Periodically correlated solutions to a class of stochastic difference equations

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1 Introduction.

The class of the periodically correlated processes sets up one of the possible frameworks for description and modelling of time series having pseudo-periodic behaviour. The mean and the autocovariance functions of the processes from this class are periodic. Many of the concepts of the stationary theory admit generalisation to the periodic case. There is a duality between the multivariate stationary processes and the periodically correlated processes which makes the investigation of these two classes theoretically equivalent. A survey on these questions (mainly from algorithmic point of view) and a lot of references may be found in Boshnakov [1].

We investigate the stochastic difference equation which is used to define the periodic autoregression model. We determine conditions for existence, uniqueness and causality of the periodically correlated solutions to that equation and investigate some properties of the corresponding autocovariance function. Our approach is based on a non-stationary Markovian representation of the model which in some respects gives more intuitive description of the properties of the periodic autoregression than its familiar representation as stationary multivariate autoregression. Proofs are generally omitted. They may be found in [1].

2 Deterministic and causal processes

Definition 1 *A process $\{X_t\}$ is said to be causal with respect to another process $\{Z_t\}$ if for every t there exists an absolutely summable sequence of constants $\{\psi_{ti}\}$, such that*

$$X_t = \sum_{i=0}^{\infty} \psi_{ti} Z_{t-i}.$$

The set of these sequences $\{\psi_{ti}\}$ is called a linear filter (with time-varying coefficients) and we say that $\{X_t\}$ is obtained by filtering $\{Z_t\}$. The following proposition shows that in typical situations the process $\{X_t\}$ obtained in this way is well defined both in mean square and almost sure sense. It is a straightforward generalization of the corresponding result for time invariant filters, for which $\psi_{ti} = \psi_{si}$ for all t, s, and i.

Proposition 1 *Let the time series $\{X_t\}$ be such that $\sup_t E|X_t| < \infty$, and let $\{\psi_{ti}\}$ be absolutely summable sequences. Then*

1. *for each t the series $\sum_{i=-\infty}^{\infty} \psi_{ti} X_{t-i}$ converges absolutely with probability one;*
2. *if $\sup_t E|X_t|^2 < \infty$ then the mean square limit of $\sum_{i=-\infty}^{\infty} \psi_{ti} X_{t-i}$ exists and is the same as in 1;*
3. *$P(A) = 0$, where A is the event $A = \{\omega \in \Omega :$ There exists at least one t, such that the series $\sum_{i=-\infty}^{\infty} \psi_{ti} X_{t-i}$ is not absolutely convergent$\}$;*
4. *if $\sup_t E|X_t|^2 < \infty$ and the series $\sum_i |\psi_{ti}| < \infty$ are uniformly convergent then both limits are uniform.*

The assertion *3* shows that almost all trajectories of the process $\{Y_t\}$, where $Y_t = \sum_{i=-\infty}^{\infty} \psi_{ti} X_{t-i}$, are well-defined.

3 Periodically correlated processes

It is convenient to say occassionally that a function f is d-periodic, instead of the longer expression "f is periodic with period d". A d-periodic function of two arguments is one for which $f(x+d, y+d) = f(x, y)$. Similarly a sequence of sequences $\{\psi_{ti}\}$ is said to be d-periodic if $\psi_{t+d,i} = \psi_{t,i}$ for each t and i.

Definition 2 *A process $\{X_t\}$ is said to be periodically correlated (d-periodically correlated) if its mean $\mu_t = EX_t$ and autocovariance function $R(t,s) = E(X_t - \mu_t)(X_s - \mu_s)$ are finite and there exists a positive integer d such that both μ_t and $R(t,s)$ are periodic with period d (i.e. $\mu_{t+d} = \mu_t$, and $R(t+d, s+d) = R(t,s)$ for all t, s).*

The number d is called period of the process. Usually (but not always) the smallest possible d is selected.

The stationary white noise is the simplest stationary process and it is used as a building block to generate other stationary processes. The same role in the periodic case plays the periodic white noise. It reduces to stationary white noise when σ_t^2 (see below) is constant (or equivalently, $d = 1$).

Definition 3 *The process $\{\varepsilon_t\}$ is said to be periodic white noise $PWN(0, \sigma_t^2,$ $d)$ if and only if it has the following properties (1) $E\varepsilon_t = 0$; (2) $\sigma_t^2 = E\varepsilon_t^2$ is d-periodic; (3) $E\varepsilon_t\varepsilon_s = 0$ for $t \neq s$.*

4 The periodic autoregression model

The periodic autoregression models are introduced by Jones and Brelsford [3]. The basic properties of these models, including the asymptotic theory of the autocorrelation based estimators, are established by Pagano [5]. He has fully exploited the dual process, giving the definition of a covariance stationary periodic autoregression in terms of it and expressing the properties of a periodic autoregressive process entirely through the dual process.

To avoid confusion we should point out that in Pagano's paper the notion covariance stationary multivariate autoregression means covariance stationary and causal multivariate autoregression. Similarly, covariance stationary periodic autoregression is what we call here causal periodic autoregression.

Our treatment of the periodic autoregression is slightly different from that of Pagano. First, we give a definition which is in terms of the properties of the process itself (plus a difference equation), without referring to the multivariate representation. On the other hand, we define it from the beginning to be periodically correlated. This is in accordance with the definition of the stationary autoregression which requires the process to be stationary and to satisfy a difference equation (see e.g., [2, Chapter 3]).

Definition 4 *The process $\{X_t\}$ is said to be periodic autoregression PAR(p_1, $\dots$, p_d) with period $d > 0$, and orders $p = (p_1, \dots, p_d)$ if it is periodically correlated and if for every t,*

$$X_t - \sum_{i=1}^{p_t} \phi_{ti} X_{t-i} = \varepsilon_t, \tag{1}$$

where $\{\varepsilon_t\}$ is periodic white noise $PWN(0, \sigma_t^2, d)$ and all parameters are d-periodic too, i.e. $p_{t+d} = p_t$, $\phi_{t+di} = \phi_{ti}$, $\sigma_{t+d}^2 = \sigma_t^2$.

It is common to write equation (1) in operator form by using the backward shift operator B ($BX_t = X_{t-1}$):

$$\phi_t(B)X_t = \varepsilon_t, \qquad \text{where} \qquad \phi_t(z) = 1 - \sum_{i=1}^{p_t} \phi_{ti} z^i. \tag{2}$$

The constants in equation (1) are called parameters, the process $\{\varepsilon_t\}$ — innovation series (or simply innovations). One can think of the process $\{X_t\}$ as being a solution of equation (1) (or (2)). Given a periodic autoregressive model $(\phi_t(z), \{\varepsilon_t\})$ it does not necessarily follow that there exists a periodic autoregressive process following this model. We will see that there exists at most one periodically correlated process corresponding to a given periodic autoregression model. On the other side, if a time series is periodic autoregression then, in general, it can be represented by several periodic autoregressive models. The most important among them is the causal one.

Definition 5 *A PAR($p_1, \ldots, p_d$) model for the time series $\{X_t\}$ given by the equation $\phi_t(B)X_t = \varepsilon_t$ is said to be causal if and only if $\{X_t\}$ is causal with respect to $\{\varepsilon_t\}$.*

The following theorems give relations between the autocovariances and the crosscorrelations between X_t and ε_s for the causal periodic autoregression.

Theorem 2 (Periodic Yule-Walker equations, ([5])) *Let the periodic autoregression process $\{X_t\}$ have the causal representation*

$$X_t - \sum_{i=1}^{p_t} \phi_{ti} X_{t-i} = \varepsilon_t.$$

Then for every t the autocovariance function of $\{X_t\}$ satisfies the periodic Yule-Walker equations

$$R(t,s) - \sum_{i=1}^{p_t} \phi_{ti} R(t-i,s) = \sigma_t^2 \delta_{t,s}$$

for every $s \leq t$, where $\delta_{ts} = 1$ if $t = s$ and $\delta_{ts} = 0$ otherwise.

Proposition 3 *If the PAR($p_1, \ldots, p_d$) model $\phi_t(B)X_t = \varepsilon_t$ is causal then for every t the crosscorrelation $EX_t\varepsilon_{t-k}$ between X_t and ε_{t-k} equals 0, σ_t^2, or $\sum_{i=1}^{\min(p_t,k)} \phi_{ti} E(X_{t-i}\varepsilon_{t-k})$, when $k < 0$, $k = 0$ and $k > 0$, respectively.*

The following lemma holds.

Lemma 4 *Let the PAR($p_1, \ldots, p_d$) model $\phi_t(B)X_t = \varepsilon_t$ for the time series $\{X_t\}$ be given. Suppose that $\sigma_t^2 > 0$, $t = 1, \ldots, d$. The model is causal if and only if the sequences $\{\psi_{ti}\}$, $t = 1, \ldots, d$ are absolutely summable.*

4.1 The Markovian dual model

Let us introduce the notations: $m = \max_{1 \le i \le d} p_i$, $E_t = (\varepsilon_t, 0, \dots, 0)'$,

$$A_t = \begin{pmatrix} -\phi_{t1} & -\phi_{t2} & \dots & -\phi_{tm-1} & -\phi_{tm} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix},$$

$$\alpha_t = A_t A_{t-1} ... A_{t-d+1}, \qquad t = 1, \dots, d.$$

Definition 6 *Let $\{X_t\}$ be e periodic autoregressive process. Its Markovian dual process $\{Z_t\}$ is defined to be the vector of m consecutive X-es*

$$Z_t = (X_t, X_{t-1}, \dots, X_{t-m+1})'.$$

Clearly $\{Z_t\}$ is nonstationary. Its properties are collected in the following proposition.

Proposition 5 *The Markovian dual process $\{Z_t\}$ of a causal periodic autoregressive process $\{X_t\}$ has the following properties:*

1. $Z_t = A_t Z_{t-1} + E_t$, *where E_t is orthogonal to Z_s for $s < t$.*
2. $M(Z_t | Z_s, s < t) = A_t Z_{t-1}$. *(linear Markovian property of $\{Z_t\}$)*
3. $Z_t = \alpha_t Z_{t-d} + V_t,$, *where $\alpha_t = A_t A_{t-1} ... A_{t-d+1}$, V_t is orthogonal to Z_s, $s \le t - d$, and is given by the formula*

$$V_t = (A_t A_{t-1} ... A_{t-d+2}) E_{t-d+1} + \dots + A_t E_{t-1} + E_t.$$

4. *For every i the process $\{z_t^{(i)}\}$, obtained by taking every d-th Z_t, $z_t^{(i)} = Z_{i+td}$, is stationary multivariate AR(1) process.*

From *4* it follows, in particular, that for $k > 0$,

$$R_z(t, t - kd) = \alpha_t R_z(t - d, t - kd) = \alpha_t R_z(t, t - (k-1)d). \tag{3}$$

The autocovariance function of the dual Markovian process is closely related to that of the process $\{X_t\}$ and its properties are studied to the end of this section.

Let $S_t = E(E_t E_t^T)$. Using standard arguments we obtain that for every t the following equations hold for the autocovariance function $R_z(t, s)$ of $\{Z_t\}$

$$\begin{array}{lcl} R_z(t, t-1) & = & A_t R_z(t-1, t-1) \\ R_z(t, t) & = & A_t R_z(t-1, t) + S_t \end{array} \qquad t = 1, \dots, d \tag{4}$$

If the model is given, the solution of the system (4) gives the autocovariances of the process $\{X_t\}$. It shows also that for every t

$$R_z(t,t) \quad = \quad A_t R_z(t-1,t-1) A_t^T + S_t \tag{5}$$

Proposition 6 *$R_z(t,t)$ is a solution of the equation*

$$R_z(t,t) = \alpha_t R_z(t,t) \alpha_t^T + B_t \tag{6}$$

where

$$B_t = \sum_{i=1}^{d-1} \alpha_{ti} S_{t-i} \alpha_{ti}^T, \qquad \alpha_{ti} = \prod_{j=0}^{i-1} A_{t-j}, \qquad \alpha_t \equiv \alpha_{td-1}.$$

Note that $R_z(t,t)$ consists of autocovariances of the process $\{X_t\}$. Moreover, if $R_z(t,t)$ is known for some t then all other autocovariances of $\{X_t\}$ can be generated simply by the use of the Yule-Walker equations. Equation (6) can be rewritten in the form,

$$(I - \alpha_t \otimes \alpha_t)\mathrm{vec}R_z(t,t) = \mathrm{vec}B_t, \tag{7}$$

where the operator "vec" stacks the columns of a matrix one over another.

The second part of the following proposition generalises the corresponding result for stationary univariate and multivariate models (see [4]).

Proposition 7 *1. the eigenvalues of the matrices α_t, $t = 1, \ldots, d$, are the same.*

2. If for each pair of eigenvalues λ_i, λ_j of α_t the condition $|\lambda_i \lambda_j| \neq 1$ holds, then equation (6) has unique solution for every right-hand matrix B_t, and it is given by the formula

$$\mathrm{vec}R_z(t,t) = (I - \alpha_t \otimes \alpha_t)^{-1} \mathrm{vec}B_t.$$

If the eigenvalues of A lie inside the unit circle the solution can be expressed in the form

$$\mathrm{vec}R_z(t,t) = \sum_{k=0}^{\infty} \alpha_t^k B {\alpha_t'}^k = \sum_{i=0}^{\infty} (\alpha_t \otimes \alpha_t)^i \mathrm{vec}B_t.$$

From equation (3) we can see that $R_z(t, t-kd) = \alpha_t^k R_z(t-d, t-d)$. Therefore the following proposition holds.

Proposition 8 *$R_z(t, t-kd) \to 0$ when $k \to \infty$ if and only if all eigenvalues of α_t have modulus less than one.*

Since the α's are products of companion matrices it follows that the α's are nonsingular if and only if $p_i = m$ for $i = 1, \ldots, d$. It is natural to call the nonzero eigenvalues of α_t characteristic or eigennumbers of the periodic autoregression model. There are at most m of them.

5 Solutions of the PAR(p_1,..., p_d) equation

Using the results and notations from the previous sections we can establish the following theorem.

Theorem 9 *Let a PAR(p_1,..., p_d) model be given. If the eigenvalues of α_t are less than one in absolute value then the equation*

$$R_z(t,t) = \alpha_t R_z(t,t)\alpha_t' + B_t,$$

has unique solution. This solution gives the autocovariance function of a periodic autoregressive process.

Corollary 10 *If the eigenvalues of α_t are inside the unit circle then the equation $\phi_t(B)X_t = \varepsilon_t$, has unique solution and it is causal.*

Theorem 11 *The periodic autoregression model $(\phi_t(B), \{\varepsilon_t\})$ is causal if and only if the moduli of all eigenvalues of α_t are less than one.*

Theorem 12 *The equation $\phi_t(B)X_t = \varepsilon_t$ has periodically correlated solution if and only if the moduli of all eigenvalues of α_t are different from one. When such solution exists, it is unique.*

Proof. This result follows from the corresponding result for multivariate processes. But since this seems not to be easily available we give here a proof, which in addition throws additional light on the required solution.

Consider the Markovian dual process of $\{X_t\}$, as defined above

$$Z_t = \alpha_t Z_{t-d} + V_t.$$

Let the process $\{z_t\}$ be obtained by setting $z_t = Z_{dt}$. Similarly, $v_t = V_{dt}$. The process z_t is a multivariate AR(1) process. Below we will omit the index of α_t.

Let UJU^{-1} be the Jordan decomposition of α. Consider the equations

$$z_t = \alpha z_{t-1} + v_t, \qquad \text{and} \qquad y_t = Jy_{t-1} + w_t,$$

where $y_t = U^{-1}z_t$, $w_t = U^{-1}v_t$. If $\{z_t\}$ is a stationary solution of the first equation then $\{y_t\}$ is a stationary solution of the second one. For every eigenvalue λ of J there is a row, say i, in the last equation such that $y_{ti} = \lambda y_{t-1,i} + w_{ti}$, i.e. y_{ti} is (possibly complex) univariate autoregression. There exists a stationary solution for y_{ti} if and only if $|\lambda| \neq 1$. Since the components of a multivariate stationary process are themselves stationary, the same condition applies to z.

If the size of the Jordan cell for λ is $k > 1$, we must check that the condition is sufficient. For such a cell we have $y_{ti} = \lambda y_{t-1,i} + w_{ti}$, and

$$y_{t,i-j} = \lambda y_{t-1,i-j} + y_{t-1,i-j+1} + w_{t,i-j}$$

for $j = 0, \ldots, k-1$. From these equations it is easy to see that $y_{t,i-j}$, $j = 0, 1, \ldots, k-1$, are ARMA(j+1,j) processes with autoregressive parts $(1 - \lambda B)^{j+1}$, in which the root λ cannot be cancelled by the moving average parts. Therefore the condition $|\lambda| < 1$ is necessary and sufficient for the existence of the required solution. Q.E.D.

The above proof is a constructive one. Namely, we have for each Jordan cell $y_{ti} = \sum_{j=0}^{\infty} \lambda^j w_{t-j,i}$ for $|\lambda| < 1$, $y_{ti} = \sum_{j=0}^{\infty} \lambda^{-j} w_{t+j,i}$ for $|\lambda| > 1$, and if the size of the Jordan cell is $k > 1$, the remaining $y_{t,i-l}$, $l = 1, \ldots, k-1$ can be obtained recurrently from the formulas $y_{t,i-l} = \sum_{j=0}^{\infty} \lambda^j (y_{t-l+j,i-l+1} + w_{t-j,i-l})$ for $|\lambda| < 1$, $y_{t,i-l} = \sum_{j=0}^{\infty} \lambda^{-j} (y_{t-l+j,i-l+1} + w_{t+j,i-l})$ for $|\lambda| > 1$.

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