Examples of rational maps

For simplicity, in all these examples the field is taken to be $\mathbb{C}$, but they work over any algebraically closed field of characteristic other than 2.

1. Let $t$ be the co-ordinate on $\mathbb{A}^1$ and $x, y$ the co-ordinates on $\mathbb{A}^2$. Let $V = \mathbb{A}^1$ and let $W \subset \mathbb{A}^2$ be the circle $x^2 + y^2 - 1 = 0$. Consider the intersections of the line $y = t(x - 1)$ with the circle.

After substituting $y = t(x - 1)$ into $x^2 + y^2 - 1 = 0$ we get a quadratic for $x$, one solution is always $x = 1$, the other is $x = \frac{t^2 - 1}{t^2 + 1}$. The first one gives the point $(1, 0)$, the other $(\frac{t^2 - 1}{t^2 + 1}, -\frac{2t}{t^2 + 1})$.

$\frac{t^2 - 1}{t^2 + 1}$ and $-\frac{2t}{t^2 + 1}$ are rational functions in $t$, so they are elements of $\mathbb{C}(V)$, and $(\frac{t^2 - 1}{t^2 + 1}, -\frac{2t}{t^2 + 1}) \in W$ by its construction whenever it is defined. Therefore $\phi(t) = (\frac{t^2 - 1}{t^2 + 1}, -\frac{2t}{t^2 + 1})$ defines a rational map $V \dashrightarrow W$.

It can be verified purely algebraically, too, that the image lies in $W$, since

$$\left(\frac{t^2 - 1}{t^2 + 1}\right)^2 + \left(-\frac{2t}{t^2 + 1}\right)^2 - 1 = 0.$$  

To find an inverse for $\phi$, we need solve $x = \frac{t^2 - 1}{t^2 + 1}$, $y = -\frac{2t}{t^2 + 1}$ for $t$ as a rational function in $x$ and $y$.

From the geometric construction it is clear that $t$ can be recovered as the slope of the line through $(x, y)$ and $(1, 0)$, therefore $t = y/(x - 1)$. $\psi(x, y) = \ldots$
\( y/(x - 1) \) is a rational map \( W \rightarrow V \), since \( y/(x - 1) \in K(W) \) and because the codomain is \( \mathbb{A}^1 \), there are no equations to check. (As it is clear in all the examples that the maps are written in terms of polynomial or rational functions, this will not be mentioned specifically again.)

It can be verified purely algebraically that \( \psi \) is the inverse of \( \phi \) as a rational map.

\[
(\psi \circ \phi)(t) = \psi\left(\frac{t^2 - 1}{t^2 + 1} - \frac{2t}{t^2 + 1}\right) = \frac{-2t}{t^2 - 1} = \frac{1 + t^2}{-2} = t \quad (t \neq \pm i),
\]

\[
(\phi \circ \psi)(x, y) = \phi\left(\frac{y}{x - 1}\right) = \left(\frac{-1 + \left(\frac{y}{x - 1}\right)^2}{1 + \left(\frac{y}{x - 1}\right)^2}, -\frac{2\left(\frac{y}{x - 1}\right)}{1 + \left(\frac{y}{x - 1}\right)^2}\right).
\]

In \( K[W] \) we have \((x - 1)^2 + y^2 = x^2 - 2x + 1 + y^2 = (x^2 - 2x + 1 + y^2) - (x^2 + y^2 - 1) = 2(1 - x)\) and \(- (x - 1)^2 + y^2 = -x^2 + 2x - 1 + y^2 = (x^2 - 2x + 1 + y^2) - (x^2 + y^2 - 1) = 2x(1 - x)\), therefore

\[
(\phi \circ \psi)(x, y) = \left(\frac{2x(1 - x)}{2(1 - x)}, -\frac{2y(x - 1)}{2(1 - x)}\right) = (x, y) \quad (x, y) \neq (1, 0)
\]

Hence \( \phi \) and \( \psi \) mutually inverse birational maps between \( V \) and \( W \), therefore \( V \) and \( W \) are birationally equivalent and \( W \) is a rational variety.

Neither \( \phi \), nor \( \psi \) is a morphism. \( \phi \) is already written in terms of rational functions with coprime numerator and denominator and \( K[V] \cong K[t] \) is a unique factorisation domain, so \( \phi \) is not defined at \( t = \pm i \), where \( 1 + t^2 = 0 \). \( \psi \) is not defined at \((1, 0)\). \( y/(x - 1) \) is a 0/0 type fraction, but \( \frac{y}{x - 1} = -\frac{x + 1}{y} \) in \( K(W) \), since \( y^2 = 1 - x^2 \) in \( K[W] \), the numerator \( x + 1 \) is not 0 at \((1, 0)\), but denominator \( y \) is 0.

A further point worth noting is that \((x - 1)/y \) is not defined at \((1, 0)\) as a rational map on \( \mathbb{A}^2 \), but it is defined as a rational map on \( W \), since \( \frac{y}{x + 1} = \frac{y}{x + 1} \) on \( W \) and this latter fraction is defined and has value 0 at \((1, 0)\).
2. Let $V = \mathbb{A}^1$ and let $W \subset \mathbb{A}^2$ be the cuspidal cubic $y^2 - x^3 = 0$. We already saw that $\phi : V \to W$, $\phi(t) = (t^2, t^3)$ is a morphism $V \to W$. $\psi(x, y) = y/x$ is a rational map which is the inverse of $\phi$ considered as a rational map. 

$(\psi \circ \phi)(t) = \psi(t^2, t^3) = t^3/t^2 = t \ (t \neq 0)$ and $(\phi \circ \psi)(x, y) = \psi(y/x) = \left(\left(\frac{y}{x}\right)^2, \left(\frac{y}{x}\right)^3\right) = \left(\frac{x^3}{x^2}, \frac{y^3}{y^2}\right) = (x, y) \ ((x, y) \neq (0, 0))$. Geometrically, for $t \neq 0$, $\phi(t)$ is the unique intersection point of $W$ with the line $y = tx$ other than the origin for $t \neq 0$, therefore the inverse is clearly $t = y/x$.

\[ \psi \text{ is defined at every point of } W \text{ except } (0, 0), \text{ but it is not defined at } (0, 0), \text{ since if it were, it would be a morphism (Question 5 on Problem Sheet 4) but we proved earlier that } \phi \text{ has no inverse morphism.} \]

The existence of the mutually inverse rational maps between $V$ and $W$ shows that they birationally equivalent, so $W$ is rational. We can also prove this algebraically. We saw that $K[W]$ is isomorphic to the subring $S$ of $K[t]$ consisting of polynomials with no degree 1 term. This implies that $K(W)$ is isomorphic to a subfield of $K(t)$. However, $t^2, t^3 \in S$, therefore $t = t^3/t^2$ is in the field of fractions of $S$, so the field of fractions is the whole of $K(t)$ and therefore $K(W) \cong K(t) \cong K(V)$.

3. Let $V = \mathbb{A}^1$ and let $W \subset \mathbb{A}^2$ be the nodal cubic $y^2 - x^3 - x^2 = 0$. $\phi : V \to W$, $\phi(t) = (t^2 - 1, t(t^2 - 1))$ is a morphism $V \to W$, since $(t(t^2 - 1))^2 - (t^2 - 1)^3 - (t^2 - 1)^2 = (t^2 - 1)^2(t^2 - (t^2 - 1) - 1) = 0.$
The geometric description of $\phi(t)$ is the same as in the previous example, $\phi(t)$ is the intersection point of $W$ with the line $y = tx$ other than the origin for $t \neq \pm 1$, therefore the inverse is clearly $t = y/x$.

$\psi(x,y) = y/x$ is a rational map which is the inverse of $\phi$ considered as a rational map since $(\psi \circ \phi)(t) = \psi(t^2 - 1, t(t^2 - 1)) = t(t^2 - 1)/(t^2 - 1) = t$ ($t \neq \pm 1$) and $(\phi \circ \psi)(x,y) = \left(\frac{y^2}{x^2} - 1, \frac{y}{x}((\frac{y}{x})^2 - 1)\right) = \left(\frac{y^2 - x^2}{x^2}, \frac{y(y^2 - x^2)}{x^3}\right) = \left(\frac{x^2}{x^2}, \frac{y^3}{x^3}\right) = (x,y)$ ($(x,y) \neq (0,0)$). This shows that $V$ and $W$ are birationally equivalent, therefore $W$ is rational.

$\psi$ is defined at every point of $W$ except $(0,0)$, but it is not defined at $(0,0)$, since if it were, it would be a morphism (Question 4 on Problem Sheet 4), but $\phi$ is not injective, $\phi(-1) = \phi(1) = (0,0)$, so it cannot have an inverse morphism.

4. Let $x$, $y$, $z$ be the co-ordinates on $\mathbb{A}^3$ and let $u$, $v$ be co-ordinates on $\mathbb{A}^2$. Let $V = V((x^2 + y^2 + z^2 - 1)) \subset \mathbb{A}^3$ be the sphere of radius 1 centred at the origin and let $W = \mathbb{A}^2$. The stereographic projection maps each point $(x,y,z)$ of the sphere to the intersection point of the line through $(x,y,z)$ and $(0,0,1)$ with the plane $z = 0$. The points of the line can be written in parametric form as $(\lambda x, \lambda y, \lambda z + (1 - \lambda))$, so the intersection point with the plane $z = 0$ corresponds to $\lambda = 1/(1 - z)$ and the intersection point itself is $(x/(1 - z), y/(1 - z), 0)$. Hence $\phi(x,y,z) = (x/(1 - z), y/(1 - z))$. 

4
Similarly, the points of the line through \((u, v, 0)\) and \((0, 0, 1)\) can be written as \((\lambda u, \lambda v, 1 - \lambda)\), by substituting these values into \(x^2 + y^2 + z^2 - 1 = 0\) we obtain a quadratic equation with roots \(\lambda = 0\) and \(\lambda = \frac{2}{1 + u^2 + v^2}\), hence the intersection point with the sphere is

\[
\psi(u, v) = \left(\frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{-1 + u^2 + v^2}{1 + u^2 + v^2}\right)
\]

From the geometric description it is clear that \(\psi(u, v) \in V\) whenever it is defined and that \(\phi\) and \(\psi\) are inverses of each other. This can also be verified algebraically, similarly to Example 1.

\(\psi\) is written in terms of rational functions with coprime numerator and denominator and \(K[W] \cong K[u, v]\) is a unique factorisation domain, so \(\psi\) is not defined at the points with \(1 + u^2 + v^2 = 0\). \(\phi\) is clearly defined at the points of \(V\) where \(z \neq 1\). We shall prove that it is not defined at the points with \(z = 1\), which consist of the lines \(z = 1, y = ix\) and \(z = 1, y = -ix\). At the points with \(z = 1\) and \(x \neq 0\), the numerator of \(x/(1 - z)\) is not 0, but the denominator is 0, so \(x/(1 - z)\) cannot be defined there. This leaves the point \((0, 0, 1)\). If \(x/(1 - z)\) could be written in an equivalent form \(f/g\) with \(g(0, 0, 1) \neq 0\), then \(g(x, y, z)\) would only be 0 at finitely many points of the lines \(z = 1, y = \pm ix\) so \(f/g\) could also be used to define this rational function at all but finitely many points of these lines, but we know that this is not possible. (The key idea here is that the set of points at which a rational function is defined is Zariski open, that is, the complement of a subvariety.)
5. $C = V((x^2 + y^2 + z^2 - 1, x^2 - y^2 + (z - 1)^2)) \subset \mathbb{A}^3$ (a “figure of 8” curve) and $D = V(u^2 - v^2 + 1) \subset \mathbb{A}^2$ (a hyperbola).

$\phi|_C$ and $\psi|_D$ give rational maps between $C$ and $D$. Assume that $(x, y, z) \in C$ and let $(u, v) = \phi(x, y, z)$. Then

$$u^2 - v^2 + 1 = \left(\frac{x}{z - 1}\right)^2 - \left(\frac{y}{z - 1}\right)^2 + 1 = \frac{x^2 - y^2 + (z - 1)^2}{(z - 1)^2} = 0,$$

so $\phi(x, y, z) \in D$. Conversely, assume that $(u, v) \in D$ and let $(x, y, z) = \psi(u, v)$. Then

$$x^2 - y^2 + (z - 1)^2 = \left(\frac{2u}{1 + u^2 + v^2}\right)^2 - \left(\frac{2v}{1 + u^2 + v^2}\right)^2 + \left(\frac{-1 + u^2 + v^2}{1 + u^2 + v^2} - 1\right)^2$$

$$= \frac{(2u)^2 - (2v)^2 + (-2)^2}{(1 + u^2 + v^2)^2} = \frac{4(u^2 - v^2 + 1)}{(1 + u^2 + v^2)^2} = 0,$$

so $\psi(u, v) \in C$, since we already know $x^2 + y^2 + z^2 - 1 = 0$ from Example 4.

We shall show that $\phi|_C$ is not defined at $(0, 0, 1)$. This is the only point of $C$ with $z = 1$, so $\phi$ is clearly defined at the other points of $C$. We have $x^2 + z(z - 1) = (x^2 + y^2 + z^2 - 1)/2 + (x^2 - y^2 + (z - 1)^2)/2 \in I(C)$, therefore $x/(1 - z) = z/x$ in $K(C)$, and this latter fraction cannot be defined at $(0, 0, 1)$.

The only points at which $\psi|_D$ may not be defined are the points of $D$ with $u^2 + v^2 - 1 = 0$. The solutions of $u^2 + v^2 + 1 = u^2 - v^2 + 1 = 0$ are $u = \pm i$, $v = 0$. At these points the numerator of $2u/(1 + u^2 + v^2)$ is not 0, while the denominator is 0, so $2u/(1 + u^2 + v^2)$ is not defined there and therefore $\psi|_D$ is not defined either.