Nash Equilibrium in Nonzero-Sum Games of Optimal Stopping for Brownian Motion

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We present solutions to nonzero-sum optimal stopping games for Brownian motion in [0, 1] absorbed at either 0 or 1. The approach used is based on the double partial superharmonic characterisation of the value functions derived in [1]. In this setting this characterisation of the value functions has a transparent geometrical interpretation of ‘pulling two ropes’ above ‘two obstacles’ which must however be constrained to pass through certain regions. This is an extension of the analogous result derived by Peskir in [16] and [17] (semiharmonic characterisation) for the value function in zero-sum games of optimal stopping. To derive the value functions we transform the game into a free-boundary problem. The latter is then solved by making use of the double smooth-fit principle which was also observed in [1]. Martingale arguments based on Itô-Tanaka formula will then be used to verify that the solution to the free-boundary problem coincide with the value functions of the game and this will establish Nash equilibrium.

1. Introduction

The purpose of this work is to derive Nash equilibrium in two-player nonzero-sum games of optimal stopping for Brownian motion in [0, 1], absorbed at either 0 and 1. For this we shall use the results obtained in [1], in particular the double partial superharmonic characterisation of the value functions of the two players and the principle of double smooth fit. This probabilistic approach for studying the value functions and corresponding Nash equilibrium is in line with the results derived by Peskir in [16] and [17] for zero-sum games. In the case of absorbed Brownian motion in [0,1] the results of Peskir show that the value function in zero-sum games is equivalent to ‘pulling a rope’ between ‘two obstacles’ (semiharmonic characterisation) which in turn establishes Nash equilibrium (by ‘pulling a rope’ between ‘two obstacles’ we mean finding the shortest path between the graphs of two functions). In nonzero-sum games, under certain assumptions on the payoff functions, we will show that the value functions are equivalent to ‘pulling two ropes’ above ‘two obstacles’ which must however be constrained to pass through
certain regions. As in the case of zero-sum games this geometric explanation of the value function will establish Nash equilibrium.

Literature on nonzero-sum games of optimal stopping are mainly concerned with existence of Nash equilibrium. Initial studies in discrete time date back to Morimoto [12] wherein a fixed point theorem for monotone mappings is used to derive sufficient conditions for the existence of a Nash equilibrium point. Ohtsubo [14] derived equilibrium values via backward induction and in [15] the same author considers nonzero-sum games with the smaller gain processes having a monotone structure and gives sufficient conditions for a Nash equilibrium point to exist. Shmaya and Solan in [20] proved that every two player nonzero-sum game in discrete time admits an $\varepsilon$-equilibrium in randomised stopping times. In continuous time Bensoussan and Friedman [3] showed that, for diffusions, Nash equilibrium exists if there exists a solution to a system of quasi-variational inequalities. However, the regularity and uniqueness of the solution remain open problems. Nagai [13] studies a nonzero-sum stopping game of symmetric Markov processes. A system of quasi-variational inequalities is introduced in terms of Dirichlet forms and the existence of extremal solutions of a system of quasi-variational inequalities is discussed. Nash equilibrium is then established from these extremal solutions. Cattiaux and Lepeltier [4] study right processes and they prove existence of a quasi-Markov Nash Equilibrium point. The authors follow Nagai’s idea but use probabilistic tools rather than the theory of Dirichlet forms. Moreover they complete Nagai’s result (whose construction of the extremal solutions of the quasi-variational inequalities is not complete) and extend it to non-symmetric processes. Huang and Li in [10] prove the existence of a Nash equilibrium point for a class of nonzero-sum noncyclic stopping games using the martingale approach. Laraki and Solan [11] proved that every two-player nonzero-sum Dynkin game in continuous time admits an $\varepsilon-$equilibrium in randomised stopping times whereas Hamadène and Zhang in [9] prove existence of Nash equilibrium points using the martingale approach, for processes with positive jumps.

The structure of this paper is as follows: In Section 2 we introduce the game and review the double partial superharmonic characterisation of the value functions and the double smooth fit principle (cf. [1]), when the underlying process is assumed to be absorbed Brownian motion in [0,1]. In Section 3 we formulate and solve an equivalent free-boundary problem for a certain class of payoff functions. Under additional assumptions on the payoff functions we then show that the solution is also unique. In Section 4 we use martingale arguments based on Itô-Tanaka formula to verify that the solution to the free-boundary problem coincides with the value functions of the game. In Section 5 we provide a counterexample to show that if the original assumptions imposed on the payoff functions are relaxed then Nash equilibrium may not be established via the double partial superharmonic characterisation of the value functions. In Section 6 we explain how these results can be extended to one-dimensional regular diffusions and in Section 7 we conclude by giving some remarks for future research.
2. Double partial superharmonic characterisation of the value functions

Let $X$ be Brownian motion in $[0, 1]$, started at $x \in [0, 1]$ and absorbed at either 0 or 1 and let $G_i, H_i : [0, 1] \to \mathbb{R}$ for $i = 1, 2$ be $C^2$ functions such that $G_i \leq H_i$. Assume also that $G_i(0) = H_i(0)$ and $G_i(1) = H_i(1)$. Consider the nonzero-sum game of optimal stopping in which player one wants to choose a stopping time $\tau_*$ and player two a stopping time $\sigma_*$ such that their total average gains, which are respectively given by

$$\mathcal{M}_1^2(\tau, \sigma) = E_x [G_1(X_\tau) I(\tau \leq \sigma) + H_1(X_\tau) I(\sigma < \tau)]$$

$$\mathcal{M}_2^2(\tau, \sigma) = E_x [G_2(X_\sigma) I(\sigma < \tau) + H_2(X_\tau) I(\tau \leq \sigma)]$$

are maximized. For a given strategy $\sigma$ chosen by player two, we shall define the value function of player one by

$$V_\sigma^1(x) = \sup_\tau \mathcal{M}_1^2(\tau, \sigma). \tag{2.1}$$

Similarly, for a given strategy $\tau$ chosen by player one, we shall define the value function of player two by

$$V_\tau^2(x) = \sup_\sigma \mathcal{M}_2^2(\tau, \sigma). \tag{2.2}$$

In this context a saddle point of stopping times is characterized via Nash equilibrium. Formally, a pair of stopping times $(\tau_*, \sigma_*)$ is a saddle point if $\mathcal{M}^1_2(\tau, \sigma_*) \leq \mathcal{M}^1_2(\tau_*, \sigma_*)$ and $\mathcal{M}^2_2(\tau_*, \sigma) \leq \mathcal{M}^2_2(\tau_*, \sigma_*)$ for all stopping times $\tau$ and $\sigma$ and for all $x \in [0, 1]$.

Under the mentioned assumptions on $G_i$ and $H_i$, for $i = 1, 2$, the result on the double partial superharmonic characterisation of the value functions of player one and player two with the underlying process $X$ introduced above become applicable (cf. [1]). It is well known that superharmonic/subharmonic functions of $X$ are equivalent to concave/convex functions and that continuity in the fine topology is equivalent to continuity in the familiar Euclidean topology. Thus in this setting, the double partial superharmonic characterisation of the value functions can be explained rigorously as finding two continuous functions $u$ and $v$ such that

$$u = \inf_{F \in \text{Sup}_{\alpha}^1(G_1, K_1)} F \text{ and } v = \inf_{F \in \text{Sup}_{\alpha}^2(G_2, K_2)} F \tag{2.3}$$

where

$$\text{Sup}_{\alpha}^1(G_1, K_1) = \{ F : [0, 1] \to [G_1, K_1] : F \text{ is continuous}, F = H_1 \text{ in } D_2, F \text{ is concave in } D_2^2 \}$$

and

$$\text{Sup}_{\alpha}^2(G_2, K_2) = \{ F : [0, 1] \to [G_2, K_2] : F \text{ is continuous}, F = H_2 \text{ in } D_1, F \text{ is concave in } D_1^2 \}$$

with $D_1 = \{ u = G_1 \}, D_2 = \{ v = G_2 \}$ and $K_i$, for $i = 1, 2$, is the smallest concave function majorizing $H_i$. Indeed if the boundaries $\partial D_1$ and $\partial D_2$ of $D_1$ and $D_2$ are regular for their respective sets then the functions $u$ and $v$ solve the optimal stopping game, that is

$$u(x) = V_{\sigma_*}^1(x) = \sup_{\tau} \mathcal{M}^1_2(\tau, \sigma_*) \text{ and } v(x) = V_{\tau_*}^2(x) = \sup_{\sigma} \mathcal{M}^2_2(\tau_*, \sigma) \tag{2.4}$$
where \( \tau_* = \inf\{ t \geq 0 : X_t \in D_1 \} \) and \( \sigma_* = \inf\{ t \geq 0 : X_t \in D_2 \} \).

We initiate this study by showing that if \( D_1 \) is of the form \([m, n] \cup \{0, 1\}\) and \( D_2 \) of the form \([r, l] \cup \{0, 1\}\) for some points \( 0 \leq m \leq n \leq 1 \) and \( 0 \leq r \leq l \leq 1 \) then the functions \( u \) and \( v \) are contained in the sets \( \text{Sup}^1_v(G_1, K_1) \) and \( \text{Sup}^2_v(G_2, K_2) \) respectively. We will prove this claim for \( u \) as the result for \( v \) follows by symmetry. Clearly we have that \( u(x) = H_1(x) \) for all \( x \in D_2 \) and that \( u \) is bounded above by \( K_1 \). By definition of the infimum we also have that \( u(x) \geq G_1(x) \) for all \( x \in [0, 1] \). Since the infimum of concave functions is concave it follows that \( u \) is concave in \( D_2 \) and so \( u \) is continuous in \( \text{int}(D_2^c) \) the interior of \( D_2^c \) (recall that concave functions defined on open sets are continuous). Continuity of \( u \) in \( D_2 \) follows from the continuity of \( H_1 \). So it remains to show that \( u \) is continuous at the boundary of \( D_2 \).
To prove this we shall follow the line of thought of Ekström and Villeneuve in [7]. Without loss of generality we prove that \( u \) is lower semi-continuous at \( l \). Upper semi-continuity of \( u \) holds from the fact that \( u \) is the infimum of continuous functions. So suppose for contradiction that \( u \) is not right-lower-semicontinuous at \( l \) (note that \( u \) is left continuous at \( l \) by continuity of \( H_1 \)). This means that there exists \( \bar{\epsilon} > 0 \) such that \( \lim_{x \downarrow l} u(x) < u(l) - \bar{\epsilon} \). For given \( \delta > 0 \), let \( L \) be the line segment joining the points \( (l, u(l) - \bar{\epsilon}) \) and \( (l + \delta, u(l + \delta)) \). By continuity of \( L \) it follows that there exists \( y \in (l, l + \delta) \) such that \( L(y) > u(y) \). By definition of \( u \) this means that there exists \( F \in \text{Sup}^1_v(G_1, K_1) \) such that \( F(y) < L(y) \). Since \( F \) is continuous in \([0, 1]\) and concave in \((l, l + \delta)\) we have that

\[
F(l) \left( \frac{l + \delta - y}{\delta} \right) + L(l + \delta) \left( \frac{y - l}{\delta} \right) = F(l) \left( \frac{l + \delta - y}{\delta} \right) + u(l + \delta) \left( \frac{y - l}{\delta} \right)
\]

\[
\leq F(l) \left( \frac{l + \delta - y}{\delta} \right) + F(l + \delta) \left( \frac{y - l}{\delta} \right)
\]

\[
\leq F(y) < L(y) = (u(l) - \bar{\epsilon}) \left( \frac{l + \delta - y}{\delta} \right) + L(l + \delta) \left( \frac{y - l}{\delta} \right)
\]

This implies that \( F(l) < u(l) - \bar{\epsilon} \), which contradicts the fact that \( F \geq u \). Thus \( u \) is right-lower-semi-continuous at \( l \).

3. Free-boundary problem

In this section we shall formulate a free-boundary problem by making use of the double partial superharmonic characterisation of the value functions. For this we will assume that there exist thresholds \( a, b \) with \( 0 \leq a < b \leq 1 \), such that

\[
(3.1) \quad G_2^a(x) < 0 \text{ for } x \in [0, a)
\]

\[
(3.2) \quad G_1^a(x) = 0 \text{ for } x = a
\]

\[
(3.3) \quad G_1^a(x) > 0 \text{ for } x \in (a, 1]
\]

and

\[
(3.4) \quad G_2^b(x) > 0 \text{ for } x \in [0, b)
\]

\[
(3.5) \quad G_2^b(x) = 0 \text{ for } x = b
\]
In this setting the double partial superharmonic characterisation of the value functions can be explained geometrically as follows: Suppose that two ropes are pulled above two obstacles $G_1$ and $G_2$ with their endpoints pulled to the ground. Let $D'_1$ be the region where the first rope touches the first obstacle and let $D'_2$ be the region where the second rope touches the second obstacle. Then on $D'_2$ the first rope is constrained to pass through a certain region (this region corresponds to the value of $H_1$ on $D'_2$) and so creates a (new) contact region with its obstacle $G_1$, say $D''_1$. Similarly, on $D'_1$ the second rope is also constrained to pass through a certain region (as specified by the value of $H_2$ on $D'_1$) and thus creates a (new) contact region with its obstacle $G_2$, say $D''_2$. All points of contact are then altered until both ropes touch their respective obstacles smoothly (it may also happen that the new regions coincide with the boundary points 0 and 1 in which case smoothness will break down). However, this must be done in such a way that the point of contact of the first rope with its obstacle $G_1$ must coincide with the point of contact of the second rope with $H_2$ and vice versa. With this intuitive explanation, we will search for a saddle point $(\tau_*, \sigma_*)$ of optimal stopping times of the form

\[
\begin{align*}
\tau_* &= \inf\{t \geq 0 : X_t \leq A_*\} \wedge \rho_{0,1} \\
\sigma_* &= \inf\{t \geq 0 : X_t \geq B_*\} \wedge \rho_{0,1}
\end{align*}
\]

where $0 \leq A_* < B_* \leq 1$ are optimal stopping boundaries that need to be determined and $\rho_{0,1} = \inf\{t \geq 0 : X_t \in \{0,1\}\}$. Prior to formulating the free-boundary problem we note that if there exists such optimal stopping boundaries then we must have that $A_* < a$ and $B_* \geq b$. This is a consequence of the double partial superharmonic characterisation of the value functions (which requires the value function of player one to be concave in $(0, B_*)$ and that of player two to be concave in $(A_*, 1)$).

We are now in a position to formulate the free-boundary problem for unknown points $0 \leq A_* \leq a < b \leq B_* \leq 1$ and unknown functions $u, v : [0, 1] \rightarrow \mathbb{R}$:

\[
\begin{align*}
u''(x) &= 0 \text{ and } v''(x) = 0 \text{ for } x \in (A_*, B_*) \\
u(A_*) &= G_1(A_*) \text{ and } v(B_*) = G_2(B_*) \\
u(B_*) &= H_1(B_*) \text{ and } v(A_*) = H_2(A_*) \\
u(x) &= G_1(x) \text{ for } x \in [0, A_*) \text{ and } v(x) = G_2(x) \text{ for } x \in (B_*, 1] \\
u(x) > G_1(x) \text{ and } v(x) > G_2(x) \text{ for } x \in (A_*, B_*) \\
u(x) &= H_1(x) \text{ for } x \in (B_*, 1] \\
v(x) &= H_2(x) \text{ for } x \in [0, A_*)
\end{align*}
\]

By means of straightforward calculations one can show that the solution of system (3.9)-(3.11) takes the form

\[
\begin{align*}
u(x) &= \frac{H_1(B_*) - G_1(A_*)}{B_* - A_*} x + G_1(A_*) - \frac{(H_1(B_*) - G_1(A_*))A_*}{B_* - A_*} \\
v(x) &= \frac{H_2(A_*) - G_2(B_*)}{A_* - B_*} x + G_2(B_*) - \frac{(H_2(A_*) - G_2(B_*))B_*}{A_* - B_*}
\end{align*}
\]
for all \( x \in (A_*, B_*) \). In certain cases (which shall be specified below) the double smooth fit principle (cf. [1]) will also be satisfied, that is

\[
(3.18) \quad u'(A_*) = G'_1(A_*) \text{ and } v'(B_*) = G'_1(B_*).
\]

If (3.16)-(3.17) and (3.18) hold we get that the optimal boundary points \( A_* \) and \( B_* \) must solve the system of non-linear equations

\[
(3.19) \quad G'_1(A_*)(B_* - A_*) - H_1(B_*) + G_1(A_*) = 0
\]

\[
(3.20) \quad G'_2(B_*)(A_* - B_*) - H_2(A_*) + G_2(B_*) = 0.
\]

For given \( A, B \in [0, 1] \) we shall denote the right hand side expressions in (3.19) and (3.20) by \( \Theta(A, B) \) and \( \Gamma(A, B) \) respectively. Note that since \( G_i \) and \( H_i \) for \( i = 1, 2 \) are \( C^2 \) functions, \( \Theta \) and \( \Gamma \) are continuous functions on \( [0, 1] \times [0, 1] \).

We next study existence of points \( 0 \leq A_* < a < b < B_* \leq 1 \), and corresponding functions \( u \) and \( v \) which solve the free-boundary problem (3.9)-(3.15). For the functions \( u \) and \( v \) to coincide with the value functions of the game (as shall be seen in the verification theorem in Section 4) we further need to study solutions to the system of equations (3.19)-(3.20). For the latter we shall make use of the following result from convex analysis. The proof of this result can be found in [2].

**Proposition 3.1** Let \( \mathcal{U} \) be a non-empty convex subset of \( \mathbb{R}^n \) and \( f : \mathcal{U} \to \mathbb{R} \) a differentiable strictly convex (resp. strictly concave) function. Then

\[
(3.21) \quad f(x) > (\text{ resp. } <) f(\bar{x}) + \nabla f(\bar{x})^T(x - \bar{x})
\]

for each \( \bar{x} \in \text{int}(\mathcal{U}) \) and for all \( x \in \mathcal{U} \) with \( x \neq \bar{x} \), where \( \nabla f(\bar{x}) \) is the vector of partial derivatives of \( f \) at \( \bar{x} \).

We will also need the following preliminary result.

**Lemma 3.2**

i. Let \( B \in [b, 1] \) be given and fixed. Then \( \Theta(A, B) < 0 \) for all \( A \in [a, 1] \) such that \( A \neq B \). Similarly if \( A \in [0, a] \) is given and fixed, then \( \Gamma(A, B) < 0 \) for all \( B \in [0, b] \) such that \( B \neq A \).

ii. Let \( B \in [b, 1] \) be given and fixed. If there exists \( A_*^{g, B} \in (0, a) \) such that \( \Theta(A_*^{g, B}, B) = 0 \) then \( A_*^{g, B} \) is unique. Similarly, suppose that \( A \in [0, a] \) is given and fixed. If there exists \( B_*^{g, A} \in (b, 1] \) such that \( \Gamma(A, B_*^{g, A}) = 0 \) then \( B_*^{g, A} \) is unique.

iii. Suppose that for each \( B \in [b_1, b_2] \), where \( b \leq b_1 < b_2 \leq 1 \), there exists a unique \( A_*^{g, B} \in (0, a) \) such that \( \Theta(A_*^{g, B}, B) = 0 \). Then there exists a unique continuously differentiable function \( \phi : [b_1, b_2] \to [0, a] \) such that \( \Theta(\phi(B), B) = 0 \) for all \( B \in [b_1, b_2] \). Similarly, suppose that for each \( A \in [a_1, a_2] \), where \( 0 \leq a_1 < a_2 \leq a \), there exists a unique \( B_*^{g, A} \in (b, 1] \) such that \( \Gamma(B_*^{g, A}, A) = 0 \). Then there exists a unique continuously differentiable function \( \psi : [a_1, a_2] \to (b, 1] \) such that \( \Gamma(A, \psi(A)) = 0 \) for all \( A \in [a_1, a_2] \).
Proof.

i. We only prove that \( \Theta(A, B) < 0 \) for all \( A \in [a, 1] \) as for \( \Gamma \) the result follows by
symmetry. Consider first the case when \( A \in (a, 1) \). Since \( G''_1 \) is strictly convex in \((a, 1]\)
we see from Proposition 3.1 that \( G_1(B) > G_1(A) + G'_1(A)(B - A) \). From this we get that

\[
\Theta(A, B) = G'_1(A)(B - A) - H_1(B) + G_1(A) \\
\leq G'_1(A)(B - A) - G_1(B) + G_1(A) \\
< G'_1(A)(B - A) - G_1(A) - G'_1(A)(B - A) + G_1(A) \\
= 0
\]

(3.22)

for \( A \neq B \), where the first inequality follows from the fact that \( G_1 \leq H_1 \). Now suppose
that \( A = a \). From (3.22) we have that \( \Theta(a + \varepsilon, B) < 0 \) for any \( \varepsilon > 0 \) sufficiently small.
Since \( \Theta \) is continuous on \([0, 1] \times [0, 1]\) and the mapping \( A \mapsto \Theta(A, B) \) is increasing in
\((a, a + \varepsilon)\) it follows that \( \Theta(a, B) < 0 \). When \( A = 1 \) we get, from Taylor’s theorem with
the mean value form of the remainder, that \( G_1(B) = G_1(1) + G'_1(1)(B - 1) + \frac{G''_1(\xi)}{2}(B - 1)^2 \)
where \( \xi \in (B, 1) \). Since \( G''_1(\xi) > 0 \) it follows that \( G_1(B) > G_1(1) + G'_1(1)(B - 1) \).
From this we can deduce, in a similar way to (3.22), that \( \Theta(1, B) < 0 \) and this proves
the required result.

ii. This follows from the fact that for each \( B \in [b, 1] \) the mapping \( A \mapsto \Theta(A, B) \) is strictly
decreasing in \([0, a]\) whereas for each \( A \in [0, a] \) the mapping \( B \mapsto \Gamma(A, B) \) is strictly
increasing in \((b, 1]\).

iii. The existence and uniqueness of \( \phi \) and \( \psi \) follows from the Implicit Function theorem
(note that for a given \( B \in [b_1, b_2] \), \( \Theta_A(A, B) \neq 0 \) for all \( A \in [0, a] \) and similarly for a
given \( A \in [a_1, a_2] \), \( \Gamma_B(A, B) \neq 0 \) for all \( B \in (b, 1] \)).

We are now in a position to determine the points \( A_* \) and \( B_* \) in the free-boundary problem
(3.9)-(3.15) by studying the system of equations (3.19) - (3.20). For this we shall consider only
the case when there exists a unique (cf. Lemma 3.2 (ii)) \( A_{\Theta}^{G,1} \in [0, a] \) such that \( \Theta(A_{\Theta}^{G,1}, 1) = 0 \)
and a \( B_{\Theta}^{G,0} \in [b, 1] \) such that \( \Theta(0, B_{\Theta}^{G,0}) = 0 \). All other cases can be dealt with in the same
way. In particular, one of the following cases will happen: (i.) either (3.19)-(3.20) has a solution in \([0, a] \times (b, 1] \), or (ii.) \( \Theta(A_*, B_*) = 0 \) and \( \Gamma(A_*, B_*) < 0 \) where \( A_* \in [0, a] \) and
\( B_* = 1 \), or (iii.) \( \Theta(A_*, B_*) < 0 \) and \( \Gamma(A_*, B_*) = 1 \) with \( A_* = 0 \) and \( B_* \in (b, 1] \), or (iv.)
\( \Theta(A_*, B_*) < 0 \) and \( \Gamma(A_*, B_*) < 0 \) with \( A_* = 0 \) and \( B_* = 1 \). In Section 4. we shall see
that whenever \( \Theta(A_*, B_*) < 0 \) and/or \( \Gamma(A_*, B_*) < 0 \), condition (3.13) in the free-boundary
problem will always be satisfied. To this end let us introduce the following notation:

(1.) If there exists at least one \( A_{\Gamma}^{G,1} \in [0, a] \) such that \( \Gamma(A_{\Gamma}^{G,1}, 1) = 0 \) we will set

\[
(3.23) \quad \delta_{\min}^{\Gamma,1} = \min\{A_{\Gamma}^{G,1} : \Gamma(A_{\Gamma}^{G,1}, 1) = 0\}.
\]
Moreover we will assign

\[(3.24) \quad \bar{a}^{\Gamma,1} = \max\{A_s^{b,1} : A_s^{b,1} \leq A_s^{a,1}\} \text{ and } \bar{a}^{\Gamma,1} = \min\{A_s^{b,1} : A_s^{b,1} \geq A_s^{a,1}\}\]

whenever the sets \{A_s^{b,1} : A_s^{b,1} \leq A_s^{a,1}\} and \{A_s^{b,1} : A_s^{b,1} \geq A_s^{a,1}\} are nonempty. If on the other hand \{A_s^{b,1} : A_s^{b,1} \leq A_s^{a,1}\} = \emptyset we will assign \(\bar{a}^{\Gamma,1} = 0\) whereas if \{A_s^{b,1} : A_s^{b,1} \geq A_s^{a,1}\} = \emptyset we shall set \(\bar{a}^{\Gamma,1} = a\).

(II.) We shall assign

\[(3.25) \quad \tilde{b}_{\max}^{\Theta,0} = \max\{B_s^{\Theta,0} : \Theta(0, B_s^{\Theta,0}) = 0\}.\]

If in addition, there exists a unique \(B_s^{\Gamma,0} \in (b, 1]\) such that \(\Gamma(0, B_s^{\Gamma,0}) = 0\) we set

\[(3.26) \quad \tilde{b}^{\Theta,0} = \max\{B_s^{\Theta,0} : B_s^{\Theta,0} \leq B_s^{\Gamma,0}\} \text{ and } \hat{b}^{\Theta,0} = \min\{B_s^{\Theta,0} : B_s^{\Theta,0} \geq B_s^{\Gamma,0}\}\]

whenever the sets \{\(B_s^{\Theta,0} : B_s^{\Theta,0} \leq B_s^{\Gamma,0}\)\} and \{\(B_s^{\Theta,0} : B_s^{\Theta,0} \geq B_s^{\Gamma,0}\)\} are nonempty. In the case when \{\(B_s^{\Theta,0} : B_s^{\Theta,0} \leq B_s^{\Gamma,0}\)\} = \emptyset we assign \(\hat{b}^{\Theta,0} = b\) whereas if \{\(B_s^{\Theta,0} : B_s^{\Theta,0} \geq B_s^{\Gamma,0}\)\} = \emptyset we set \(\tilde{b}^{\Theta,0} = 1\).

We now explain how the points \(A_\ast\) and \(B_\ast\) are constructed by first considering the case when \(\Gamma(0, B_s^{\Gamma,0}) = 0\) for some unique \(B_s^{\Gamma,0} \in (b, 1]\) and by then considering the case when such \(B_s^{\Gamma,0}\) does not exist.

(1\(^\circ\)). Assume that there exists a unique (cf. Lemma 3.2 (ii)) \(B_s^{\Gamma,0} \in (b, 1]\) satisfying \(\Gamma(0, B_s^{\Gamma,0}) = 0\), so that \(\Gamma(0, B_s) \geq 0\) for all \(B_s \in (B_s^{\Gamma,0}, 1]\) (cf. Lemma 3.2 (i)). Let \(\tilde{b}^{\Theta,0}\) and \(\hat{b}^{\Theta,0}\) be defined as in point (II.) above and consider first the case \(B_s^{\Gamma,0} = \tilde{b}^{\Theta,0}\). It is clear that \(A_\ast = 0\) and \(B_\ast = B_s^{\Gamma,0}\) solve the system of equations (3.19) - (3.20). If \(\Theta(0, B_s^{\Gamma,0}) < 0\) we shall set \(A_\ast = 0\) and \(B_\ast = B_s^{\Gamma,0}\) in the free-boundary problem (3.9)-(3.15). Now suppose that \(\Theta(0, B_s^{\Gamma,0}) > 0\). Since \(B_s^{\Gamma,0} \in (\tilde{b}^{\Theta,0}, \hat{b}^{\Theta,0})\) we have, by definition of \(\tilde{b}^{\Theta,0}\) and \(\hat{b}^{\Theta,0}\), that \(\Theta(0, B_s^{\Gamma,0}) > 0\) for all \(B_s \in (\tilde{b}^{\Theta,0}, \hat{b}^{\Theta,0})\). Moreover, from Lemma 3.2 (iii) we see that there exists a unique continuously differentiable function \(\phi : [\tilde{b}^{\Theta,0}, \hat{b}^{\Theta,0}] \rightarrow [0, a]\) such that \(\Theta(\phi(B), B) = 0\) for all \(B \in [\tilde{b}^{\Theta,0}, \hat{b}^{\Theta,0}]\).

i. Suppose that \(\Gamma(A, 1) > 0\) for all \(A \in [0, a]\). Again from Lemma 3.2 we have the existence of a unique continuously differentiable function \(\psi : [a, 0] \rightarrow (b, 1]\) such that \(\Gamma(A, \psi(A)) = 0\) for all \(A \in [a, 0]\). Since \(B_s^{\Gamma,0} \in (\tilde{b}^{\Theta,0}, \hat{b}^{\Theta,0})\) it follows that the curves \(B \mapsto \Theta(\phi(B), B)\) and \(A \mapsto \Gamma(A, \psi(A))\) must intersect in \([0, a] \times [\tilde{b}^{\Theta,0}, \hat{b}^{\Theta,0}]\). From this we conclude that there exists \(A_\ast, B_\ast \in [0, a] \times [\tilde{b}^{\Theta,0}, \hat{b}^{\Theta,0}]\) solving (3.19) - (3.20).

ii. Suppose that there exists at least one \(A_s^{\Gamma,1} \in [0, a]\) such that \(\Gamma(A_s^{\Gamma,1}, 1) = 0\). Let \(a_{\min}^{\Gamma,1}\) be defined as in (3.23). Since \(B_s^{\Gamma,0} \in (\tilde{b}^{\Theta,0}, \hat{b}^{\Theta,0})\) we have that \(a_{\min}^{\Gamma,1} > 0\) and so \(\Gamma(A, 1) > 0\) for all \(A \in [0, a_{\min}^{\Gamma,1}]\). Again by using Lemma 3.2 we see that there exists a unique continuously differentiable function \(\psi : [0, a_{\min}^{\Gamma,1}] \rightarrow (b, 1]\) such that \(\Gamma(A, \psi(A)) = 0\) for all \(A \in [0, a_{\min}^{\Gamma,1}]\). If either \(\tilde{b}^{\Theta,0} < 1\), or \(\hat{b}^{\Theta,0} = 1\) and \(\Theta(0, \tilde{b}^{\Theta,0}) = 0\), the curves \(A \mapsto \Gamma(A, \psi(A))\) and \(B \mapsto \Theta(\phi(B), B)\) intersect in \([0, a_{\min}^{\Gamma,1}] \times [\tilde{b}^{\Theta,0}, \hat{b}^{\Theta,0}]\)
and hence we conclude that there exists \((A_*, B_*) \in [0, a_{\text{min}}^{\Gamma,1}] \times [\hat{b}^{\Theta,0}, \hat{b}^{\Theta,0}]\) such that 
\[\Theta(A_*, B_*) = \Gamma(A_*, B_*) = 0\]. Now suppose that \(\hat{b}^{\Theta,0} = 1\) and \(\Theta(0, \hat{b}^{\Theta,0}) > 0\) (note that \(\Theta(0, \hat{b}^{\Theta,0})\) cannot be negative under the assumption that \(\Theta(A_*, B_*) = 0\)). If \(a_{\text{min}}^{\Gamma,1} \geq A_\Theta^{\ast,1}\) then the curves \(A \mapsto \Gamma(A, \psi(A))\) and \(B \mapsto \Theta(\phi(B), B)\) intersect in \([0, a_{\text{min}}^{\Gamma,1}] \times [\hat{b}^{\Theta,0}, \hat{b}^{\Theta,0}]\) and so we conclude that there exists \((A_*, B_*) \in [0, a_{\text{min}}^{\Gamma,1}] \times [\hat{b}^{\Theta,0}, \hat{b}^{\Theta,0}]\) solving (3.19) - (3.20). If \(a_{\text{min}}^{\Gamma,1} < A_\Theta^{\ast,1}\) we shall consider three cases. In the case \(\Gamma(A_\Theta^{\ast,1}, 1) < 0\) we shall set \(A_* = A_\Theta^{\ast,1}\) and \(B_* = 1\) in the free-boundary problem (3.9)-(3.15). If \(\Gamma(A_\Theta^{\ast,1}, 1) = 0\) then the pair \((A_\Theta^{\ast,1}, B_*)\) with \(B_* = 1\) solves (3.19) - (3.20). Finally suppose that \(\Gamma(A_\Theta^{\ast,1}, 1) > 0\) and let \(a^{\Gamma,1}\) and \(b^{\Gamma,1}\) be defined as in point (II) above. Then we see (cf. Lemma 3.2), that there exists a unique continuously differentiable function \(\psi : [\hat{a}^{\Gamma,1}, \hat{a}^{\Gamma,1}] \to (b, 1]\) such that \(\Gamma(A, \psi(A)) = 0\) for all \(A \in [a^{\Gamma,1}, a^{\Gamma,1}]\). Since \(A_\Theta^{\ast,1} \in (a^{\Gamma,1}, a^{\Gamma,1}]\) it follows that the curves \(B \mapsto \Theta(\phi(B), B)\) and \(A \mapsto \Gamma(A, \psi(A))\) intersect in \([a^{\Gamma,1}, a^{\Gamma,1}] \times [\hat{b}^{\Theta,0}, \hat{b}^{\Theta,0}]\) and hence we conclude that there exists \((A_*, B_*) \in [a^{\Gamma,1}, a^{\Gamma,1}] \times [\hat{b}^{\Theta,0}, \hat{b}^{\Theta,0}]\) such that \(\Theta(A_*, B_*) = \Gamma(A_*, B_*) = 0\).

It remains to consider the case \(B_*^{\Gamma,0} = \hat{b}^{\Theta,0}\). If \(\hat{b}^{\Theta,0} < 1\), or \(\hat{b}^{\Theta,0} = 1\) and \(\Theta(0, \hat{b}^{\Theta,0}) = 0\) we can set \(A_* = 0\) and \(B_* = B_*^{\Gamma,0}\) in the free-boundary problem (3.9) - (3.15) and again we have that the system of equations (3.19) - (3.20) are satisfied at \(A_*\) and \(B_*\). If \(\hat{b}^{\Theta,0} = 1\) and \(\Theta(0, \hat{b}^{\Theta,0}) > 0\) then upon using Lemma 3.2 again we get the existence and uniqueness of a continuously differentiable function \(\phi : [\hat{b}^{\Theta,0}, 1] \to [0, a]\) such that \(\Theta(\phi(B), B) = 0\) for all \(B \in [\hat{b}^{\Theta,0}, 1]\). Similar to part 1° (ii) above we shall consider three different cases. If \(\Gamma(A_\Theta^{\ast,1}, 1) < 0\) we set \(A_* = A_\Theta^{\ast,1}\) and \(B_* = 1\). If on the other hand \(\Gamma(A_\Theta^{\ast,1}, 1) = 0\) we have that (3.19)-(3.20) are satisfied at \(A_* = A_\Theta^{\ast,1}\) and \(B_* = 1\). Finally suppose that \(\Gamma(A_\Theta^{\ast,1}, 1) > 0\). Again we see that there exists a unique continuously differentiable function \(\psi : [\hat{a}^{\Gamma,1}, \hat{a}^{\Gamma,1}] \to (b, 1]\) such that \(\Gamma(A, \psi(A)) = 0\) for all \(A \in [\hat{a}^{\Gamma,1}, \hat{a}^{\Gamma,1}]\). Moreover it is easily seen that \(\phi\) and \(\psi\) intersect in \([\hat{b}^{\Theta,0}, 1] \times [\hat{a}^{\Gamma,1}, \hat{a}^{\Gamma,1}]\) and so it follows that there exists \((A_*, B_*) \in [\hat{b}^{\Theta,0}, 1] \times [\hat{a}^{\Gamma,1}, \hat{a}^{\Gamma,1}]\) satisfying (3.19)-(3.20).

(2°). Let us now assume that \(\Gamma(0, B) < 0\) for all \(B \in [b, 1]\). If there exists no \(a_{\text{min}}^{\Gamma,1}\) such that \(\Gamma(A_{\Theta}^{\ast,1}, 1) = 0\) then \(\Gamma(A, B) < 0\) in \([0, a] \times [b, 1]\) (cf. Lemma 3.2 (i)). In this case we set \(A_* = A_\Theta^{\ast,1}\) and \(B_* = 1\) in the free boundary problem. Suppose on the other hand that there exists such \(a_{\text{min}}^{\Gamma,1}\). If \(\Gamma(A_{\Theta}^{\ast,1}, 1) \leq 0\) then we shall set \(A_* = A_\Theta^{\ast,1}\) and \(B_* = 1\). If on the otherhand \(\Gamma(A_{\Theta}^{\ast,1}, 1) > 0\) then there exists a unique continuously differentiable function \(\psi : [\hat{a}^{\Gamma,1}, \hat{a}^{\Gamma,1}] \to (b, 1]\) such that \(\Gamma(A, \psi(A)) = 0\) for all \(A \in [\hat{a}^{\Gamma,1}, \hat{a}^{\Gamma,1}]\). Let us define \(b_*^{\Theta,0}\) as in (3.25). Again by using Lemma 3.2 we see that there exists a unique continuously differentiable function \(\phi : [\hat{b}^{\Theta,0}, 1] \to [0, a]\) such that \(\Theta(\phi(B), B) = 0\) for all \(B \in [b_*^{\Theta,0}, 1]\). The fact that \(A_\Theta^{\ast,1} \in (a_{\text{min}}^{\Gamma,1}, \hat{a}^{\Gamma,1})\) implies that the curves \(B \mapsto \Theta(\phi(B), B)\) and \(A \mapsto \Gamma(A, \psi(A))\) intersect in \([\hat{a}^{\Gamma,1}, \hat{a}^{\Gamma,1}] \times [b_*^{\Theta,0}, 1]\) and hence we conclude that there exists \((A_*, B_*) \in [\hat{a}^{\Gamma,1}, \hat{a}^{\Gamma,1}] \times [b_*^{\Theta,0}, 1]\) solving (3.19) - (3.20).

3.1. Uniqueness of solution to the free-boundary problem

Having proved that there exists a solution to the free-boundary problem we now consider some special cases in which the solution to the system of equations (3.19)-(3.20) is unique. For
simplicity we shall only consider the case when there exist unique continuously differentiable functions \( \phi : [b, 1] \to [0, a] \) and \( \psi : [0, a] \to (b, 1] \) satisfying \( \Theta(\phi(B), B) = 0 \) for all \( B \in [b, 1] \) and \( \Gamma(A, \psi(A)) = 0 \) for all \( A \in [0, a] \), and such that the mappings \( B \mapsto \Theta(\phi(B), B) \) and \( A \mapsto \Gamma(A, \psi(A)) \) intersect in \([0, a] \times (b, 1] \). Let \( \mathcal{A}_\Theta \) denote the range of the function \( \phi \). By continuity of \( \phi \) it follows that \( \mathcal{A}_\Theta \) is a closed interval in \([0, a] \). Similarly if we set \( \mathcal{A}_\Gamma \) to be the range of the function \( \psi \), then by continuity of \( \psi \) it follows that \( \mathcal{A}_\Gamma \) is a closed interval in \((b, 1] \).

**Proposition 3.3** Suppose that

i. \( H'_1(B) > G'_1(A) \) and \( H'_2(A) > G'_2(B) \) for all \( (A, B) \in \mathcal{A}_\Theta \times \mathcal{A}_\Gamma \).

ii. \( H'_1(B) < G'_1(A) \) and \( H'_2(A) < G'_2(B) \) for all \( (A, B) \in \mathcal{A}_\Theta \times \mathcal{A}_\Gamma \).

Then the solution to (3.19)-(3.20) is unique.

**Proof.** We shall only prove (i.) as for (ii.) the result follows analogously. For this we note that for any \( A \in \mathcal{A}_\Theta \) given and fixed, \( \Theta_B(A, B) < 0 \) for all \( B \in \mathcal{A}_\Gamma \) and so the mapping \( B \mapsto \Theta(A, B) \) is decreasing in \( \mathcal{A}_\Gamma \). Similarly for \( B \in \mathcal{A}_\Gamma \) given and fixed \( \Gamma_A(A, B) < 0 \) for all \( A \in \mathcal{A}_\Theta \) and so the mapping \( A \mapsto \Gamma(A, B) \) is decreasing in \( \mathcal{A}_\Theta \). Suppose, for contradiction, that there exist two pairs \((A^1_1, B^1_1)\) and \((A^2_1, B^2_1)\) in \( \mathcal{A}_\Theta \times \mathcal{A}_\Gamma \), such that \((A^1_1, B^1_1) \neq (A^2_1, B^2_1)\), which solve (3.19)-(3.20). Suppose first that \( A^1_1 < A^2_1 \). If \( B^1_1 \leq B^2_1 \) we have that \( 0 = \Theta(A^1_1, B^1_1) > \Theta(A^2_1, B^1_1) \geq \Theta(A^2_1, B^2_1) = 0 \), where the first inequality follows from the fact that for \( B \in [b, 1] \) the mapping \( A \mapsto \Theta(A, B) \) is decreasing in \([0, a] \) (cf. Lemma 3.2 (i)). So we must have that \( B^1_1 > B^2_1 \) whenever \( A^1_1 < A^2_1 \). But if this is the case we get that \( 0 = \Gamma(A^1_1, B^1_1) > \Gamma(A^2_1, B^1_1) \geq \Gamma(A^2_1, B^2_1) = 0 \). The second inequality follows from the fact that the mapping \( B \mapsto \Gamma(A, B) \) is increasing for any given \( A \in [0, a] \) (cf. Lemma 3.2 (i)). From this it follows that \( A^1_1 \geq A^2_1 \). By symmetry one can see that this case is not possible either and so uniqueness of \( A_\ast \) and \( B_\ast \) follows.

**Proposition 3.4** Suppose that \( H'_1(B) > G'_1(A) \) and \( H'_2(A) < G'_2(B) \) for all \( (A, B) \in \mathcal{A}_\Theta \times \mathcal{A}_\Gamma \). Then if \( G'_1(\ast) \) is increasing in \( \mathcal{A}_\Theta \), \( G'_2(\ast) \) is decreasing in \( \mathcal{A}_\Gamma \), \( H'_1(\ast) \) is concave in \( \mathcal{A}_\Gamma \) and \( H'_2(\ast) \) is concave in \( \mathcal{A}_\Theta \), the system of equations (3.19)-(3.20) is unique.

Prior to proving Proposition 3.4 we need the following simple fact from convex analysis.

**Lemma 3.5** Let \( f, g \) be differentiable functions on some closed interval \([l, m]\). Suppose that there exists \( A \in [l, m] \) such that \( f(A) = g(A) \). If \( f \) is convex, \( g \) is concave and \( f(m) < g(m) \), then there exists no other point \( B \in [l, m] \) such that \( f(B) = g(B) \).

**Proof.** We first show that \( f(B) < g(B) \) for any \( B \in (A, m) \). For this consider the lines \( L_1(x) \) joining the points \((A, g(A))\) and \((m, g(m))\) and \( L_2(x) \) joining the points \((A, f(A))\) and \((m, f(m))\). By concavity of \( g \) and convexity of \( f \) we get that \( g(B) \geq L_1(B) > L_2(B) \geq f(B) \). We next show that \( f(B) > g(B) \) for any \( B \in [l, A] \). For this we note, by convexity of \( f \) and concavity of \( g \) (recall Proposition 3.1), that

\[
(3.27) \quad f'(A) \leq \frac{f(m) - f(A)}{m - A} < \frac{g(m) - g(A)}{m - A} \leq g'(A).
\]
Again by convexity of $f$, concavity of $g$ and Proposition 3.1, we have that

$f(B) \geq f(A) + f'(A)(B - A)$
$= g(A) + f'(A)(B - A)$
$\geq g(B) - g'(A)(B - A) + f'(A)(B - A)$
$= g(B) + (f'(A) - g'(A))(B - A) > g(B)$

for all $B \in [l, A]$, where the last inequality follows from (3.27).

**Proof of Proposition 3.4.** Since the functions $\phi$ and $\psi$ are continuously differentiable we can take the partial derivatives on both sides of the equations $\Theta(\phi(B), B) = 0$ and $\Gamma(A, \psi(A)) = 0$ and rearranging terms to get

\begin{align}
\phi'(B) &= \frac{\Theta_B(\phi(B), B)}{\Theta_A(\phi(B), B)} - \frac{G'_1(\phi(B)) - H'_1(B)}{G''_1(\phi(B))(B - \phi(B))} < 0 \\
\psi'(A) &= -\frac{\Gamma_A(A, \psi(A))}{\Gamma_B(A, \psi(A))} - \frac{G'_2(\psi(A)) - H'_2(A)}{G''_2(\psi(A))(A - \psi(A))} < 0.
\end{align}

The inequalities follow from the concavity properties of $G_1$ and $G_2$ and from the fact that $G'_1(\phi(B)) < H'_1(B)$ and $G'_2(\psi(A)) > H'_2(A)$. From this we conclude that $\phi$ and $\psi$ are decreasing on $A_\Gamma$ and $A_\Theta$ respectively. Take any $B_1, B_2 \in A_\Gamma$ such that $B_1 < B_2$. From the monotonicity property of $\phi$ together with the facts that $G''_1 < 0$ and is monotonic increasing on $A_\Theta$, and that $B_1 - \phi(B_1) < B_2 - \phi(B_2)$ we have that

\begin{equation}
\frac{-1}{G''_1(\phi(B_1))(B_1 - \phi(B_1))} > \frac{-1}{G''_1(\phi(B_2))(B_2 - \phi(B_2))} > 0.
\end{equation}

Using again the concavity property of $G_1$ on $A_\Theta$ and that of $H_1$ on $A_\Gamma$ we get that

\begin{equation}
G'_1(\phi(B_1)) - H'_1(B_1) < G'_1(\phi(B_2)) - H'_1(B_2) < 0
\end{equation}

where the last inequality follows by recalling that $G'_1(A) < H'_1(B)$ for all $(A, B) \in A_\Theta \times A_\Gamma$. Combining (3.30) and (3.31) we see that

\begin{equation}
\phi'(B_1) = \frac{G'_1(\phi(B_1)) - H'_1(B_1)}{G''_1(\phi(B_1))(B_1 - \phi(B_1))} < \frac{G'_1(\phi(B_2)) - H'_1(B_2)}{G''_1(\phi(B_2))(B_2 - \phi(B_2))} = \phi'(B_2)
\end{equation}

from which the strict convexity property of $\phi$ on $A_\Gamma$ follows. Analogously, one can show that $\psi$ is strictly concave on $A_\Gamma$. Since $\phi$ is continuously differentiable and $\phi' < 0$ in $[b, 1]$, it follows that the inverse function $\phi^{-1} : A_\Theta \to [b, 1]$ is a decreasing continuously differentiable function. Moreover, using the fact that the inverse of convex decreasing functions is also convex we deduce that $\phi^{-1}$ is convex on $A_\Theta$. Since $A_\Theta$ is a closed interval, we can use Lemma 3.5 to deduce that the functions $\phi^{-1}$ and $\psi$ intersect only once on $A_\Theta$ and so we can conclude that there exists only one point $(A_s, B_s) \in A_\Theta \times A_\Gamma$ which solve the system of equations (3.19)-(3.20).
4. Verification theorem

We initiate this section by showing that if there exist $0 \leq A_1 < a < b < B_1 \leq 1$ solving (3.19)-(3.20) then the functions $u$ and $v$ in the free-boundary problem (3.9)-(3.15) coincide with the value functions of the nonzero-sum game (2.1)-(2.2).

**Theorem 4.1** Let $X$ be Brownian motion in $[0,1]$, started at $x \in [0,1]$ and absorbed at either 0 or 1. Suppose that $G_i, H_i$, for $i = 1, 2$, are $C^2$ functions on $[0,1]$ such that $G_i \leq H_i$. Assume also that $G_i(0) = H_i(0)$ and $G_i(1) = H_i(1)$. If $G_1, G_2$ satisfy assumptions (3.1)-(3.6) then the functions

\[
(4.1) \quad u(x) = \begin{cases} 
G_1(x) & \text{if } 0 \leq x \leq A_1 \\
u_+(x; A_1, B_1) & \text{if } A_1 < x < B_1 \\
H_1(x) & \text{if } B_1 \leq x \leq 1
\end{cases}
\]

and

\[
(4.2) \quad v(x) = \begin{cases} 
H_2(x) & \text{if } 0 \leq x \leq A_1 \\
v_+(x; A_1, B_1) & \text{if } A_1 < x < B_1 \\
G_2(x) & \text{if } B_1 \leq x \leq 1
\end{cases}
\]

where $u_+(x; A_1, B_1)$ takes the form (3.16) and $v_+(x; A_1, B_1)$ is given by (3.17), coincide with the value functions $V^1_{\tau_1}(x) = \sup_\sigma M^1_1(\tau_1, \sigma)$ and $V^2_{\tau_1}(x) = \sup_\sigma M^2_1(\tau_1, \sigma)$ respectively, where $\tau_1 = \inf\{t \geq 0 : X_t \in [A_1, B_1]\} \wedge \rho_{0,1}$ and $\sigma = \inf\{t \geq 0 : X_t \in [B_1, 1]\} \wedge \rho_{0,1}$ with the optimal stopping boundaries $A_1$ and $B_1$ solving the system of nonlinear equations (3.19)-(3.20).

**Proof.** We first show that $V^1_{\tau_1}(x) \leq u(x)$ for all $x \in [0,1]$. Since $G_1, H_1$ and $u_+$ are $C^1$ functions on $[0,1]$ it follows that $u$ is absolutely continuous on $[0,1]$ and that $u'$ (which exists a.e.) is of bounded variation. But this implies that $u$ can be written as the difference of two convex functions. So we can apply Itô-Tanaka formula (cf. [21]) to $u(X_t)$ to get that

\[
u(X_t) = u(x) + \int_0^t u'_+(X_s) dX_s + \frac{1}{2} \int_0^t \mu^x d\mu(x)
\]

\[
u(X_t) = u(x) + \int_0^t u'_+(X_s) dX_s + \frac{1}{2} \int_0^t \mu^x u''(x) I(x \neq A_1, x \neq B_1) dx
\]

\[
u(X_t) = u(x) + \int_0^t \mu^x I(x = A_1) du'_+(x) + \frac{1}{2} \int_0^t \mu^x I(x = B_1) du'_+(x)
\]

\[
u(X_t) = u(x) + M_t
\]

\[
u(X_t) = u(x) + \frac{1}{2} \int_0^t [G^\mu_1(X_s) I(0 \leq X_s < A_1) + 0 I(A_1 < X_s < B_1) + H^\mu_1(X_s) I(B_1 < X_s \leq 1)] I(X_s \neq A_1, X_s \neq B_1) ds
\]

\[
u(X_t) = u(x) + M_t
\]

\[
u(X_t) = u(x) + \int_0^t [G^\mu_1(X_s) I(0 \leq X_s < A_1) + 0 I(A_1 < X_s < B_1) + H^\mu_1(X_s) I(B_1 < X_s \leq 1)] ds
\]

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\[
\frac{1}{2} l^B_t (H'_1(B_s) - G'_1(A_s))
\]

where \((l^B_t)_{t\geq 0}\) is the local time of \(X\) at the point \(B_s\), defined by \(l^B_t = P_x - \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t I(B_s < X_t < B_s + \varepsilon) ds\) and \((M_t)_{t\geq 0}\) is a local martingale, given by \(\int_0^t u(X_s)dX_s\). The third equality follows from the occupation time space formula (cf. [8]) together with the definition of \(u\) and the fact that \(u\) is smooth at \(A_s\). The last equality follows again from the definition of \(u\).

Since \(\sigma_* = \inf \{ t \geq 0 : X_t \geq B_s \} \) we have that

\[
G_1(X_t) I(t \leq \sigma_* ) + H_1(X_{\sigma_* }) I(\sigma_* < t) \leq u(X_t) I(t \leq \sigma_* ) + H_1(X_{\sigma_* }) I(\sigma_* < t)
\]

\[
= u(X_t) I(t \leq \sigma_* ) + u(X_{\sigma_* }) I(\sigma_* < t)
\]

\[
= u(X_t \land \sigma_* )
\]

\[
\leq u(x) + M_{t \land \sigma_*}
\]

for any given \(t \geq 0\). The first inequality can be seen by noting that since \(G_1\) is concave in \([0,a]\) then the line \(u_* (x; A_s, B_s)\) supports the hypograph of \(G_1\) in \([A_s, a]\) and so \(u \geq G_1\) in \([A_s, a]\). On the other hand since \(u(B_s) \geq G_1(B_s)\) it follows that \(u\) majorises the line joining the points \((a, G_1(a))\) and \((B_s, G_2(B_s))\) in the interval \([a, B_s]\), which in turn, by convexity of \(G_1\) in \([a, B_s]\), majorises \(G_1\) in \([a, B_s]\). The first equality follows from the fact that \(u(X_{\sigma_* }) \in \{0\} \cup [B_s, 1]\) and by definition \(u = H_1\) in \([0, B_s]\). The second inequality follows from (4.3) upon noting that \(G_1'' \leq 0\) in \([0, A_s]\) and that \(l^B_t\) increases only when the process is at \(B_s\).

Now suppose that \((\tau_n)_{n=1}^{\infty}\) is a localizing sequence of stopping times for \(M\). Then from (4.4) we get that

\[
G_1(X_{\tau \land \tau_n}) I(\tau \land \tau_n \leq \sigma_* ) + H_1(X_{\sigma_* }) I(\sigma_* < \tau \land \tau_n) \leq u(x) + M_{\tau \land \tau_n \land \sigma_*}
\]

for every stopping time \(\tau\) of \(X\). Taking the \(P_x\)-expectation we conclude, by the optional sampling theorem, that

\[
\mathbb{E}_x [G_1(X_{\tau \land \tau_n}) I(\tau \land \tau_n \leq \sigma_* ) + H_1(X_{\sigma_* }) I(\sigma_* < \tau \land \tau_n)] \leq u(x)
\]

for all stopping times \(\tau\). Letting \(n \to \infty\) we conclude by Fatou's lemma that

\[
M^1_x (\tau, \sigma_*) \leq u(x)
\]

for all \(\tau\). Taking the supremum over all \(\tau\) it follows that \(V^1_{\sigma_*}(x) \leq u(x)\). It remains to prove that (4.7) holds with equality if \(\tau\) is replaced by \(\tau_*\). Indeed, from (4.3) and the structure of the stopping times \(\tau_*\) and \(\sigma_*\) we get that

\[
u(X_{\tau_* \land \tau_n \land \sigma_*}) = u(x) + M_{\tau_* \land \tau_n \land \sigma_*}.
\]

Taking the \(P_x\)-expectation on both sides of (4.8) and the limit as \(n \to \infty\) we have, by Lebesgue's dominated convergence theorem, that

\[
\lim_{n \to \infty} \mathbb{E}_x u(X_{\tau_* \land \sigma_* \land \tau_n}) = \mathbb{E}_x [\lim_{n \to \infty} u(X_{\tau_* \land \sigma_* \land \tau_n})] = \mathbb{E}_x u(X_{\tau_* \land \sigma_*}) = u(x)
\]

Since \(u(X_{\tau_* \land \sigma_*}) = G_1(X_{\tau_*}) I(\tau_* \leq \sigma_*) + H_1(X_{\sigma_*}) I(\sigma_* < \tau_*)\) we conclude that \(M^1_x (\tau_*, \sigma_*) = u(x)\) and so \(V^1_{\sigma_*}(x) \leq M^1_x (\tau_*, \sigma_*)\). By definition \(V^1_{\sigma_*}(x) \geq M^1_x (\tau_*, \sigma_*)\) and so equality of \(V^1_{\sigma_*}\) and \(u\).
follows. By symmetry one can also show that $V_\alpha^2(x) = v(x)$. ■

We next provide three results to link the solution of the free-boundary problem with the value functions of the game in the case when $A_*$ and $B_*$ do not solve the system of equations (3.19)-(3.20). The proofs can be carried out using similar arguments to the proof of Theorem 4.1 and therefore shall be omitted.

**Proposition 4.2** Consider the assumptions given in Theorem 4.1. Let

$$u(x) = \begin{cases} G_1(x) & \text{if } 0 \leq x \leq A_* \\ u_*(x; A_*, 1) & \text{if } A_* < x \leq 1 \end{cases}$$

and

$$v(x) = \begin{cases} H_2(x) & \text{if } 0 \leq x \leq A_* \\ v_*(x; A_*, 1) & \text{if } A_* < x \leq 1 \end{cases}$$

where $u_*(x; A_*, 1)$ takes the form (3.16) and $v_*(x; A_*, 1)$ is given by (3.17), with $B_*$ = 1 and $A_*$ being the solution of the nonlinear equation (3.19). If $\Gamma(A_*, 1) < 0$ then $u$ and $v$ coincide with the value functions $V_\tau^1(x) = \sup_\gamma M_2^1(\tau, \sigma_\gamma)$ and $V_\sigma^2(x) = \sup_\sigma M_2^2(\tau_\sigma, \sigma)$ respectively, where $\tau_* = \inf\{t \geq 0 : X_t \leq A_*\} \land \rho_{0,1}$ and $\sigma_* = \rho_{0,1}$.

**Proposition 4.3** Consider the assumptions given in Theorem 4.1. Let

$$u(x) = \begin{cases} u_*(x; 0, B_*) & \text{if } 0 \leq x < B_* \\ H_1(x) & \text{if } B_* \leq x \leq 1 \end{cases}$$

and

$$v(x) = \begin{cases} v_*(x; 0, B_*) & \text{if } 0 \leq x < B_* \\ G_2(x) & \text{if } B_* \leq x \leq 1 \end{cases}$$

where $u_*(x; 0, B_*)$ takes the form (3.16) and $v_*(x; 0, B_*)$ is given by (3.17) with $A_* = 0$ and $B_*$ being the solution of the nonlinear equation (3.20). If $\Theta(0, B_*) < 0$ then $u$ and $v$ coincide with the value functions $V_\tau^1(x) = \sup_\gamma M_2^1(\tau, \sigma_\gamma)$ and $V_\sigma^2(x) = \sup_\sigma M_2^2(\tau_\sigma, \sigma)$ respectively, where $\tau_* = \rho_{0,1}$ and $\sigma_* = \inf\{t \geq 0 : X_t \geq B\} \land \rho_{0,1}$.

**Proposition 4.4** Consider the assumptions given in Theorem 4.1. Let

$$u(x) = u_*(x; 0, 1)$$

and

$$v(x) = v_*(x; 0, 1)$$

where $u_*(x; 0, 1)$ takes the form (3.16) and $v_*(x; 0, 1)$ is given by (3.17) with $A_* = 0$ and $B_* = 1$. If $\Theta(0, 1) < 0$ and $\Gamma(0, 1) < 0$ then the functions $u$ and $v$ coincide with the value functions $V_\tau^1(x) = \sup_\gamma M_2^1(\tau, \sigma_\gamma)$ and $V_\sigma^2(x) = \sup_\sigma M_2^2(\tau_\sigma, \sigma)$ respectively, where $\tau_* = \sigma_* = \rho_{0,1}$.
5. Counterexample

In this section we shall give a counterexample to show that if assumptions (3.1)-(3.6) are relaxed then the functions \(u\) and \(v\) may not exist. For this consider the payoff functions \(G_i\) and \(H_i\) given in figure 1. Suppose that the pair \((\tau_s, \sigma_s)\) is a Nash equilibrium point so that

\[
V_{\sigma_s}^1(x) = \sup_{\tau} M_x^1(\tau, \sigma_s) = M_x^1(\tau_s, \sigma_s)
\]

\[
V_{\tau_s}^2(x) = \sup_{\sigma} M_x^2(\tau_s, \sigma) = M_x^2(\tau_s, \sigma_s).
\]

Suppose also that \(V_{\sigma_s}^1\) and \(V_{\tau_s}^2\) are continuous in \(x\).

\((1^0)\). Let \(V_1(x) = \sup_{\tau} E_x G_1(X_{\tau})\). From the theory of optimal stopping (cf. [18]) it is known that \(V_1(x) = E_x G_1(X_{\tau_{D_1}})\) where \(\tau_{D_1} = \inf\{t \geq 0 : X_t \in D_1\}\) with \(D_1 = \{V_1 = G_1\}\). We show that \(\tau_s \geq \tau_{D_1}\) \(P_x\)-a.s. for each \(x \in [0,1]\). For this we first prove that \(V_{\sigma_s}^1(x) \geq V_1(x)\). Again from the theory of optimal stopping we know that \(V_1\) coincides with the smallest concave function majorizing \(G_1\) (which is known to be continuous by the continuity of \(G_1\)). Since \(H_1\) is concave and \(H_1 \geq G_1\) we have that

\[
M_x^1(\tau_{D_1}, \sigma_s) = E_x[G_1(X_{\tau_{D_1}}) I(\tau_{D_1} \leq \sigma) + H_1(X_{\sigma_s}) I(\sigma_s < \tau_{D_1})]
\]

\[
\geq E_x[G_1(X_{\tau_{D_1}}) I(\tau_{D_1} \leq \sigma_s) + V_1(X_{\sigma_s}) I(\sigma_s < \tau_{D_1})]
\]

\[
= E_x[V_1(X_{\tau_{D_1}}) I(\tau_{D_1} \leq \sigma_s) + V_1(X_{\sigma_s}) I(\sigma_s < \tau_{D_1})]
\]

\[
= E_x V_1(X_{\tau_{D_1} \wedge \sigma_s})
\]

\[
= E_x E_{X_{\tau_{D_1} \wedge \sigma_s}} V_1(X_{\tau_{D_1}})
\]

\[
= E_x E_x (V_1(X_{\tau_{D_1}}) \circ \theta_{\tau_{D_1} \wedge \sigma_s}) |\mathcal{F}_{\tau_{D_1} \wedge \sigma_s}
\]

\[
= E_x E_x (V_1(X_{\tau_{D_1} \wedge \sigma_s + \theta_{\tau_{D_1} \wedge \sigma_s}}) |\mathcal{F}_{\tau_{D_1} \wedge \sigma_s})
\]

\[
= E_x E_x (V_1(X_{\tau_{D_1}}) |\mathcal{F}_{\tau_{D_1} \wedge \sigma_s})
\]

\[
= E_x V_1(X_{\tau_{D_1}}) = V_1(x)
\]

where \(\theta\) is the shift operator (cf. [18, Chapter II]). The second equality follows from the fact that \(V_1(X_{\tau_{D_1}}) = G_1(X_{\tau_{D_1}})\) (since both \(V_1\) and \(G_1\) are continuous). The fourth and last
equalities follows from the fact that \( V_1(X_{\tau_{D_1}}) = G_1(X_{\tau_{D_1}}) \) and that \( V_1(x) = E_x G_1(X_{\tau_{D_1}}) \). The fifth equality follows from the strong Markov property of \( X \) whereas the seventh equality follows from the fact that \( \tau_{D_1} = \tau_D \wedge \sigma_1 \circ \theta_{\tau_{D_1} \wedge \sigma_1} \) upon noting that \( \tau_{D_1} \wedge \sigma_1 \leq \tau_{D_1} \) \( P_x \)-a.s. Thus we have that

\[
V_{\sigma_1}^1(x) \geq M_{X_{\tau_{D_1}}}^1(\tau_1, \sigma_1) \geq V_1(x). 
\]

To prove the required result we let \( \tau_{D_1} = \inf\{t \geq 0 : X_t \in D_1\} \) where \( D_1 = \{V_{\sigma_1}^1 = G_1\} \) (since we are assuming that \( V_{\sigma_1}^1 \) is continuous we must have that \( D_1 \) is closed). From (5.1) and the Markov property of \( X \) we have

\[
V_{\sigma_1}^1(X_{\tau_{D_1}}) = M_{X_{\tau_{D_1}}}^1(\tau_1, \sigma_1) = E_x[G_1(X_{\tau_{D_1}}) \circ \theta_{\tau_{D_1}} I(\tau_1 \circ \theta_{\tau_{D_1}} \leq \sigma_1 \circ \theta_{\tau_{D_1}})
+ H_1(X_{\sigma_1}) \circ \theta_{\tau_{D_1}} I(\sigma_1 \circ \theta_{\tau_{D_1}} < \tau_1 \circ \theta_{\tau_{D_1}}) | F_{\tau_{D_1}}
] = E_x[G_1(X_{\tau_{D_1}}) | F_{\tau_{D_1}}] = G_1(X_{\tau_{D_1}})
\]

(5.4) For this we used the fact that \( \tau_1 \circ \theta_{\tau_{D_1}} = 0 \). From (5.4) and by definition of \( \tau_{D_1} \) (upon recalling that \( D_1 \) is closed) it follows that \( \tau_1 \geq \tau_{D_1} \) \( P_x \)-a.s. Finally, since \( V_{\sigma_1}^1(x) \geq V_1(x) \) (cf. (5.3)) we see that \( D_1 \subseteq D_1 \) and so \( \tau_{D_1} \geq \tau_{D_1} \) \( P_x \)-a.s. from which the result follows.

(2?) We show that the pair \((\rho_0, 1, \sigma)_1\), with \( \rho_0, 1 = \inf\{t \geq 0 : X_t \in \{0, 1\}\} \), cannot be a Nash equilibrium point. So suppose for contradiction that \((\rho_0, 1, \sigma)_1\) is a Nash equilibrium point. Consider first the optimal stopping problem \( V_2(x) = \sup_\sigma E_x G_2(X_\sigma) \) and let \( \sigma_{D_2} = \inf\{t \geq 0 : X_t \in D_2\} \) where \( D_2 = \{V_2 = G_2\} \). From the theory of optimal stopping we know that \( V_2 \) is the smallest concave function majorizing \( G_2 \) and \( \sigma_{D_2} \) is an optimal stopping time (as for \( V_1 \) one can show that \( V_2 \) is continous by continuity of \( G_2 \)). We first show that if \( \tau_1 = \rho_0, 1 \) then \( \sigma_1 \wedge \rho_0, 1 \geq \sigma_{D_2} \). Since

\[
M_{X_{\tau_{D_1}}}(\rho_0, 1, \sigma) = E_x G_2(X_{\tau_{D_1} \wedge \sigma})
\]

for any stopping time \( \sigma \), in particular for \( \sigma_1 \) we have that

\[
V_{\rho_0, 1}^2(x) = E_x G_2(X_{\sigma_1 \wedge \rho_0, 1}) \geq E_x G_2(X_{\sigma_1 \wedge \rho_0, 1})
\]

(5.5) Taking the supremum over all \( \sigma \) we get that

\[
E_x G_2(X_{\sigma_1 \wedge \rho_0, 1}) \geq \sup_\sigma E_x G_2(X_{\sigma_1 \wedge \rho_0, 1}) \geq E_x G_2(X_{\sigma_{D_2} \wedge \rho_0, 1}) = E_x G_2(X_{\sigma_{D_2}}) = V_2(x).
\]

(5.6) Note that the penultimate inequality follows from the fact that \( V_2(0) = G_2(0) \) and \( V_2(1) = G_2(1) \) and so \( D_2 \supseteq \{0, 1\} \). On the other hand \( V_2(x) \geq E_x G_2(X_{\sigma_1 \wedge \rho_0, 1}) = V_{\rho_0, 1}^2(x) \) by definition of \( V_2 \) and so we conclude that \( E_x G_2(X_{\sigma_1 \wedge \rho_0, 1}) = V_2(x) \). From this it follows that \( \sigma_1 \wedge \rho_0, 1 \) is an optimal stopping time for problem \( V_2(x) = \sup_\sigma E_x G_2(X_\sigma) \). Moreover, by the Markov property of \( X \) we have that

\[
V_2(X_{\sigma_1 \wedge \rho_0, 1}) = E_{X_{\sigma_1 \wedge \rho_0, 1}} G_2(X_{\sigma_1 \wedge \rho_0, 1}) = E_x [G_2(X_{\sigma_1 \wedge \rho_0, 1}) \circ \theta_{\sigma_1 \wedge \rho_0, 1} | F_{\sigma_1 \wedge \rho_0, 1}] = G_2(X_{\sigma_1 \wedge \rho_0, 1})
\]

(5.7)
and so, by continuity of $V_2$ and $G_2$ we get that $\sigma_s \land \rho_{0,1} \geq \sigma_{D_2}$ $P_x$-a.s. for each $x \in [0, 1]$. From this, together with the fact that $H_1$ is concave (that is superharmonic relative to $X$), it follows that

$$(5.8) \quad M^1_x(\rho_{0,1}, \sigma_s) = E_x H_1(X_{\sigma_s \land \rho_{0,1}}) \leq E_x H_1(X_{\sigma_{D_2}}).$$

Now if $(\rho_{0,1}, \sigma_s)$ were to be a Nash equilibrium point then we must have that $M^1_x(\rho_{0,1}, \sigma_s) \geq G_1(x)$ for all $x \in [0, 1]$. But it is clear (cf. Figure 2 below) that there exists $x \in [0, 1]$ for which $M^1_x(\rho_{0,1}, \sigma_s) < E_x H_1(X_{\sigma_{D_2}}) < G_1(x)$ and thus we get a contradiction.

Figure 2: A drawing of the functions $x \mapsto V_2(x)$ and $x \mapsto E_x H_1(X_{\sigma_{D_2}})$
hand, since \( V_{\tau_A^2} \geq V_{\tau_D^2} \), we can conclude that \( \tilde{D}_2 \subseteq D_2^* \) and so \( \sigma_{D_2^*} \leq \sigma_{\tilde{D}_2} \) \( P_x \)-a.s. From this one can see that \( D_1^* \setminus \{0, 1\} = \emptyset \) which contradicts the fact that \( D_1^* \supset \{0, 1\} \).

### 6. Regular Diffusions

We shall now link nonzero-sum games of optimal stopping for one-dimensional regular diffusions with nonzero-sum games of optimal stopping for Brownian motion. In doing so one can then use the results in the previous sections to show that for a certain class of payoff functions, nonzero-sum optimal stopping games for one-dimensional regular diffusions admit a Nash equilibrium point. So let \( X \) be a one-dimensional regular diffusion in \([0, 1]\), absorbed at either 0 or 1 and suppose that \( \alpha \geq 0 \) is a given constant. Let us assume that the fine topology coincides with the Euclidean topology and let \( L_X \) be the infinitesimal generator of \( X \). It is well known that under regularity conditions (cf. example [18]), \( L_X F = \frac{\sigma^2(x)F_{xx}}{2} + \mu(x)F_x \) for \( x \in (0, 1) \) where \( \mu(x) \in \mathbb{R} \) is the drift and \( \sigma^2(x) \) is the diffusion coefficient of \( X \). Moreover, the second order \( L_X F = \alpha F \) admits two linearly independent solutions \( \psi \) and \( \varphi \) such that \( \psi(0), \varphi(1) > 0 \) and that \( \psi \) is increasing and \( \varphi \) is decreasing. These solutions are uniquely determined up to a multiplicative constant. In the case when \( \alpha = 0 \) we can take \( \psi = S \) and \( \varphi \equiv 1 \) where \( S \) is the scale function of \( X \).

Consider the nonzero-sum game of optimal stopping in which player one chooses a stopping time \( \tau_* \) and player two a stopping time \( \sigma_* \) in order to maximize their expected payoffs, which are respectively given by

\[
\begin{align*}
E_x \left[ e^{-\alpha(\tau \wedge \sigma)}(G_1(X_{\tau})I(\tau \leq \sigma) + H_1(X_{\sigma})I(\sigma < \tau)) \right] \\
E_x \left[ e^{-\alpha(\tau \wedge \sigma)}(G_2(X_{\sigma})I(\sigma < \tau) + H_2(X_{\tau})I(\tau \leq \sigma)) \right]
\end{align*}
\]

(6.1)
where \( G_i, H_i : [0, 1] \to \mathbb{R} \), for \( i = 1, 2 \), are continuous functions such that \( G_i \leq H_i \) with \( G_i(0) = H_i(0) \) and \( G_i(1) = H_i(1) \). For a given stopping time \( \sigma \) chosen by player two, let

\[
V_{\sigma}^{1,\alpha}(x) = \sup_{\tau} \mathbb{E}_x \left[ e^{-\alpha(\tau \wedge \sigma)} (G_1(X_{\tau}) I(\tau \leq \sigma) + H_1(X_{\sigma}) I(\sigma < \tau)) \right]
\]

be the value function of player one and for a given stopping time \( \tau \) chosen by player one let

\[
V_{\tau}^{2,\alpha}(x) = \sup_{\sigma} \mathbb{E}_x \left[ e^{-\alpha(\tau \wedge \sigma)} (G_2(X_{\sigma}) I(\sigma < \tau) + H_2(X_{\tau}) I(\tau \leq \sigma)) \right]
\]

be the value function of player two. Suppose that there exist continuous functions \( u, v : [0, 1] \to \mathbb{R} \) such that

\[
u = \inf_{F \in \text{Sup}_u^2(G_2, K_2)} F \]

and

\[
u = \inf_{F \in \text{Sup}_v^1(G_1, K_1)} F
\]

where

\[
\text{Sup}_u^1(G_1, K_1) = \{ F : [0, 1] \to [G_1, K_1] : F \text{ is continuous, } F = H_1 \text{ in } D_2, F \text{ is } \alpha\text{-superharmonic in } D_2^c \}
\]

and

\[
\text{Sup}_u^2(G_2, K_2) = \{ F : [0, 1] \to [G_2, K_2] : F \text{ is continuous, } F = H_2 \text{ in } D_1, F \text{ is } \alpha\text{-superharmonic in } D_1^c \}
\]

with \( K_i \), for \( i = 1, 2 \), being the smallest \( \alpha \)-superharmonic function (relative to \( X \)) majorizing \( H_i \), \( D_1 = \{ u = G_1 \} \) and \( D_2 = \{ v = G_2 \} \) (recall that a measurable function \( F : \mathbb{R} \to \mathbb{R} \) is \( \alpha \)-superharmonic if \( \mathbb{E}_x [e^{-\alpha \tau} F(X_{\tau})] \leq F(x) \) for all stopping times \( \tau \) of \( X \) and all \( x \in [0, 1] \)). From the double partial superharmonic characterisation of the value functions we have that

\[
u(x) = V_{\sigma_{D_2}}^{1,\alpha}(x) \text{ and } v(x) = V_{\tau_{D_1}}^{2,\alpha}(x)
\]

for all \( x \in [0, 1] \), where \( \tau_{D_1} = \inf \{ t \geq 0 : X_t \in D_1 \} \) and \( \sigma_{D_2} = \inf \{ t \geq 0 : X_t \in D_2 \} \).

\((2^c)\). Let \( I : [0, 1] \to \mathbb{R} \) be a strictly increasing continuous function and \( J : [0, 1] \to \mathbb{R} \) a Borel measurable function. \( J \) is said to be \( I \)-concave if

\[
J(x) \geq J(c) \left( \frac{I(d) - I(x)}{I(d) - I(c)} \right) + J(d) \left( \frac{I(x) - I(c)}{I(d) - I(c)} \right)
\]

for \( 0 \leq c < x < d \leq 1 \). It is known (cf. for example [6, Chapter 16] or [17, proof of Theorem 3.2]) that a Borel measurable function \( J \) is \( \alpha \)-superharmonic if and only if \( \frac{J}{\psi} \) is \( I \)-concave or equivalently if and only if \( \frac{J}{\psi} \) is \( \hat{I} \)-concave where \( I \) and \( \hat{I} \) are strictly increasing continuous
functions given by $I = \frac{\psi}{\varphi}$ and $\tilde{I} = -\frac{\varphi}{\psi}$. From this it follows that the collections of functions in (6.6)-(6.7) are equivalent to

$$\text{Sup}_1^\psi(G_1, K_1) = \{F : [0, 1] \to [G_1, K_1] : F \text{ is continuous,}$$

$$F = H_1 \text{ in } D_2, \frac{F}{\varphi} \text{ is } I\text{-concave in } D_2^c\}$$

and

$$\text{Sup}_2^\psi(G_2, K_2) = \{F : [0, 1] \to [G_2, K_2] : F \text{ is continuous,}$$

$$F = H_2 \text{ in } D_1, \frac{F}{\varphi} \text{ is } I\text{-concave in } D_1^c\}$$

where $K_i$, for $i = 1,2$, is the smallest function majorizing $H_i$ such that $\frac{K_i}{\varphi}$ is $I$-concave.

(3°). We show that the sets in (6.9)-(6.10) are equivalent to collections involving ordinary concave functions. For this let $B$ be a Brownian motion in $[I(0), I(1)]$, absorbed at either $I(0)$ or $I(1)$ and consider the non-zero-sum game of optimal stopping in which player one chooses a stopping time $\gamma^*$ and player two a stopping time $\beta^*$ in order to maximize their expected payoffs, which are respectively given by

$$E_y[\tilde{G}_1 (B_\gamma) I(\gamma \leq \beta) + \tilde{H}_1 (B_\beta) I(\beta < \gamma)]$$

and

$$E_y[\tilde{G}_2 (B_\beta) I(\beta < \gamma) + \tilde{H}_2 (B_\gamma) I(\gamma \leq \beta)]$$

for $y \in [I(0), I(1)]$, where $\tilde{G}_i := \frac{G_i}{\varphi} \circ I^{-1}$ and $\tilde{H}_i := \frac{H_i}{\varphi} \circ I^{-1}$, for $i = 1,2$. Given stopping time $\beta$ chosen by player two let

$$W^{1,\alpha}_\beta(y) = \sup_{\gamma} E_y[\tilde{G}_1 (B_\gamma) I(\gamma \leq \beta) + \tilde{H}_1 (B_\beta) I(\beta < \gamma)]$$

be the value function of player one and similarly, given stopping time $\gamma$ chosen by player one let

$$W^{2,\alpha}_\gamma(y) = \sup_{\beta} E_y[\tilde{G}_2 (B_\beta) I(\beta < \gamma) + \tilde{H}_2 (B_\gamma) I(\gamma \leq \beta)]$$

be the value function of player two. Suppose that there exist continuous functions $\tilde{u}, \tilde{v} : [I(0), I(1)] \to \mathbb{R}$ such that

$$\tilde{u} = \inf_{F \in \text{Sup}_1^\psi(G_1, \tilde{K}_1)} F$$

and

$$\tilde{v} = \inf_{F \in \text{Sup}_2^\psi(G_2, \tilde{K}_2)} F$$

where

$$\text{Sup}_1^\psi(G_1, \tilde{K}_1) = \{F : [I(0), I(1)] \to [\tilde{G}_1, \tilde{K}_1] : F \text{ is continuous,}$$

$$\text{Sup}_2^\psi(G_2, \tilde{K}_2) = \{F : [I(0), I(1)] \to [\tilde{G}_2, \tilde{K}_2] : F \text{ is continuous,}$$
\begin{equation}
F = \tilde{H}_1 \text{ in } \tilde{D}_2, F \text{ is concave in } \tilde{D}_2^c
\end{equation}
and
\begin{equation}
\overline{\text{Sup}}^2(\tilde{G}_2, \tilde{K}_2) = \{ F : [I(0), I(1)] \rightarrow [\tilde{G}_2, \tilde{K}_2] : F \text{ is continuous,}
\end{equation}
\begin{equation}
F = \tilde{H}_2 \text{ in } \tilde{D}_1, F \text{ is concave in } \tilde{D}_1^c
\end{equation}
with \( \tilde{K}_i \), for \( i = 1, 2 \), being the smallest concave function majorizing \( \tilde{H}_i \). \( \tilde{D}_1 = \{ \tilde{u} = \tilde{G}_1 \} \) and \( \tilde{D}_2 = \{ \tilde{v} = \tilde{G}_2 \} \). Again from the double partial superharmonic characterisation of the value functions (note that \( \tilde{G}_i(I(0)) = \tilde{H}_i(I(0)) \) and \( \tilde{G}_i(I(1)) = \tilde{H}_i(I(1)) \) since \( G_i(0) = H_i(0) \) and \( G_i(1) = H_i(1) \)), we have that
\begin{equation}
\tilde{u}(y) = W_{\beta_{\tilde{D}_2}}^{1,\alpha}(y) \text{ and } \tilde{v}(y) = W_{\gamma_{\tilde{D}_1}}^{2,\alpha}(y)
\end{equation}
for all \( y \in [I(0), I(1)] \), where \( \gamma_{\tilde{D}_1} = \inf \{ t \geq 0 : B_t \in \tilde{D}_1 \} \) and \( \beta_{\tilde{D}_2} = \inf \{ t \geq 0 : B_t \in \tilde{D}_2 \} \).

\section*{(4*.)} We now link the value functions in (6.2) - (6.3) with those in (6.11) - (6.12) via the collections of functions in (6.9) - (6.10) and (6.15) - (6.16). It is easy to see that \( \varphi(x)\tilde{u}(I(x)) \geq G_1(x) \) for all \( x \in [0,1] \) and that \( \varphi(x)\tilde{u}(I(x)) = H_1(x) \) for all \( x \in \{ x \in [0,1] : \varphi(x)(\tilde{v} \circ I)(x) = G_2(x) \} \). Since we know that \( \tilde{u} \) is concave in \( \{ y \in [I(0), I(1)] : \tilde{v}(y) > \tilde{G}_2(y) \} \) then by writing \( \tilde{u} = (\tilde{u} \circ I) \circ I^{-1} \) and by making use of the fact that a Borel measurable function \( F \) on \( D \subseteq [0,1] \) is \( I \)-concave if and only if \( F \circ I^{-1} \) is concave on \( I(D) = \{ I(x) : x \in [0,1] \} \) we get that \( \tilde{u} \circ I \) is \( I \)-concave in \( \{ x \in [0,1] : \varphi(x)(\tilde{v} \circ I)(x) > G_2(x) \} \). This fact can also be used to show that \( \varphi(x)\tilde{K}_1(I(x)) = K_1(x) \) for all \( x \in [0,1] \). Repeating the above arguments for \( \tilde{v} \) and comparing the functions \( \varphi(\tilde{u} \circ I) \) and \( \varphi(\tilde{v} \circ I) \) with the functions \( u \) and \( v \) in (6.4) - (6.5) it follows that
\begin{align}
\varphi(x)\tilde{u}(I(x)) &= V_{\sigma_{\tilde{D}_2}}^{1,\alpha}(x) \\
\varphi(x)\tilde{v}(I(x)) &= V_{\gamma_{\tilde{D}_1}}^{2,\alpha}(x)
\end{align}
for \( x \in [0,1] \), where \( D_1 = \{ x \in [0,1] : \varphi(x)\tilde{u}(I(x)) = G_1(x) \} \) and \( D_2 = \{ x \in [0,1] : \varphi(x)\tilde{v}(I(x)) = G_2(x) \} \). From (6.17) we can deduce that
\begin{align}
V_{\sigma_{\tilde{D}_2}}^{1,\alpha}(x) &= \varphi(x)W_{\beta_{\tilde{D}_2}}^{1,\alpha}(I(x)) \\
V_{\gamma_{\tilde{D}_1}}^{2,\alpha}(x) &= \varphi(x)W_{\beta_{\tilde{D}_1}}^{2,\alpha}(I(x))
\end{align}
for \( x \in [0,1] \).

\section*{7. Concluding Remarks}
We conclude this study by pointing out some remarks and directions for future research.

1. In general, given only assumptions (3.1) - (3.6), there may exist stopping boundaries \( 0 \leq B_x \leq A_x \leq 1 \) such that the first entry times \( \tau_* = \inf \{ t \geq 0 : X_t \leq A_x \} \wedge \rho_{0,1} \) and \( \sigma_* = \inf \{ t \geq 0 : X_t \geq B_x \} \wedge \rho_{0,1} \) form a Nash equilibrium point. This may happen whenever there exists \( y \in (0, 1) \) such that \( G_i(y) = H_i(y) \), where \( i = 1, 2 \). Consider for example the
payoff functions in Figure 4 and let \( \tau_* = \inf \{ t \geq 0 : X_t \leq A_* \} \). Moreover, if \( \sigma_* = \inf \{ t \geq 0 : X_t \geq B_* \} \), then we must have that \( M^1_x(\tau_*, \sigma_*) = H_1(x) \). On the other hand

\[
M^1_x(\tau, \sigma) = E_x[G_1(X_\tau)I(\tau \leq \sigma) + H_1(X_{\sigma \wedge \tau})I(\sigma < \tau)]
= E_x[G_1(X_{\tau \wedge \sigma})I(\tau \leq \sigma) + H_1(X_{\sigma \wedge \tau})I(\sigma < \tau)]
\leq E_x[H_1(X_{\tau \wedge \sigma})I(\tau \leq \sigma) + H_1(X_{\sigma \wedge \tau})I(\sigma < \tau)]
\]

(7.1)

for all stopping times \( \tau \) and for all \( x \in [0, 1] \), where the second inequality follows from the fact that \( H_1 \) is concave in \([0, B_*)\). By symmetry we can show that \( M^2_x(\tau_*, \sigma_*) \geq M^2_x(\tau, \sigma) \) for all stopping times \( \sigma \) and for all \( x \in [0, 1] \), so that \( (\tau_*, \sigma_*) \) is a Nash equilibrium point. In general if \( B_* < A_* \) then we must have that \( A_* \geq b \) and \( B_* \leq a \). This assertion can be easily proved by contradiction. We shall only prove that \( B_* \leq a \). The fact that \( A_* \geq b \) follows by symmetry.

So suppose, for contradiction, that \( B_* > a \) and let \( \tau_{B_*} = \inf \{ t \geq 0 : X_t \leq B_* - \varepsilon \} \), where \( \varepsilon > 0 \) sufficiently small. It is easy to see that

\[
M^1_x(\tau_{B_*}, \sigma_*) \geq M^1_x(\tau_*, \sigma_*)
\]

(7.2)

Moreover, if \( x \in (B_* - \varepsilon, B_*) \) then (7.2) holds with a strict inequality and this shows that \( (\tau_*, \sigma_*) \) cannot be optimal for player one in the case \( B_* > a \). We next show that if \( A_*, B_* \in (0, 1) \), then the case \( A_* \geq B_* \) cannot occur if \( G_1(x) < H_1(x) \) for all \( x \in (0, 1) \). So suppose, for contradiction, that \( A_* \geq B_* \). Consider first the case \( A_* > B_* \) and let \( \tau_{B_*} = \inf \{ t \geq 0 : X_t \leq B_* \} \). Then it is easy to see that

\[
M^1_x(\tau_{B_*}, \sigma_*) = G_1(x)I(x \leq B_*) + H_1(x)I(x > B_*) > M^1_x(\tau_*, \sigma_*)
\]

for all \( x \in (0, 1) \) and this contradicts optimality of \( \tau_* \) for player one. In the case \( A_* = B_* \) one can show that if \( \tau_{B_*} = \inf \{ t \geq 0 : X_t \leq B_* - \varepsilon \} \), for \( \varepsilon > 0 \) sufficiently small then \( M^1_x(\tau_*, \sigma_*) \leq M^1_x(\tau_{B_*}, \sigma_*) \) for all \( x \in (B_* - \varepsilon, B_*) \). In particular one can see that \( M^1_x(\tau_*, \sigma_*) < M^1_x(\tau_{B_*}, \sigma_*) \)
if \( x \in (B_* - \varepsilon, B_*) \).

2. Going back to the question of uniqueness, we have seen that in the case \( \Theta(0, B) > 0 \) for all \( B \in [b, 1] \) and \( \Gamma(A, 1) > 0 \) for all \( A \in [0, a] \) additional properties on the payoff functions can guarantee uniqueness of the value functions. However in the case when there exists at least one \( B^{\Theta,0}_* \) and/or \( A^{\Gamma,1}_* \) such that \( \Theta(0, B^{\Theta,0}_*) = \Gamma(A^{\Gamma,1}_*, 1) = 0 \) the admissible regions \( A_{\Theta} \) and \( A_{\Gamma} \) may happen to be a union of disjoint intervals and uniqueness in this case is a more challenging property to deal with.

4. Another interesting problem would be to analyse the case when \( G_1 \) is not necessarily convex in \((a, 1]\) and similarly when \( G_2 \) is not necessarily convex in \([0, b)\) while retaining assumptions (3.1) and (3.6).

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