) we will further assume that

**Assumption 5.6**

\[
(5.23) \quad l(y) > W_1(y)
\]

for all \( y \in \left[ \frac{K_p}{p}, B \right] \) where \( l \) is the line segment joining the points \( \left( \frac{K_p}{p}, \tilde{H}_1\left( \frac{K_p}{p} \right) \right) \) and \( \left( B, \tilde{H}_1(B) \right) \) and \( W_1 \) is the value function for problem (3.6).

If \( \tilde{H}_2\left( \frac{K_p}{p} \right) > l\left( \frac{K_p}{p} \right) \) then we formulate the same free boundary problem presented in the case \( A_* \leq B \). If on the other hand \( \tilde{H}_2\left( \frac{K_p}{p} \right) \leq l\left( \frac{K_p}{p} \right) \) we can formulate the following free-boundary problem for unknown functions \( \tilde{u} \) and \( \tilde{v} \) and unknown points \( B_*^1 \leq B_*^2 < A_* \leq \frac{L_p}{p} \):

\[
(5.24) \quad \mathbb{L}_B \tilde{u} = 0 \quad \text{and} \quad \mathbb{L}_B \tilde{v} = 0 \quad \text{for} \quad y \in \left( \frac{K_p}{p}, B_*^1 \right)
\]

\[
(5.25) \quad \lim_{y \downarrow 0} \tilde{u}(y) = \lim_{y \downarrow 0} \tilde{G}_1(y) \quad \text{and} \quad \lim_{y \downarrow 0} \tilde{v}(y) = \lim_{y \downarrow 0} \tilde{H}_2(y)
\]

\[
(5.26) \quad \tilde{u}(y) = \tilde{H}_1(y) \quad \text{and} \quad \tilde{v}(y) = \tilde{G}_2(y) \quad \text{for} \quad y = \frac{K_p}{p}
\]

\[
(5.27) \quad \tilde{u}(A_*) = \tilde{G}_1(A_*) \quad \text{and} \quad \tilde{v}(A_*) = \tilde{H}_2(A_*)
\]

\[
(5.28) \quad \tilde{u}(B_*^1) = \tilde{H}_1(B_*^1) \quad \text{and} \quad \tilde{v}(B_*^1) = \tilde{G}_2(B_*^1)
\]

\[
(5.29) \quad \tilde{u}(y) = \tilde{G}_1(y) \quad \text{and} \quad \tilde{v}(y) = \tilde{H}_2(y) \quad \text{for} \quad y \in (A_*, \infty)
\]
\[ \bar{\gamma}(\Lambda) \] and \( \bar{\gamma}(\Lambda) \) for \( y \in (0, \frac{K_{\theta}}{\mu}) \), so we have that
\[ \bar{v}'(B) = \bar{G}'_n(B) \]}

Moreover, by Lemma 5.4 we see that \( A_* \) and \( B_*^2 \) in (5.17)-(5.23) can be identified with the limits of the sequences \( A_n \) and \( B_n \) and so
\[ \bar{u}'(A) \leq \bar{G}'_n(A) \]
\[ \bar{v}'(B^2) = \bar{G}'_n(B^2). \]

**Remark 5.7** If assumption (5.5) holds, then in the case \( \Theta(\frac{L_{\theta}}{\mu}, \frac{K^2_{\theta}}{\mu}) > 0 \) and \( \gamma_1 \geq \gamma_2 \) the structure of the optimal stopping time of the writer of the option is of the form (5.1). This follows from the fact that whenever \( \gamma_1 \geq \gamma_2 \) if there exists \( A_* > B_* \) such that \( \Theta(A_*, B_*) = 0 \) then \( \Gamma(A_*, B_*) < 0 \). The case \( \gamma_1 > \gamma_2 \) can be treated by using similar arguments to the case \( \gamma_1 < \gamma_2 \) in section (4.0.1). To prove the result for \( \gamma_1 = \gamma_2 := \gamma \) assume for contradiction, that
\[ \Theta(A_*, B_*) = 0 \]
\[ \Gamma(A_*, B_*) \geq 0. \]

Multiplying (5.27) by \( -e^{-\gamma_2((pB_*)^{\frac{1}{p}} - K + \delta)} \) and (5.28) by \( -e^{-\gamma_1((pA_*)^{\frac{1}{p}} - K)} \) we obtain
\[ 1 - e^{\gamma((pA_*)^{\frac{1}{p}} - (pB_*)^{\frac{1}{p}} - \delta) + \gamma((pA_*)^{\frac{1}{p}} - (pB_*)^{\frac{1}{p}} - \delta)} = 0 \]
\[ 1 - e^{\gamma((pA_*)^{\frac{1}{p}} - (pB_*)^{\frac{1}{p}} - \delta) + \gamma_1((pB_*)^{\frac{1}{p}} - (pA_*)^{\frac{1}{p}} - \delta)} \leq 0 \]

Since \( A_* > B_* \) then (4.41) must be satisfied for \( \Theta(A_*, B_*) = 0 \). So from (5.29) and (5.30) we get that \( A_*^{\frac{1}{p} - 1} \geq B_*^{\frac{1}{p} - 1} \) which contradicts the fact that \( A_* > B_* \) (note that \( p > 1 \)).

**5.1. Verification Theorems**

We are now in a position to state the verification results when \( \mu < 0 \). As in the case \( 0 \leq \mu < \frac{\sigma^2}{2} \), to avoid confusion in notation, we shall set \( a_*^0 \) to be the optimal stopping boundary for the optimal stopping problem \( \sup_{x} E_x[G_2(X_\tau)] \).

**Theorem 5.8** Suppose that \( \mu < 0 \) so that \( S(x) = x^p \). Set \( \bar{l} \) to be the line defined in the text following (5.3) and \( \bar{l} \) to be the line defined in the text following the proof of Lemma 5.4. Let \( f = \bar{l} \circ S \) and \( \bar{f} = \bar{l} \circ S \). If one of the following assertions hold
\[ H_2(a_*^0) > \max(\bar{f}(a_*^0), \bar{f}(a_*^0)) \]
\[ H_2(a_*^0) > \bar{f}(a_*^0) \]
\[ a_*^0 < \bar{b} \]
where \( \bar{b} = S^{-1}(\bar{B}) \), the value functions in (2.7)-(2.8) take the form (4.70)-(4.71). The optimal stopping time \( \tau_* \) is of the form (4.1) with \( a_* = a_*^0 \) whereas \( \sigma_* = \infty \).
Proof. This theorem can be proved in the same way as the proof of theorem 4.7 upon using (2.9) and (4.72), and by noting that if assertions (5.31)-(5.32) hold the expression on the right hand side of (4.71) majorises $G_2$. ■

**Theorem 5.9** Suppose that $\mu < 0$. Let $\bar{f}$ and $\tilde{f}$ be defined as in theorem 5.8. Set $\bar{b} = S^{-1}(\bar{B})$ and $a_{\ast}^{S(K)} = S^{-1}(A_{\ast}^{S(K)})$ where $A_{\ast}^{S(K)}$ is the point defined in Lemma 5.4. Suppose that assertions (5.31) and (5.32) do not hold, that $\bar{f} < \tilde{f}$ and assumption (5.5) holds.

\begin{enumerate}[(A)]
    
    \item If $a_{\ast}^{S(K)} \leq \bar{b}$ then the value functions (2.7)-(2.8) take the form (4.75)-(4.76) with $A_{\ast} = A_{\ast}^{S(K)}$ and $a_{\ast} = a_{\ast}^{S(K)}$. The optimal stopping times $\tau_\ast$ and $\sigma_\ast$ are of the form (4.1) and (4.3) respectively with $b_{\ast}^1 = b_{\ast}^2 = K$.
    
    \item Suppose that $a_{\ast}^{S(K)} > \bar{b}$ and assume that (5.13) holds. If $H_2(a_{\ast}^{S(K)}) > \tilde{f}(a_{\ast}^{S(K)})$ then the value functions (2.7)-(2.8) take the form presented in part (A) above. If on the other hand $H_2(a_{\ast}^{S(K)}) \leq \tilde{f}(a_{\ast}^{S(K)})$ then the value functions (2.7)-(2.8) take the form

\begin{align}
    V^1_{\sigma_\ast}(x) = \begin{cases}
        \left( \frac{H_1(K) - G_1(0+)}{K} S(x) + G_1(0+) \right) & \text{if } 0 < x < K \\
        H_1(x) & \text{if } K = x \\
        \tilde{u}_1(S(x); S(K), B_1^1) & \text{if } K < x < b_1^1 \\
        H_1(x) & \text{if } b_1^1 \leq x \leq b_2^1 \\
        \tilde{u}_2(S(x); A_{\ast}, B_2^1) & \text{if } b_2^1 < x < a_{\ast} \\
        G_1(x) & \text{if } a_{\ast} \leq x
    \end{cases}
\end{align}

and

\begin{align}
    V^2_{\tau_\ast}(x) = \begin{cases}
        \left( \frac{G_2(K) - H_2(0+)}{K} S(x) + H_2(0+) \right) & \text{if } 0 < x < K \\
        G_2(x) & \text{if } K = x \\
        \tilde{v}_1(S(x); S(K), B_1^1) & \text{if } K < x < b_1^1 \\
        G_2(x) & \text{if } b_1^1 \leq x \leq b_2^1 \\
        \tilde{v}_2(S(x); A_{\ast}, B_2^1) & \text{if } b_2^1 < x < a_{\ast} \\
        H_2(x) & \text{if } a_{\ast} \leq x
    \end{cases}
\end{align}

\end{enumerate}

where $A_{\ast}$ and $B_2^2$ can be identified with the limits of the sequences defined in Lemma 5.4, $b_{\ast}^1 = \bar{b}$, $b_{\ast}^2 = S^{-1}(B_2^2)$ and $a_{\ast} = S^{-1}(A_{\ast})$. The functions $\tilde{u}_1(y; B_1, B_2)$ and $\tilde{v}_2(y; B_1, B_2)$ for given points $B_1 < B_2$ and $y \in (B_1, B_2)$ take the form $m_1 y + n_1$ and $m_2 y + n_2$ respectively where

\begin{align}
    m_1 &= \frac{\tilde{H}_1(B_1) - \tilde{H}_1(B_2)}{B_1 - B_2}, \quad n_1 = \tilde{H}_1(B_1) - m_1 B_1 \\
    m_2 &= \frac{\tilde{G}_2(B_1) - \tilde{G}_2(B_2)}{B_1 - B_2}, \quad n_2 = \tilde{G}_2(B_1) - m_2 B_1
\end{align}

The functions $\tilde{u}_2(y; B_1, A_{\ast})$ and $\tilde{v}_2(y; B_1, A_{\ast})$, for given points $B_1 < A_{\ast}$ and $y \in (B_1, A_{\ast})$ are given by the expressions on the right hand side of (4.17) and (4.18) respectively. The optimal stopping times $\tau_\ast$ and $\sigma_\ast$ are of the form (4.1) and (5.1) respectively (see figure 11).
Proof. The proof can be carried out in the same way as the proof of theorem (4.7). If we consider case (B) for example, and denote by \( u \) and \( v \) the expressions on the right hand side of (5.33) and (5.34) then the theorem can be proved by using (2.9) together with the fact that \( \lim_{t \to \infty} X_t = 0 \) for \( \mu < \frac{\nu^2}{2} \) and by noting that assumption (5.13) guarantees that \( u \) majorises \( G_1 \).

![Figure 11](image.png)

**Figure 11:** Case B: A graphical representation of the payoff functions and the value functions when \( \mu < 0 \) and \( \delta < L - K \)

References


