On Structured Condition Numbers in Scalar Product Spaces

Françoise Tisseur
School of Mathematics
The University of Manchester

`ftisseur@ma.man.ac.uk`
`http://www.ma.man.ac.uk/~ftisseur/`

Joint work with Stef Graillat, Michiel Karow, and Daniel Kressner
Thanks to Nil and Steve Mackey

GAMM-SIAM Conf. Applied Linear Algebra, DĄsseldorf, July 24–27, 2006.

Structured Condition Number

Let $\mathbb{S}$ be a class of structured matrices. Define structured condition number of a simple e'val λ of $A \in \mathbb{S}$ by

$$\kappa(A, \lambda; \mathbb{S}) = \lim_{\epsilon \to 0} \sup \Big\{ \frac{|\widehat{\lambda} - \lambda|}{\epsilon} : \widehat{\lambda} \in \mathrm{Sp}(A + E), \\ A + E \in \mathbb{S},\ \|E\| \le \epsilon \Big\}.$$

Clearly, $\kappa(A, \lambda; \mathbb{S}) \le \kappa(A, \lambda) \equiv \kappa(A, \lambda; \mathbb{C}^{n \times n})$.

For normalized right and left e'vecs, x and y, resp.,

$$\kappa(A, \lambda; \mathbb{S}) = \frac{1}{|y^* x|} \lim_{\epsilon \to 0} \sup \Big\{ \frac{|y^* E x|}{\epsilon} :\ A + E \in \mathbb{S},\ \|E\| \le \epsilon \Big\}.$$

Structure Forms a Smooth Manifold

Recall

$$\kappa(A,\lambda;\mathbb{S}) = \frac{1}{|y^*x|}\limsup_{\epsilon\to 0}\left\{\frac{|y^*Ex|}{\epsilon} :\ A+E\in\mathbb{S},\ \|E\|\le\epsilon\right\}.$$

Theorem (Karow, Kressner, T. 2006)

Suppose $\mathbb{S}$ *is a smooth submanifold of* $\mathbb{K}^{n\times n}$ $(\mathbb{K}=\mathbb{R}$ *or* $\mathbb{C})$*. Then for any* $\|\cdot\|$ *on* $\mathbb{K}^{n\times n}$*,*

$$\kappa(A,\lambda;\mathbb{S}) = \frac{1}{|y^*x|}\max\left\{|y^*Hx| :\ H\in T_A\mathbb{S},\ \|H\|=1\right\},$$

where $T_A\mathbb{S}$ *is the tangent space of* $\mathbb{S}$ *at* A*.*

Tangent Space $T_A\mathbb{S}$

Suppose $A \in \mathbb{S} \iff F(A) = 0$ for some smooth F. Then

$$T_A\mathbb{S} = \{\, X \in \mathbb{K}^{n\times n} :\ J_A(X) = 0 \,\},$$

where J_A is the Fréchet derivative of F at A.

$T_A\mathbb{S}$ is a linear vector space.

Hence

$$\kappa(A,\lambda;\mathbb{S}) = \frac{1}{|y^*x|} \max\{|y^*Hx| :\ H \in T_A\mathbb{S},\ \|H\| = 1\}$$

is a linearly constrained optimization problem.

Kronecker Product Approach

$T_A\mathbb{S}$ linear vector space of dim $m \le n^2 \Rightarrow$
there is **pattern matrix** $B \in \mathbb{K}^{n^2\times m}$ s. t.

$$\boxed{E \in T_A\mathbb{S} \iff \mathrm{vec}(E) = B\,p, \quad \|E\|_F = \|p\|_2.}$$

Parameter vector $p \in \mathbb{K}^m$ uniquely defined.

When $\mathbb{K} = \mathbb{C}$ and $\|\cdot\| = \|\cdot\|_F$,

$$\begin{aligned}\kappa_F(A,\lambda;\mathbb{S}) &= \frac{1}{|y^*x|}\max\Big\{|y^*Ex| : \; E \in T_A\mathbb{S}, \; \|E\| = 1\Big\}\\ &= \frac{1}{|y^*x|}\|(\overline{x}\otimes y)^*B\|_2.\end{aligned}$$

Kronecker Product Formula

For smooth manifold $\mathbb{S}$,

$$\kappa_F(A,\lambda;\mathbb{S}) = \frac{1}{|y^*x|}\|(\overline{x}\otimes y)^*B\|_2,$$

where B is a pattern matrix for $T_A\mathbb{S}$.

Difficulties:

- Characterize tangent space $T_A\mathbb{S}$.
- Build pattern matrix B.
- Compare $\kappa_F(A,\lambda;\mathbb{S})$ to Wilkinson's $\kappa_F(A,\lambda) = \frac{1}{|y^*x|}$.

Noschese & Pasquini (2006) show that for
$\mathbb{S} =$ set of matrices with some zero structure,

$$\kappa_F(A,\lambda;\mathbb{S}) = \|(yx^*)|_{\mathbb{S}}\|_F/|y^*x|.$$

Scalar Product Spaces

Given nonsingular M and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$,

$$\langle x, y \rangle_{M} = \begin{cases} x^T M y, & \text{real or complex bilinear forms,} \\ x^* M y, & \text{sesquilinear forms.} \end{cases}$$

Recall adjoint $A^\star$ of $A \in \mathbb{K}^{n\times n}$ wrt $\langle \cdot, \cdot \rangle_{M}$ defined by

$$\langle Ax, y \rangle_{M} = \langle x, A^\star y \rangle_{M} \quad \forall x, y \in \mathbb{K}^{n\times n}.$$

Can show: $$A^\star = \begin{cases} M^{-1} A^T M, & \text{for bilinear forms,} \\ M^{-1} A^* M, & \text{for sesquilinear forms.} \end{cases}$$

Structured Classes Associated with $\langle \cdot, \cdot \rangle$

- $\langle Ax, Ay \rangle = \langle x, y \rangle, \forall x, y \in \mathbb{K}^n$.
  Automorphisms or isometries of $\langle \cdot, \cdot \rangle$.
- $\langle Kx, y \rangle = -\langle x, Ky \rangle, \forall x, y \in \mathbb{K}^n$.
  Skew-adjoint with respect to $\langle \cdot, \cdot \rangle$.
- $\langle Sx, y \rangle = \langle x, Sy \rangle, \forall x, y \in \mathbb{K}^n$.
  Self-adjoint with respect to $\langle \cdot, \cdot \rangle$.

In terms of adjoint:

$$\begin{array}{lcll} \mathbb{G} & = & \{A \in \mathbb{K}^{n\times n} : A^\star = A^{-1}\} & \text{Automorphism group,} \\ \mathbb{L} & = & \{K \in \mathbb{K}^{n\times n} : K^\star = -K\} & \text{Lie algebra,} \\ \mathbb{J} & = & \{S \in \mathbb{K}^{n\times n} : S^\star = S\} & \text{Jordan algebra.} \end{array}$$

Familiar Classes

Space	$\langle x, y\rangle$	$\mathbb{G}$	$\mathbb{J}$	$\mathbb{L}$
Bilinear forms				
$\mathbb{R}^n$	$x^T y$	Real orthog	Symm	Skew-symm
$\mathbb{C}^n$	$x^T y$	Cplx orthog	Cplx symm	Cplx skew-symm
$\mathbb{R}^n$	$x^T R y$	Real perplectics	Persymmetrics	Perskew-symm
$\mathbb{R}^n$	$x^T \Sigma_{p,q} y$	Pseudo-orthog	Pseudo symm	Pseudo skew-symm
$\mathbb{R}^{2n}$	$x^T J y$	Real symplectics	Skew-Hamil	Hamiltonians

Sesquilinear forms

$\mathbb{C}^n$	$x^* y$	Unitaries	Herm	Skew-Herm
$\mathbb{C}^n$	$x^* \Sigma_{p,q} x$	Pseudo unitaries	Pseudo Herm	Pseudo skew-Herm
$\mathbb{C}^{2n}$	$x^T J y$	Conj symplectics	J-skew-Herm	J-Herm

$$R = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \qquad \Sigma_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$$

Orthosymmetric and Unitary Scalar Products

Scalar products defining structures in Table are all *orthosymmetric* and *unitary*.

A scalar product $\langle\cdot,\cdot\rangle_{M}$ is **unitary** if αM is unitary for some $\alpha > 0$.

A scalar product is said to be **orthosymmetric** if

$$M = \begin{cases} \beta M^T, \ \beta = \pm 1, & \text{for bilinear forms,} \\ \beta M^*, \ |\beta| = 1, & \text{for sesquilinear forms.} \end{cases}$$

Refer to D. S. Mackey, N. Mackey & T. (2006) for a list of equivalent properties.

Jordan Algebra $\mathbb{J}$ and Lie Algebra $\mathbb{L}$

Let $\mathbb{S} = \mathbb{J}$ or $\mathbb{L}$ with $\langle \cdot, \cdot \rangle_{M}$ orthosymm. and unitary.

Theorem (D. S. Mackey, N. Mackey, T. 2006)

Let $x \neq 0$, $b \in \mathbb{K}^n$ of unit 2-norm s.t. the conditions below hold:

$\mathbb{S}$	**Bilinear forms**		**Sesquilinear forms**
	Symmetric	*Skew-symm.*	*Hermitian*
$\mathbb{J}$	*always*	$b^T M x = 0$	$b^* M x \in \mathbb{R}$
$\mathbb{L}$	$b^T M x = 0$	*always*	$b^* M x \in \imath\mathbb{R}$

Then there exists $E \in \mathbb{S}$ such that $Ex = b$ with $\|E\|_2 = 1$ and $\|E\|_F \leq \sqrt{2}$.

Jordan Algebras $\mathbb{J}$

Assume $\langle\cdot,\cdot\rangle_{\mathrm{M}}$ is orthosymm. and unitary on $\mathbb{K}^n$.
Note that tangent space $T_A\mathbb{J} = \mathbb{J}$.

Theorem

For a simple e'val λ of $A \in \mathbb{J}$ we have

$$\kappa_2(A,\lambda;\mathbb{J}) = \kappa_2(A,\lambda).$$

This holds for symm., complex symm., persymm., pseudosymm., Hermitian, Pseudo-Hermitian and J-skew-Hermitian structures.

Lie Algebras $\mathbb{L}$

Assume $\langle\cdot,\cdot\rangle_{\mathrm{M}}$ is orthosymm. and unitary on $\mathbb{C}^n$.

Theorem

Let λ be a simple e'val of $A \in \mathbb{L}$. Then

- *for symm. bilinear forms $\langle\cdot,\cdot\rangle_{\mathrm{M}}$,*

$$\kappa_2(A,\lambda;\mathbb{L}) = \Big(\max_{\substack{b\in(\overline{Mx})^{\perp} \\ \|b\|_2=1}} |y^* b| \Big)\, \kappa_2(A,\lambda),$$

- *for skew-symm. bilinear forms or sesquilinear forms,*

$$\kappa_2(A,\lambda;\mathbb{L}) = \kappa_2(A,\lambda).$$

Lie Algebras $\mathbb{L}$ Cont.

$$\kappa_2(A,\lambda;\mathbb{L}) = \Bigl(\max_{\substack{b\in(\overline{M}x)^{\perp} \\ \|b\|_2=1}} |y^* b| \Bigr)\, \kappa_2(A,\lambda) \quad \text{(symm. bilinear).}$$

Can show that

$$|y^* x| \le \max_{\substack{b\in(\overline{M}x)^{\perp} \\ \|b\|_2=1}} |y^* b| \le 1 \quad \text{for } \lambda \ne 0$$

and

$$\kappa_2(A,0;\mathbb{L}) = 0 < \kappa_2(A,0).$$

Generalizes Rump's result on skew-symm. matrices.

Lie Group $\mathbb{G} = \{A \in \mathbb{K}^{n\times n} : A^\star = A^{-1}\}$

Structure is now nonlinear.

- Tangent space.

$$T_A\mathbb{G} = A \cdot \mathbb{L}.$$

- Pattern matrix B for $T_A\mathbb{G}$:
  - Form pattern matrix for $\mathbb{L}$.
  - $(I \otimes A)L = QR$, (QR factorization),
  - $B = (I \otimes A)LR^{-1}$.

Yields computable formula:

$$\kappa_F(A, \lambda; \mathbb{G}) = \frac{|\lambda|}{|y^* x|} \|(\overline{x} \otimes y)^* LR^{-1}\|_2.$$

Bounds for Automorphism Groups $\mathbb{G}$

Assume $\langle\cdot,\cdot\rangle_{M}$ is orthosymm. and unitary on $\mathbb{C}^n$.

Theorem

For a simple e'val λ of $A \in \mathbb{G}$ and $\nu = 2, F$,

- *for symmetric bilinear forms,*

$$\frac{|\lambda|}{\|A\|_2} \max_{\substack{b\in(\overline{Mx})^\perp \\ \|b\|_2=1}} |y^*b| \le \frac{\kappa_\nu(A,\lambda;\mathbb{G})}{\kappa_\nu(A,\lambda)} \le \max_{\substack{b\in(\overline{Mx})^\perp \\ \|b\|_2=1}} |y^*b|,$$

- *for skew-symmetric bilinear or sesquilinear forms,*

$$\frac{|\lambda|}{\|A\|_2} \le \frac{\kappa_\nu(A,\lambda;\mathbb{G})}{\kappa_\nu(A,\lambda)} \le 1.$$

Example: $\mathbb{G}$ is the Real Symplectic Group

From Thm 5, $|\lambda| \approx \|A\|_2 \Rightarrow \kappa_F(A,\lambda;\mathbb{G}) \approx \kappa_F(A,\lambda)$.

Can show that $|\lambda|^2 \kappa_F(A,1/\lambda;\mathbb{G}) \approx \kappa_F(A,1/\lambda)$.

Take $A = \left[\begin{smallmatrix} D & D \\ 0 & D^{-1} \end{smallmatrix}\right]$ with $D = \mathrm{diag}(10^4, 10^2, 2)$.

λ	10^4	10^2	2	$1/2$	10^{-2}	10^{-4}
$\kappa_F(A,\lambda;\mathbb{G})$	1.2	1.2	1.5	0.4	1.2×10^{-4}	1.2×10^{-8}
ρ	0.87	0.87	0.89	0.22	8.7×10^{-5}	8×10^{-9}
γ	0.5	5×10^{-3}	10^{-4}	2×10^{-5}	5×10^{-7}	5×10^{-9}

Here γ is a lower bound on $\rho = \kappa_F(A,\lambda;\mathbb{G})/\kappa_F(A,\lambda) \le 1$.

Comments

- ★ Our treatment extends and unifies previous work.
- ★ Considered nonlinear structures.
- ★ Extend to other problems, e.g.,
  - Solution to linear system
  - Matrix inversion
  - Distance to singularity.
- ★ Structured b'err
  - Can be infinite
  - Is within a small factor of unstructured one
  - Difficult to analyse for nonlinear structures.