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# Power of rank tests against location shift alternative

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## Abstract

We study the power of some exceedance rank tests against location shift and Lehman alternatives for distributions from the uniform, normal, exponential, gamma, and lognormal families. We give the corresponding power functions, obtained by Monte Carlo simulation and discuss the relative merits of the tests.

## 1. Introduction

Let  $X_1, \dots, X_m$  and  $Y_1, \dots, Y_n$  be two random samples with continuous distribution functions  $F$  and  $G$ , respectively. A standard problem is that of testing the hypothesis  $H_0$  that  $F$  and  $G$  are identical against the one-sided alternative  $H_A$  that the  $Y$ 's are stochastically larger than the  $X$ 's with strict inequality for some  $x$ :

$$H_A : G(x) \leq F(x). \quad (1)$$

This situation arises naturally when, for example, we wish to test whether a new manufacturing process or a new medical treatment is better than the existing one.

For this hypothesis testing problem a test can be based on the number of observations from one of the samples that exceed or precede all observations from the other sample. Such tests have been proposed by Rosenbaum

[10], Haga [6], Šidák and Vondracek [15], and Stoimenova [13], among others. These tests are a subclass of a general family of tests based on precedences and/or exceedances of a random level specified by the order statistics from the samples. Some basic references include Sen [11], van der Laan and Chakraborti [14], Balakrishnan and Ng [3], Bairamov [1], and Balakrishnan, Dembinska and Stepanov [2].

From the distribution theory point of view the null hypothesis may be considered a simple hypothesis since the distribution of the rank orders under  $H_0$  is not affected by the underlying distribution function,  $F$ . More problematic is the determination of the power of non-parametric tests due to the generality of the alternative  $H_A$ . A simple subclass of this alternative suggested by parametric theory is

$$H_1 : G(x) \geq F(x - \theta), \text{ for some } \theta > 0, \quad (2)$$

but in this case the distribution of the rank statistics will depend not only on  $\theta$ , but also on  $F$ .

The present paper deals with the power of one of the exceedance tests against location-shift alternatives  $H_1$ . The power of the exceedance tests is estimated through Monte Carlo simulations as a function of the shift parameter  $\theta$  for uniform, normal, exponential, lognormal, and gamma distributions.

We also compare the power of some exceedance tests against Lehmann alternatives  $H_{LE} : G(x) = 1 - [1 - F(x)]^{1/\eta}$ ,  $\eta > 0$ . Alternatives of this form are a subclass of stochastically ordered alternatives.

All computations were done with the statistical system R [9].

## 2. Tests based on exceedance statistics

Let  $X_{\min}$  and  $X_{\max}$  be respectively the smallest and the largest observation from the  $X$ -sample, and similarly, let  $Y_{\min}$  and  $Y_{\max}$  be respectively the smallest and the largest observation from the  $Y$ -sample. Denote

$$\begin{aligned} a &= \text{the number of } Y \text{'s larger than } X_{\max} \\ a' &= \text{the number of } X \text{'s larger than } Y_{\max} \end{aligned}$$

$b'$  = the number of  $Y$ 's smaller than  $X_{\min}$

$b$  = the number of  $X$ 's smaller than  $Y_{\min}$ .

We call these statistics exceedance statistics. Several tests for testing  $H_0$  against the stochastically ordered alternative  $H_A$  in (1) are based on the exceedance statistics.

The tests included in this study are briefly described below. The notations are adapted to the present hypothesis testing problem. The simplest test of exceedance type is the Rosenbaum test [10] which uses the test statistic

$$R = b'$$

Hájek, Šidák and Sen [7] found that some of these functions generate locally most powerful rank tests of testing  $H_0$  against a one-sided shift in the location parameter  $\theta$  if the underlying distribution is uniform. They found, in present formulation, that the statistic

$$V = a' + b$$

generates the locally most powerful rank test for the above problem, for  $\theta$  close to 0. This test has been suggested earlier by Šidák and Vondracek [15].

The two-sided version of this test uses the "symmetrized" statistic

$$H = a' - a + b' - b$$

The test based on  $H$  is more robust against differences in variance. It has been introduced by Haga [6] and it is called Haga test.

$E$ -test of Hájek and Šidák [7] is given by

$$E = \max(a', b') - \min(a, b)$$

The  $M$ -test is proposed by Stoimenova [13] with the test statistic

$$M = \max\{m - a', n - b'\}.$$

The statistic  $M$  also generates the locally most powerful rank test for  $H_0$  against a shift  $\theta$  of the uniform distribution over  $(0, 1)$  for  $\theta$  close to 1.

As Wilcoxon's rank-sum statistic  $WR$  is known to provide a good non-parametric test for the hypothesis testing problem described above, it will be used for comparison. It is based on the statistic

$$WR = \sum_{i=1}^n \text{Rank}(Y_i),$$

where  $\text{Rank}(Y_i)$  is the rank of  $Y_i$  in the ordered sample consisting of  $X_1, \dots, X_m, Y_1, \dots, Y_n$ .

For more details on Wilcoxon rank-sum test and exceedance tests, one may refer to Hájek, Šidák and Sen [7], Govindarajulu [5] and Gibbons and Chakraborti [4].

### 3. Critical values of exceedance tests

For a fixed level of significance  $\alpha$ , the critical region  $W$  of a particular exceedance test  $T$  satisfies

$$P(T \in W \mid H_0) \leq \alpha. \quad (3)$$

Under the alternative hypothesis that  $Y$  is stochastically larger than  $X$  we expect the  $X$ -sample to take on most of the smaller ranks. Hence, the critical regions of the different exceedance tests are of the following form.

For  $T = WR$  (Wilcoxon test) or  $T = M$ -test, the critical region  $W$  is specified by  $W = \{T \leq s\}$ , where the critical value,  $s$ , satisfies (3). Thus the critical region of these tests will be in the form  $W = \{0, 1, \dots, s - 1\}$ . For the remaining tests,  $H_0$  is rejected for large values of the statistics, i.e.  $W = \{T \geq s\}$ .

Using the exact null distributions of the above exceedance tests the critical value,  $s$ , for a chosen level of significance  $\alpha$  can be determined. However, in general we do not have equality in (3) since the exceedance tests are discrete random variables.



and value 0 with probability  $1 - \pi$ . Thus  $\pi$  is the proportion of times out of a fixed number of runs that each procedure in rejected the null hypothesis. Therefore, the level of significance of a test procedure with critical region

$$\{M \leq s\} \cup (\{M \leq s + 1\} \cap \{Z = 1\})$$

will be exactly  $\alpha$ . Some of the power values in the next section are fine tuned using this procedure.

#### 4. Power functions for Uniform distribution

Under  $H_1 : G(x) \geq F(x - \theta)$ ,  $\theta > 0$ , the distribution of rank statistics will depend not only on  $\theta$ , but also on  $F$ . In the alternative  $H_1$  we have a simple hypothesis if  $F(x)$  and  $\theta$  are fixed and a composite hypothesis if  $F(x)$  is held fixed and all  $\theta > 0$  are considered.

Evidently, under a location-shift model  $Y = X + \theta$ , we have

$$G(x) = P(Y \leq x) = P(X \leq x - \theta) = F(x - \theta) \leq F(x)$$

for all  $x$  if  $\theta > 0$ , and so  $Y$  is stochastically larger than  $X$  in this case.

In this section, the uniform distribution is considered in order to demonstrate the power performance of the exceedance tests against location-shift alternatives.

Due to the discreteness of the distributions of non-randomized test statistics based on ranks, the significance levels for the different exceedance tests are not the same. In order to achieve the same level of significance for all tests under study we use the randomized test procedure described in Section 3. This allows us to make meaningful comparison of their powers.

First, for any exceedance test,  $T$ , we determine two values  $\alpha_L$  and  $\alpha_R$  using (4) such that the interval  $(\alpha_L, \alpha_R)$  contain the critical level, say  $\alpha = 0.05$ . Next, we calculate the power values corresponding to the two critical values  $s$  and  $s + 1$  for the two parts of the critical region:

$$\beta_L = P(T \leq s | H_{LE}) \text{ and } \beta_R = P(T \leq s + 1 | H_{LE}),$$

where  $s$  is given  $P(T \leq s | H_0) \leq \alpha$ .

Then the empirical power of the test at exact level, say  $\alpha = 0.05$ , is estimated by

$$\beta = \pi\beta_R + (1 - \pi)\beta_L,$$

where the  $\pi$  factor is calculated using (5).

The power functions of the  $M$ -test,  $E$ -test, Haga test and Wilcoxon test were estimated through Monte Carlo simulations with shift  $\theta = 0.00, 0.05, 0.10, \dots$ , etc. until such  $\theta$  where the power function is close to 1. Three choices of sample sizes are considered:  $m = n = 10$ ,  $m = n = 20$  and  $m = 20, n = 10$ . In all cases the significance level is  $\alpha = 0.05$ . For each choice of sample sizes, we generated 100,000 sets of data from uniform distribution in order to obtain the estimated rejection rates. In Table 2 we have presented the power values of the exceedance tests under study for the case  $m = n = 10$ .

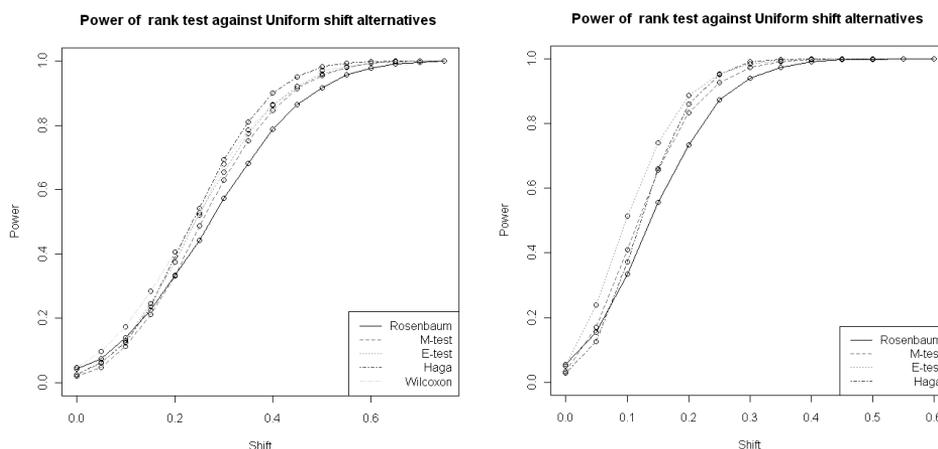
Table 2

**Power functions of exceedance tests against shift in Uniform distribution for  $m = n = 10$**

<i>shift</i>	<i>E-test</i>	<i>M-test</i>	<i>Haga</i>	<i>Wilcoxon</i>	<i>Rosenbaum</i>
0.00	0.0240	0.0189	0.0242	0.0529	0.0413
0.05	0.0663	0.0455	0.0611	0.1023	0.0786
0.10	0.1407	0.1065	0.1293	0.1794	0.1412
0.15	0.2413	0.2096	0.2397	0.2745	0.2195
0.20	0.3735	0.3446	0.3800	0.4056	0.3337
0.25	0.5170	0.4974	0.5505	0.5241	0.4474
0.30	0.6613	0.6234	0.6955	0.6608	0.5697
0.35	0.7742	0.7496	0.8116	0.7813	0.6877
0.40	0.8645	0.8446	0.8950	0.8676	0.7771
0.45	0.9237	0.9134	0.9528	0.9058	0.8605
0.50	0.9575	0.9543	0.9824	0.9732	0.9163
0.55	0.9808	0.9792	0.9932	0.9816	0.9553
0.60	0.9921	0.9914	0.9976	0.9922	0.9781
0.65	0.9969	0.9969	0.9995	0.9950	0.9914
0.70	0.9995	0.9993	0.9999	1.0000	0.9970
0.75	0.9997	0.9998	0.9999	1.0000	0.9990

In Figure 1 the power functions of  $M$  test, Rosenbaum test,  $E$ -test, Haga test and Wilcoxon test against shift in uniform distribution are compared for equal sample sizes,  $m = n = 10$ , and unequal sample sizes,  $m = 20, n = 10$ , at significance level  $\alpha = 0.05$ .

Figure 1  
**Power curves of exceedance tests against shift in Uniform distribution  
 for  $m = n = 10$  (left) and  $m = 20, n = 10$  (right)**



Though the present study is somewhat limited, by inspection of this table we may draw the following tentative conclusions:

a) for equal sample sizes,  $m = n = 10$ , Wilcoxon test seems to be better than other tests for small shift. Haga tests,  $M$ -test and  $E$ -test perform similarly. Rosenbaum test is less powerful.

b) for unequal sample sizes,  $m = 20, n = 10$   $E$ -test performs equally better than the other tests for small shift. Rosenbaum test is less powerful.

## 5. Power consistence

Due to the great generality of the alternative distributions the comparative study of the power of these tests is carried out for particular subclasses

of  $H_A$ . We require the proposed test to be consistent for a broad family of parent distributions, i.e., the power of the test should approach one as the sample increases.

In this section we will study the power of the exceedance test under the location-shift alternatives  $H_1: G(x) \geq F(x - \theta)$ ,  $\theta > 0$ .

Sen [11] has shown that the asymptotic properties of some of the exceedance tests depend on the asymptotic behaviour of the sample extreme values  $X_{\min}$ ,  $X_{\max}$  (and  $Y_{\min}$  and  $Y_{\max}$ ). In the case of continuous distributions there are three domains of attraction of the distributions of the extremes. These are (i) exponential type (or Gumbel distributions), (ii) Fréchet class of distributions, and (iii) reverse-Weibull class of distributions. Hence, the class of parent distributions of the continuous type may be divided into these three types. For distributions from the exponential type, the cumulative distribution function (cdf)  $F$ , together with its first two derivatives, is continuous everywhere and vanishes at the extremity of the range, which extends to infinity. Sen has proved that only for a sub-class of the exponential type the simple exceedance statistics are consistent against location-shift alternatives. This class is called convex exponential type and the cdfs verify  $\lim_{x \rightarrow \infty} [1 - F(x)] / F'(x) = 0$ . The class of exponential type distributions includes exponential, normal, lognormal, gamma distribution and many others.

In this section the power of the  $M$ -test is estimated through Monte Carlo simulations as a function of the shift parameter  $\theta$  for some normal, exponential, lognormal, and gamma distributions.

In order to assess the power properties of the  $M$ -test, we consider the  $M$ -test under the location-shift alternative  $H_1: F(x) = G(x + \theta)$ , for some  $\theta > 0$ , where  $\theta$  is a shift in location. The power values of the  $M$ -test were estimated through Monte Carlo simulations when  $\theta = 0.2, 0.3, 0.5, 1.0$  and  $2.0$ . Throughout the simulations, the critical level was  $\alpha = 0.05$ . The following distributions were considered in order to demonstrate the power performance of the  $M$ -test under this location-shift alternative:

1. Standard normal distribution
2. Standard exponential distribution

$$F(x; \theta) = 1 - e^{-x/\theta}, \quad x > 0, \theta > 0.$$

3. Standardised Gamma distribution. Here  $F$  is the distribution of the standardised variable  $\eta = (\xi - a)/\sqrt{a}$ , where  $\xi$  has a Gamma distribution with pdf

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} e^{-x/\theta} x^{\alpha-1}, \quad x > 0, \alpha > 0, \theta > 0.$$

4. Standardised lognormal distribution. Here  $F$  is the distribution of the standardised variable  $\eta = (\xi - e^{\sigma^2/2})/\sqrt{e^{\sigma^2}(e^{\sigma^2} - 1)}$  where  $\xi$  has a lognormal distribution with pdf

$$f(x) = \frac{1}{\sqrt{2\pi\sigma x}} \exp\left\{-\frac{1}{2\sigma^2}(\log x - \mu)^2\right\}, \quad x > 0, -\infty < \mu < \infty, \sigma > 0.$$

We estimated the rejection rates by the proportions of the rejections in samples of size 100,000 from the four distributions above.

In Tables 3-6, we have presented the power values of the  $M$ -test for two choices of sample sizes,  $m = n = 10$  and  $m = n = 20$ , for the underlying standard normal, standard exponential, standardized gamma, and standardized lognormal distributions, with location-shift being equal to 0.2, 0.3, 0.5, 1.0 and 2.0. For comparison purposes, the corresponding critical values and the exact levels of significance are also presented. The corresponding power values of the Wilcoxon's rank-sum test  $WR$  are also presented for the sake of comparison.

Table 3  
**Power of  $M$ -test and Wilcoxon's rank-sum test,  $WR$**   
**against location shift when  $m = n = 10$**

<i>Test</i>	<i>Critical value</i>	<i>Exact l.o.s.</i>	<i>Location shift</i>	<i>Distribution</i>			
				<i>Normal</i>	<i>Exponential</i>	<i>Ln(0.1)</i>	<i>Ln(0.5)</i>
<b>M</b>	8	0.050	0.2	0.0428	0.0889	0.0419	0.0547
			0.3	0.0614	0.1314	0.0595	0.0843
			0.5	0.1147	0.2121	0.1158	0.1691
			1	0.3606	0.3877	0.3605	0.4115
			2	0.8776	0.6989	0.8659	0.7440
<b>WR</b>	82	0.0446	0.5	0.2536	0.4332	0.2605	0.3508
			1.0	0.6480	0.8163	0.6554	0.7819

Table 4  
**Power of  $M$ -test and Wilcoxon's rank-sum test,  $WR$**   
**against location shift when  $m = n = 20$**

<i>Test</i>	<i>Critical value</i>	<i>Exact l.o.s.</i>	<i>Location shift</i>	<i>Distribution</i>			
				<i>Normal</i>	<i>Exponential</i>	<i>Ln(0.1)</i>	<i>Ln(0.5)</i>
<b>M</b>	18	0.044	0.2	0.0463	0.1363	0.0459	0.0684
			0.3	0.0699	0.1745	0.0695	0.1107
			0.5	0.1444	0.2305	0.1475	0.2134
			1	0.4662	0.3909	0.4673	0.4221
			2	0.9411	0.6880	0.9227	0.7277
<b>WR</b>	348	0.0482	0.5	0.4403	0.7011	0.4462	0.5941
			1.0	0.9136	0.9802	0.9179	0.9716

Table 5  
**Power of  $M$ -test and Wilcoxon's rank-sum test,  $WR$**   
**against location shift in Gamma distribution when  $m = n = 10$**

<i>Test</i>	<i>Critical value</i>	<i>Exact l.o.s.</i>	<i>Location shift</i>	<i>Distribution</i>		
				<i>Gamma(2)</i>	<i>Gamma(5)</i>	<i>Gamma(10)</i>
<b>M</b>	8	0.050	0.2	0.0571	0.0466	0.0444
			0.3	0.0898	0.0693	0.0641
			0.5	0.1753	0.1358	0.1241
			1	0.4081	0.3964	0.3784
			2	0.7477	0.8081	0.8385
<b>WR</b>	82	0.0446	0.5	0.3337	0.2822	0.2663
			1.0	0.7513	0.6964	0.6719

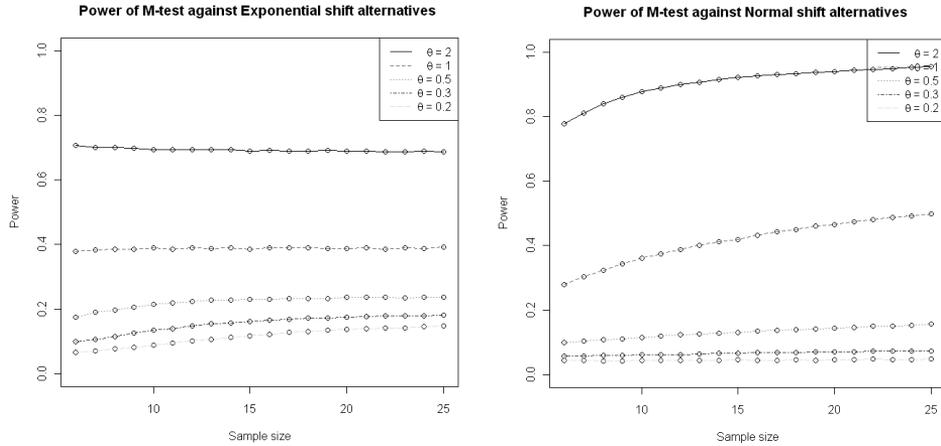
Table 6  
**Power of  $M$ -test and Wilcoxon's rank-sum test,  $WR$  against location shift in Gamma distribution when  $m = n = 20$**

<i>Test</i>	<i>Critical value</i>	<i>Exact l.o.s.</i>	<i>Location shift</i>	<i>Distribution</i>		
				<i>Gamma(2)</i>	<i>Gamma(5)</i>	<i>Gamma(10)</i>
<b>M</b>	18	0.044	0.2	0.0783	0.0542	0.0494
			0.3	0.1273	0.0873	0.0779
			0.5	0.2345	0.1818	0.1616
			1	0.4383	0.4777	0.4828
			2	0.7607	0.8397	0.8792
<b>WR</b>	348	0.0482	0.5	0.5701	0.4905	0.4646
			1.0	0.9623	0.9385	0.9275

When we compare the power values of the  $M$ -test with those of Wilcoxon's rank-sum test, we find that the Wilcoxon's rank-sum test performs better than the  $M$ -test if the underlying distributions are close to symmetry, such as the normal distribution, gamma distribution with large values of shape parameter, and lognormal distribution with small values of shape parameter. However, under some right-skewed distributions such as the exponential distribution, gamma distribution with shape parameter 2.0, and lognormal distribution with shape parameter  $\sigma = 0.5$ , the  $M$ -test has similar or higher power values than the Wilcoxon's rank-sum test.

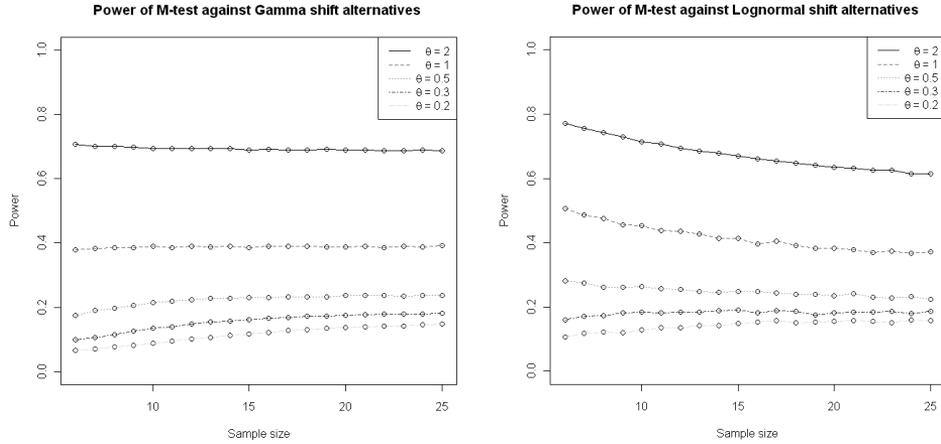
Figure 2 - 3 give plots of the power of  $M$ -test versus sample size for exponential, normal, gamma and lognormal distributions. Five values of shift are chosen:  $\theta = 0.2, 0.3, 0.5, 1.0$  and  $2.0$ .

Figure 2  
 Power curves of  $M$ -test against Exponential and Normal shift alternatives for  $\alpha = 0.05$



The plots of the power of  $M$ -test against shift in exponential, gamma and lognormal distributions do not show tendency to approach 1 with increasing the sample size. Since these distributions do not belong to the convex exponential type, we do not expect any of the exceedance tests to be consistent against these alternatives (see Sen [11]). However, normal distribution belongs to convex exponential type and we can see from the plot (Figure 2, right) that the power the  $M$ -test increases with increasing sample size. For  $\theta < 0.5$  the rate of increase is not high while for large  $\theta$  the tendency towards an increase is visible.

Figure 3  
**Power curves of  $M$ -test against Gamma and Lognormal shift alternatives for  $\alpha = 0.05$**



## 6. Power against Lehmann alternative

Lehmann [8] proposed classes of alternatives given by

$$H_{LE} : G = h(F) \tag{6}$$

where  $h$  is a differentiable non-decreasing function defined on  $[0, 1]$  with  $h(0) = 0$  and  $h(1) = 1$  and derivative  $h'$ .

Lehmann showed that under alternatives in (6), the distribution of the ranks vectors of the two samples of  $X$ 's and  $Y$ 's depends on  $h$ , but neither on  $F$  nor  $G$ . Lehmann alternative (6) not only leads to relatively simple distributions, but also has been shown to be of practical interest.

For example, under the Lehman alternative

$$H_{LE} : G(x) = 1 - [1 - F(x)]^{1/\eta} \text{ for some } \eta > 1. \tag{7}$$

the distribution of  $M$ -statistic is presented in an explicit form [12]. This allows to study power properties of the tests without choosing any particular cdfs. The power function of a test against  $H_{LE}$ , is given by

$$P(M \in W \mid H_{LE}) = \sum_{j \in W} P(M = j \mid H_{LE}), \quad (8)$$

where  $P(M = j \mid H_{LE})$  is the exact distribution of the test statistic under  $H_{LE}$ .

Clearly, the alternative hypothesis in (7) is a subclass of  $H_A$  and can be considered as a rough approximation for a shift of  $G$  to the right with respect to  $F$ . In this section we will study the power of the exceedance tests under study against Lehmann alternative (7).

Using the exact distribution under  $H_{LE}$  one can compute the power of the  $M$ -test for particular alternatives specified by  $\eta$ . However, like the null distribution, the simulated distribution under  $H_{LE}$  does not differ significantly from the exact.

For different choices of sample sizes, we generated 100,000 sets of data from the Uniform distribution in  $[0,1]$ ,  $U$ , as  $Y$ -samples and the same number of data from  $1 - (1 - U)^\eta$  as  $X$ -samples. Here  $\eta$  is an integer and take values  $\eta = 2, \dots, 7$ . Since the distribution of the  $M$ -statistic under  $H_{LE}$  is distribution free the results do not depend on this particular choice of the underlying Uniform distribution.

For any  $\eta > 1$  the power of the  $M$  test increases with increasing sample sizes, i.e. the test is consistent. For  $\eta$  close to 1 the power is not high while for  $\eta > 5$  the power is above 0.6 even for small sample sizes.

The power functions of  $M$ -test, Rosenbaum test,  $E$ -test, Haga test and Wilcoxon are estimated by Monte Carlo method for the following cases: sample size  $m = n = 6, \dots, 25$ ; significance level  $\alpha = 0.05$ ; Lehmann alternative  $\eta = 2, \dots, 7$ .

Critical values and corresponding exact levels of significance for the exceedance tests under study were estimated by generating 100,000 samples from Uniform distribution  $[0,1]$  for each test. Figure 4 - 5 give a plot of the power of the tests versus sample size for small and large  $\eta$ .

For all tests the power increases with sample size. As expected, the Wilcoxon test is the most powerful. It was followed by the Haga test. For  $\eta = 2$  the power of the  $M$ -test is good enough for small sample sizes with comparison to the other tests. For large  $\eta$  the power of the Haga test and the  $E$ -test increase quite fast in comparison to the  $M$ -test and Rosenbaum test.

Figure 4  
**Power comparison of exceedance tests for Lehmann alternative with  $\eta = 2$  and  $\eta = 3$**

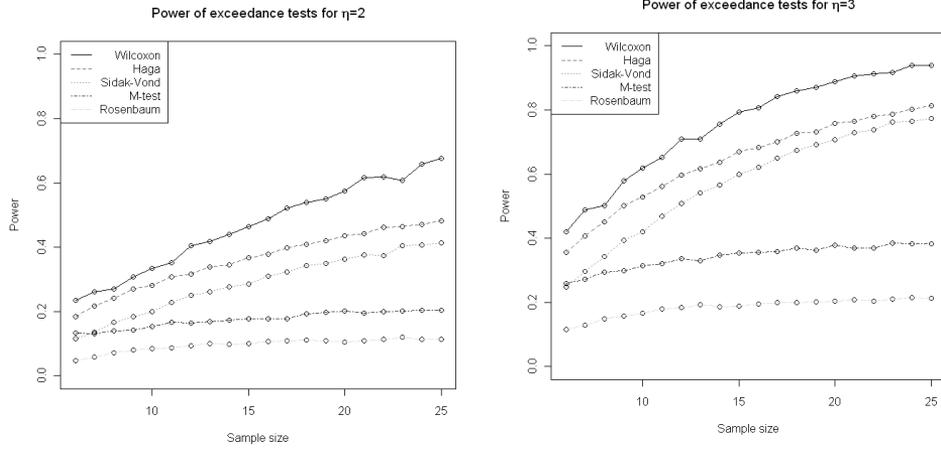
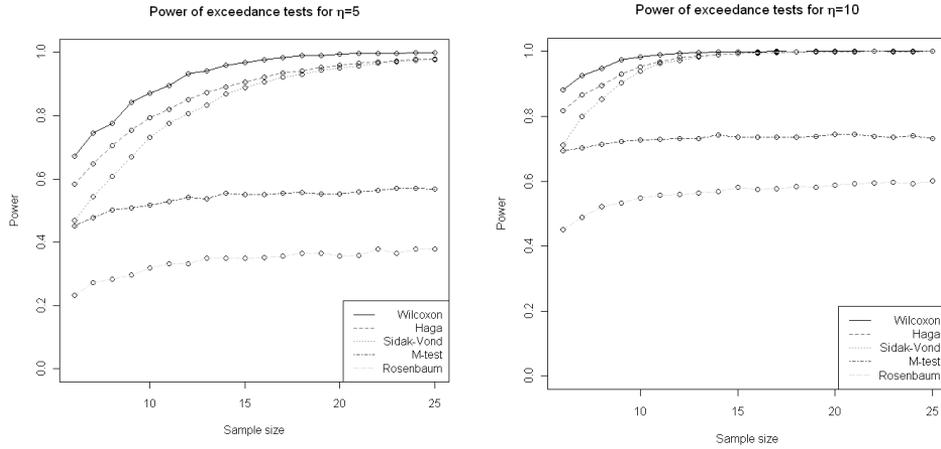


Figure 5  
**Power comparison of exceedance tests for Lehmann alternative with  $\eta = 5$  and  $\eta = 10$**



## 7. Additional remarks

1. Lehmann alternatives in (7) are of statistical interest since they include the extreme-value and exponential distributions. For instance, if  $X_i$  and  $Y_i$  are the lifetimes of two sets of similar articles the alternative in (7) corresponds to the failure rate of the second set being a constant fraction,  $\eta$ , of the first. In particular when  $F$  is exponential,

$$H_0 : F(x) = 1 - e^{-\lambda x}, x > 0,$$

and the alternative

$$H_{LE} : F(x) = 1 - e^{-\lambda x/\eta}, x > 0.$$

This example provides what is probably the most important application of the tests described here; one observes approximately exponentially distributed random variables and wishes to detect a multiplicative effect.

2. For a given cdf  $F$ , location-shift alternatives in (2) can also be considered as Lehmann alternatives in with  $h(x) = F(F^{-1}(x) - \theta)$ . However, in general this function is not easy to handle and consequently does not lead to an explicit expression for the distributions under  $H_1$ . For this reason, we have estimated power values of the tests under study through Monte Carlo simulations and evaluated power properties on the basis of these simulation results. Only for a few cdf  $F$ , exact power functions of tests based on exceedance statistics can be obtained. For example, for the Gumbel distribution with cdf  $F(x) = 1 - \exp(-e^{(x-\mu)/\sigma})$ ,  $\sigma > 0$ ,  $-\infty < \mu < \infty$ , the location-shift alternative

$$H_{LE} : F(x) = 1 - \exp(-e^{-(x-\theta-\mu)/\sigma})$$

is just the Lehmann alternative in (7) with  $\eta = e^{\theta/\sigma}$ , and so in this case the power function of a test under location-shift alternative can be immediately obtained from the power function of that test under the Lehmann alternative in (7).

Similarly, for the distribution  $F(x) = \exp(-e^{-x/\beta})$ ,  $\beta > 0$ , the Lehmann alternative

$$H_{LE} : F^k(x) = \exp(-e^{-kx/\beta}), k > 0,$$

is equivalent to a shift alternative

$$H_1 : F(x - \theta) = \exp(-ke^{-(x-\theta)/\beta})$$

where  $\theta = \beta \ln k$  and  $h(x) = x^k$  in the general Lehmann alternative given by (6).

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