Deflating Quadratic Matrix Polynomials with Structure Preserving Transformations

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Quadratic Eigenvalue Problem (QEP)

Consider

$$Q(\lambda) = \lambda^2 M + \lambda C + K,$$

with $M, C, K \in \mathbb{R}^{n\times n}$ and M **nonsingular**.

QEP: Find scalars λ and nonzero $x, y \in \mathbb{C}^n$ satisfying $Q(\lambda)x = 0$ and $y^* Q(\lambda) = 0$.

- λ is an e'val, x, y are corresponding right and left e'vecs.
- $Q(\lambda)$ has $2n$ eigenvalues, solutions of $\det(Q(\lambda)) = 0$.
- No simple canonical form analogous to Schur form

$$W^*(A - \lambda B)Z = (S - \lambda T),$$

W, Z unitary, S, T upper triangular.

Linearization

- Standard way of treating QEPs both theoretically and numerically.
- Convert $Q(\lambda) = \lambda^2 M + \lambda C + K$ into a linear pencil such as

$$L(\lambda) = \lambda \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} D & K \\ -I & 0 \end{bmatrix}.$$

- $L(\lambda)$ is a **linearization** of Q if

$$E(\lambda)L(\lambda)F(\lambda) = \begin{bmatrix} Q(\lambda) & 0 \\ 0 & I \end{bmatrix}$$

  for some $E(\lambda)$ and $F(\lambda)$ with constant, nonzero determinants.

Recent Work

- Better understanding of linearization process and effects of scaling on
  - conditioning of eigenvalues,
  - backward error of computed eigenpairs.

  [Adhikari, Alam, Betcke, Higham, Kressner,Li, Mackey, Mehl, Mehrmann, T., . . .]

- New structure preserving linearizations derived along with algorithms preserving spectral properties in finite precision arithmetic.

  [Antoniou, Higham, Lin, Mackey, Mackey, Mehl, Mehrmann, T., Vologiannidis, . . .]

Our Objective

Let (λ_j, x_j), $j = 1, 2$ be two given eigenpairs of $Q(\lambda)$.

Want to transform $n \times n$ $Q(\lambda)$ into

$$\widetilde{Q}(\lambda) = \begin{array}{c} \left[\begin{array}{cc} Q_d(\lambda) & 0 \\ 0 & q(\lambda) \end{array}\right] \end{array} \begin{array}{c} n-1 \\ 1 \end{array}$$

such that

$$\begin{aligned} \Lambda(Q) &= \Lambda(\widetilde{Q}), \text{ (same spectrum)} \\ q(\lambda_j) &= 0, j = 1, 2. \end{aligned}$$

Deflation procedure decoupling $Q(\lambda)$ into two quadratics.

Decoupling by Similarity

Suppose there exist nonsingular S_L, S_R s.t.

$$S_L Q(\lambda) S_R = \widetilde{Q}(\lambda) = \begin{bmatrix} Q_d(\lambda) & 0 \\ 0 & q(\lambda) \end{bmatrix}.$$

The roots λ_1, λ_2 of $q(\lambda)$ are e'vals of Q and $\widetilde{Q}$

$$Q(\lambda_j)x_j = 0, \quad \widetilde{Q}(\lambda_j)e_n = 0, \quad j = 1, 2$$

with e'vecs related by

$$S_R^{-1} [\, x_1 \quad x_2 \,] = [\, e_n \quad e_n \,].$$

Decoupling possible only if e'vecs x_1 and x_2 are parallel.

Main Idea

- Work with a linearization $L(\lambda)$ of $Q(\lambda)$ rather than directly with Q.
- Use similarity transformations S_L, S_R that
  - preserve block structure of $L(\lambda) \Rightarrow S_L L(\lambda) S_R$ is still a linearization of a quadratic $Q_1(\lambda)$,
  - e'vecs of $Q_1(\lambda)$ associated with λ_1, λ_2 are parallel,
  - e'vecs of $Q(\lambda)$ are easily recovered from those of $Q_1(\lambda)$.

Structure Preserving Transformation (SPT)

$$L(\lambda) = \lambda \begin{bmatrix} 0 & M \\ M & C \end{bmatrix} + \begin{bmatrix} -M & 0 \\ 0 & K \end{bmatrix}$$ is a linearization of $Q(\lambda)$.

Definition

W_L, W_R nonsingular define an **SPT** for Q with nonsingular M if

$$W_L^T L(\lambda) W_R = \widetilde{L}(\lambda) = \lambda \begin{bmatrix} 0 & \widetilde{M} \\ \widetilde{M} & \widetilde{C} \end{bmatrix} + \begin{bmatrix} -\widetilde{M} & 0 \\ 0 & \widetilde{K} \end{bmatrix}.$$

- W_L, W_R preserve the block structure of $L(\lambda)$.
- $\widetilde{L}(\lambda)$ is a linearization of $\widetilde{Q}(\lambda) = \lambda^2 \widetilde{M} + \lambda \widetilde{C} + \widetilde{K}$.

Elementary SPTs

Let $\boxed{T = I_{2n} + \begin{bmatrix} ab^T & ad^T \\ af^T & ah^T \end{bmatrix}} \in \mathbb{R}^{2n\times 2n}, \quad a, b, d, f, h \in \mathbb{R}^n.$

- Rank-2 modification of I_{2n}.
- For almost all $a \in \mathbb{R}^n$, any solution $V = [\, b \quad d \quad f \quad h\,]$ to $VA = B$ defines an SPT T, $A \in \mathbb{R}^{4\times 3}$, $B \in \mathbb{R}^{n\times 3}$ depend on a, M, C and K.
- If $(M, C, K) \xrightarrow{T} (\widetilde{M}, \widetilde{C}, \widetilde{K})$ then $\widetilde{M}, \widetilde{C}, \widetilde{K}$ are low rank modifications of M, C, K.

Constraints

(λ_1, x_1), (λ_2, x_2) to be deflated with $\lambda_1 \neq \lambda_2$ and $x_1 \neq \alpha x_2$.

Aim: construct $T = I_{2n} + \left[\begin{smallmatrix} ab^T & ad^T \\ af^T & ah^T \end{smallmatrix}\right]$ and nonzero $z \in \mathbb{R}^n$ s.t.

- $Q(\lambda) \xrightarrow{T} \widetilde{Q}(\lambda)$ with $\widetilde{Q}(\lambda_j)z = 0$, $j = 1, 2$.
  - Yields a, z and $z^T [\,b \quad d \quad f \quad h\,] = z^T V$.
- T is an SPT $\Leftrightarrow$ $VA = B$.

Existence of SPT T

Theorem

Eigenpairs (λ_1, x_1), (λ_2, x_2) with $\lambda_1 \neq \lambda_2$ either real or complex conjugate can be mapped to (λ_1, z), (λ_2, z) by elementary SPTs if

- $x_j^T Q'(\lambda_j) x_j \neq 0$, $j = 1, 2$,
- *real eigenpairs have opposite type:*

$$\mathrm{sign}(x_1^T Q'(\lambda_1) x_1) = -\mathrm{sign}(x_2^T Q'(\lambda_2) x_2).$$

Can generate a family of SPTs mapping (λ_j, x_j) to (λ_j, z), $j = 1, 2$.

Deflation of $(\lambda_1, z), (\lambda_2, z)$

Lemma

If $(\lambda_j^2 M + \lambda_j C + K)z = 0$, $j = 1, 2$ *with* $\lambda_1 \neq \lambda_2$ *then*

$$(M, C, K)z = (mp, cp, kp), \ p \in \mathbb{R}^n, \quad p^T z = 1,$$

$$c = -m(\lambda_1 + \lambda_2), \quad k = m\lambda_1\lambda_2.$$

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Let nonsingular G be such that

$$Ge_n = z, \quad G^T p = e_n.$$

Then $G^T M G e_n = G^T M z = m G^T p = m e_n$ and Lemma $\Rightarrow$

$$G^T(M, C, K)G = \left(\begin{bmatrix} \widetilde{M} & 0 \\ 0 & m \end{bmatrix}, \begin{bmatrix} \widetilde{C} & 0 \\ 0 & c \end{bmatrix}, \begin{bmatrix} \widetilde{K} & 0 \\ 0 & k \end{bmatrix} \right).$$

Example 1

$$Q(\lambda) = \lambda^2 \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

Given $\lambda_{1,2} = -0.34 \pm 1.84i$ and associated e'vecs, our deflation procedure yields

$$\lambda^2 \begin{bmatrix} 5.6 & 2.0\text{e-}16 \\ 2.0\text{e-}16 & -1.4\text{e-}1 \end{bmatrix} + \lambda \begin{bmatrix} -1.6 & -9.4\text{e-}16 \\ -9.4\text{e-}16 & -9.3\text{e-}2 \end{bmatrix} + \begin{bmatrix} 1.6 & -9.8\text{e-}17 \\ -9.8\text{e-}17 & -4.8\text{e-}1 \end{bmatrix},$$

with $\kappa_2(T) = 7.9$ and $\kappa_2(G) \approx 1$.

Decoupling accomplished to within the working precision.

Example 2: Damped Beam Problem

M, C, K generated by `nlevp('damped_beam',nele)`.

$Q(\lambda) = \lambda^2 M + \lambda C + K$ and undamped $Q_u(\lambda) = \lambda^2 M + K$ have n e'vals in common that we deflate by

- our decoupling procedure: $Q \xmapsto{S} \begin{bmatrix} Q_1(\lambda) & 0 \\ 0 & Q_2(\lambda) \end{bmatrix}$,
- using special property of $Q(\lambda)$ to orthogonally block diag'lize it and then diag'lize one block with Cholesky-QR (transformation W).

n	$\kappa_2(S)$	$\kappa_2(W)$
16	4.47e1	3.79e1
32	9.57e1	7.84e1
64	1.95e2	1.57e2

$$\kappa_2(E) = \|E\|_2 \|E^{-1}\|_2$$

$\kappa_2(S)$ not much larger that $\kappa_2(W)$.

Concluding Remarks

- Deflation procedure extends to nonsymmetric quadratics.
- First attempt at defining an SPT with a well-defined action.
- Deflation procedure finds application in
  - second-order model reduction,
  - model updating with no spill-over.

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