Structured Mapping Problems for Automorphism Groups and Lie and Jordan Algebras

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Structured Mapping Problems

Given a class of structured matrices $\mathbb{S}$,

1: For which vectors x, b is it possible to find some $A \in \mathbb{S}$ such that $Ax = b$?

2: Determine the set $\mathcal{S} = \{A \in \mathbb{S} : \; Ax = b\}$.

Motivation: *Structured backward errors for structured linear systems and structured eigenproblems.*

Structured Backward Errors

Let (x, λ) be an approximate eigenpair of A.

The backward error of (x, λ) is a measure of the smallest perturbation E such that

$$(A + E)x = \lambda x \iff Ex = r,$$

where $r = (\lambda I - A)x$.

The structured backward error is

$$\min\{\|E\| : Ex = r, E \in \mathbb{S}\}.$$

Background

Given nonsingular M and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$,

$$\langle x, y \rangle_{\mathrm{M}} = \begin{cases} x^T M y, & \text{real or complex bilinear forms,} \\ x^* M y, & \text{sesquilinear forms.} \end{cases}$$

Recall adjoint $A^\star$ of $A \in \mathbb{K}^{n\times n}$ wrt $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ defined by

$$\langle Ax, y \rangle_{\mathrm{M}} = \langle x, A^\star y \rangle_{\mathrm{M}} \quad \forall x, y \in \mathbb{K}^n.$$

The adjoint with respect to $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ is involutory if

$$(A^\star)^\star = A, \quad \forall A \in \mathbb{K}^{n\times n}.$$

Structured Matrices

Associated to $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ in $\mathbb{K}^n$ are three important classes of matrices:

- Automorphism group:

$$\mathbb{G} = \left\{ G \in \mathbb{K}^{n\times n} : \langle Gx, Gy \rangle_{\mathrm{M}} = \langle x, y \rangle_{\mathrm{M}} \right\} = \left\{ G : G^{\star} = G^{-1} \right\}.$$

- Jordan algebra

$$\mathbb{J} = \left\{ S \in \mathbb{K}^{n\times n} : \langle Sx, y \rangle_{\mathrm{M}} = \langle x, Sy \rangle_{\mathrm{M}} \right\} = \left\{ S : S^{\star} = S \right\}.$$

- Lie algebra

$$\mathbb{L} = \left\{ L \in \mathbb{K}^{n\times n} : \langle Lx, y \rangle_{\mathrm{M}} = -\langle x, Ly \rangle_{\mathrm{M}} \right\} = \left\{ L : L^{\star} = -L \right\}.$$

Consider $A \in \mathbb{G}, \mathbb{J}$ or $\mathbb{L}$ arising from orthosymmetric $\langle \cdot, \cdot \rangle_{\mathrm{M}}$.

Orthosymmetric Scalar Products

$\langle \cdot, \cdot \rangle_{\mathrm{M}}$ is orthosymmetric iff it satisfies one of the following properties:

1. $\langle x, y \rangle_{\mathrm{M}} = 0 \;\Leftrightarrow\; \langle y, x \rangle_{\mathrm{M}} = 0, \; \forall x, y \in \mathbb{K}^n$.
2. $\mathbb{K}^{n \times n} = \mathbb{L} \oplus \mathbb{J}$.
3. $M^T = \pm M$ for bilinear forms; $M^* = \alpha M$ with $\alpha \in \mathbb{C}, \; |\alpha| = 1$ for sesquilinear forms.
4. $(A^\star)^\star = A, \; \forall A \in \mathbb{K}^{n \times n}$.

Up to a scalar multiple there are

only three distinct types of orthosymmetric $\langle \cdot, \cdot \rangle_{\mathrm{M}}$:

symmetric and skew-symmetric bilinear, and Hermitian sesquilinear.

Some Structured Matrices

Space	M	**Automorphism Group**	**Jordan Algebra**	**Lie Algebra**
Symmetric bilinear forms				
$\mathbb{C}^n$	I	Complex orthog.	Complex symm.	Complex skew-symm.
$\mathbb{R}^n$	R	Real perplectic	Persymmetric	Perskew-symm.
$\mathbb{R}^n$	$\Sigma_{p,q}$	Pseudo-orthog.	Pseudo symm.	Pseudo skew-symm.
Skew-symmetric bilinear forms				
$\mathbb{R}^{2n}$	J	Real symplectic	Skew-Hamiltonian.	Hamiltonian
Hermitian sesquilinear form				
$\mathbb{C}^n$	$\Sigma_{p,q}$	Pseudo unitary	Pseudo Hermitian	Pseudo skew-Herm.
Skew-Hermitian sesquilinear form				
$\mathbb{C}^{2n}$	J	Conj. symplectic	J-skew-Hermitian	J-Hermitian

$$R = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \qquad \Sigma_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$$

Existence Problem for Lie/Jordan Algebras

Given a class of structured matrices $\mathbb{S} = \mathbb{J}, \mathbb{L}$,

For which nonzero vectors x, b is it possible to find some $A \in \mathbb{S}$ such that $Ax = b$?

Results known for:

- Symmetric and Hermitian matrices.
  [J.-G. Sun, Numer. Math. (65):1993]

- Persymmetric and skew-symmetric matrices.
  [S. Rump, SIMAX (25):2003]

Existence Results for Lie/Jordan Algebras

- Symmetric matrices: $\mathbb{J} = \{A \in \mathbb{R}^{n\times n} : A = A^T\}$,
  Persymmetric matrices: $\mathbb{J} = \{A \in \mathbb{R}^{n\times n} : RA = (RA)^T\}$,

  $$\{A \in \mathbb{J} : Ax = b\} \neq \emptyset \text{ for all } x, b \in \mathbb{R}^n,\ x \neq 0.$$

- Skew-symmetric matrices: $\mathbb{L} = \{A \in \mathbb{R}^{n\times n} : A = -A^T\}$.

  $$b^T x = 0 \Rightarrow \{A \in \mathbb{L} : Ax = b, \} \neq \emptyset.$$

- Hermitian matrices: $\mathbb{J} = \{A \in \mathbb{C}^{n\times n} : A = A^*\}$.

  $$\{A \in \mathbb{J} : Ax = b\} \neq \emptyset \Leftrightarrow x^*b \in \mathbb{R}.$$

Extend these results to all $\mathbb{L}$ and $\mathbb{J}$ of any orthosymmetric $\langle\cdot,\cdot\rangle_{\mathrm{M}}$.

Existence Theorem for Jordan Algebras

$\mathbb{J}$: Jordan algebra of any orthosymmetric $\langle\cdot,\cdot\rangle_{\mathrm{M}}$ on $\mathbb{K}^n$, x, b given vectors in $\mathbb{K}^n$, $x \neq 0$, define

$$\mathcal{J} = \{A \in \mathbb{J} : Ax = b\}.$$

Theorem 1

$$\mathcal{J} \neq \emptyset \Leftrightarrow \begin{cases} x, b \in \mathbb{K}^n,\ x \neq 0 & \textit{(symmetric bilinear)} \\ \langle b, x\rangle_{\mathrm{M}} = 0 & \textit{(skew-symmetric bilinear)} \\ \langle b, x\rangle_{\mathrm{M}} \in \mathbb{R} & \textit{(Hermitian sesquilinear)} \\ \langle b, x\rangle_{\mathrm{M}} \in i\,\mathbb{R} & \textit{(skew-Hermitian sesquilinear).} \end{cases}$$

Proof. $(\Rightarrow)$ In all cases we have

$$A \in \mathbb{J} \Rightarrow \langle b, x\rangle_{\mathrm{M}} = \langle Ax, x\rangle_{\mathrm{M}} = \langle x, Ax\rangle_{\mathrm{M}} = \langle x, b\rangle_{\mathrm{M}}.$$

Proof of Theorem 1 ($\Leftarrow$)

Let $B := \begin{cases} bw^T M + (bw^T M)^\star P & \text{(bilinear)}, \\ bw^* M + (bw^* M)^\star P & \text{(sesquilinear)}, \end{cases}$

where $P := \begin{cases} I - xw^T M & \text{(bilinear)}, \\ I - xw^* M & \text{(sesquilinear)}, \end{cases}$

and $w \in \mathbb{K}^n$ s.t. $\langle w, x \rangle_{\mathrm{M}} = 1$.

$Bx = b$ always holds.

$$\left.\begin{array}{ll} \text{Symmetric bilinear:} & x, b \in \mathbb{K}^n,\ x \neq 0 \\ \text{Skew-symmetric bilinear:} & \langle b, x \rangle_{\mathrm{M}} = 0 \\ \text{Hermitian sesquilinear:} & \langle b, x \rangle_{\mathrm{M}} \in \mathbb{R} \\ \text{Skew-Hermitian sesquilinear:} & \langle b, x \rangle_{\mathrm{M}} \in i\,\mathbb{R}. \end{array}\right\} \Rightarrow B \in \mathbb{J}.$$

Characterization Results for $\mathbb{L}$

For given $x, b \in \mathbb{K}^n$, $x \neq 0$ determine $\mathcal{S} = \{A \in \mathbb{J} : \ Ax = b\}$.

- Symmetric: $\mathbb{J} = \{A \in \mathbb{R}^{n\times n} : A = A^T\}$ [Sun,93]

$$\mathcal{S} = \left\{ \frac{bx^T}{x^Tx} + \frac{x^Tb}{x^Tx} P_x + P_x H P_x, \ \ H \in \mathbb{J} \right\}, P_x = I - \frac{xx^T}{x^Tx}.$$

Same characterization for complex symmetric matrices when x non-isotropic.

Homogeneous Problem

$\mathbb{J}$: Jordan algebra of an orthosymmetric $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ on $\mathbb{K}^n$.
x, b given vectors in $\mathbb{K}^n$, $x \neq 0$ and $w \in \mathbb{K}^n$ s.t. $\langle w, x \rangle_{\mathrm{M}} = 1$.

Lemma 1 $\{A \in \mathbb{J} : Ax = 0\} = \{P^\star S P : S \in \mathbb{J}\},$

where

$$P = \begin{cases} I - xw^T M & \textit{(bilinear)}, \\ I - xw^* M & \textit{(sesquilinear)}. \end{cases}$$

Hence, if $A, B \in \mathbb{J}$ s.t. $Ax = Bx = b$, then $(A - B)x = 0$ and Lemma 1 $\Rightarrow$

$$A = B + P^\star S P \text{ for some } S \in \mathbb{J}.$$

Mapping Problem for Jordan Algebras

$\mathbb{J}$: Jordan algebra of an orthosymmetric $\langle\cdot,\cdot\rangle_{\mathrm{M}}$ on $\mathbb{K}^n$.
x, b given vectors in $\mathbb{K}^n$, $x \neq 0$ and $w \in \mathbb{K}^n$ s.t. $\langle w, x\rangle_{\mathrm{M}} = 1$.

Theorem 2 *If* $\{A \in \mathbb{J} : Ax = b\} \neq \emptyset$ *then*

$$\{A \in \mathbb{J} : Ax = b\} = \{B + P^{\star}SP : S \in \mathbb{J}\},$$

where

$$B = \begin{cases} bw^T M + (bw^T M)^{\star} P & \textit{(bilinear)}, \\ bw^* M + (bw^* M)^{\star} P & \textit{(sesquilinear)}, \end{cases}$$

$$P = \begin{cases} I - xw^T M & \textit{(bilinear)}, \\ I - xw^* M & \textit{(sesquilinear)}. \end{cases}$$

Existence Theorem for Automorphism Groups

$\mathbb{G}$: automorphism group of an orthosymmetric $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ on $\mathbb{K}^n$.
x, b: given nonzero vectors in $\mathbb{K}^n$.

Theorem 3

$$\{A \in \mathbb{G} : Ax = b\} \neq \emptyset \Longleftrightarrow q(x) = q(b),$$

where $q(z) = \langle z, z \rangle_{\mathrm{M}}$.

Proof.
$(\Rightarrow)$ $A \in \mathbb{G}$ s.t. $Ax = b \Rightarrow q(x) = q(Ax) = q(b)$.

$(\Leftarrow)$ Nontrivial.

$\mathbb{G}$-Reflectors

A $\mathbb{G}$-reflector is a matrix in $\mathbb{G}$ of the form

$$(1) \qquad G = \begin{cases} I + \beta u u^T M & \text{(bilinear)}, \\ I + \beta u u^* M & \text{(sesquilinear)}, \end{cases}$$

where $0 \neq \beta \in \mathbb{K}$ and $0 \neq u \in \mathbb{K}^n$.

Theorem 4 ($\mathbb{G}$-Reflector Mapping Theorem) *For distinct nonzero $x, b \in \mathbb{K}^n$, there exists a $\mathbb{G}$-reflector G s.t. $Gx = b$ iff $q_{\mathrm{M}}(x) = q_{\mathrm{M}}(b)$ and $\langle b - x, x \rangle_{\mathrm{M}} \neq 0$.*
When G exists, it is unique and is specified by taking $u = b - x$ and $\beta = 1/\langle u, x \rangle_{\mathrm{M}}$ in (1).

Two $\mathbb{G}$-Reflector Mapping Theorem

Theorem 5 *For any* $0 \neq x, b \in \mathbb{K}^n$ *s.t.* $q(x) = q(b)$*, there exists* $A \in \mathbb{G}$ *such that* $Ax = b$*, where* A *is the product of at most two* $\mathbb{G}$*-reflectors.*

Proof. Define $\mathcal{F}_x := \{ y \in \mathbb{K}^n, \ \langle y - x, x \rangle = 0 \}$.

$b \notin \mathcal{F}_x$: $\exists$ single $\mathbb{G}$-reflector mapping x to b.

$b \in \mathcal{F}_x$: need a third stepping point z such that $q(z) = q(x) = q(b)$ and $z \notin \mathcal{F}_x \cup \mathcal{F}_b$.

$$x \xrightarrow{G_1} z \xrightarrow{G_2^{-1}} b.$$

Can show that there always exists such z.

Isotropic/Non-Isotropic Vectors

- x is non-isotropic $\iff x \notin x^{\perp}_{\mathrm{M}} \iff \mathbb{K}^n = \mathrm{span}\langle x\rangle \oplus x^{\perp}_{\mathrm{M}}$.
- x is isotropic $\iff x \in x^{\perp}_{\mathrm{M}} \iff \mathrm{span}\langle x\rangle \subseteq x^{\perp}_{\mathrm{M}}$,

Lemma 2 *Suppose* $A \in \mathbb{G}$ *and* $Ax = b$. *Then* $A(x^{\perp}_{\mathrm{M}}) = b^{\perp}_{\mathrm{M}}$.

$$\begin{tikzcd}
 & \mathrm{span}\langle x\rangle \arrow[r, "A"] & \mathrm{span}\langle b\rangle & \\
\mathbb{K}^n = & \oplus & \oplus & = \mathbb{K}^n \\
 & x^{\perp}_{\mathrm{M}} \arrow[r, "A"'] & b^{\perp}_{\mathrm{M}} &
\end{tikzcd}$$

Non-Isotropic Case

$\mathbb{G}$: automorphism group of an orthosymmetric $\langle\cdot,\cdot\rangle_{\mathrm{M}}$ on $\mathbb{K}^n$.
x, b: given nonzero vectors in $\mathbb{K}^n$.

Theorem 6 *If* $q(x) = q(b) \neq 0$ *then*

$$\{A \in \mathbb{G} : Ax = b\} = \{S_{b,x} + Q(I - P_x) : Q \in \mathbb{G}\ ,\ QP_x = P_bQ\},$$

where

$$P_y := \begin{cases} yy^TM/q(y) & \textit{(bilinear)}, \\ yy^*M/q(y) & \textit{(sesquilinear)}, \end{cases}$$

$$S_{b,x} := \begin{cases} bx^TM/q(x) & \textit{(bilinear)}, \\ bx^*M/q(x) & \textit{(sesquilinear)}. \end{cases}$$

Isotropic Case

Restrict to particular orthosymmetric $\langle \cdot, \cdot \rangle_{\mathrm{M}}$:
M must be either

1. real symmetric orthogonal,
2. skew-symmetric,
3. Hermitian unitary,
4. skew-Hermitian unitary.

Useful property:

$$q_{\mathrm{M}}(z) = 0 \Leftrightarrow \begin{cases} q_{\mathrm{M}^{-1}}(\bar{z}) = 0 & \text{(bilinear)}, \\ q_{\mathrm{M}^{-1}}(z) = 0 & \text{(sesquilinear)}. \end{cases}$$

Modified Structured Problem

For given nonzero $x, b \in \mathbb{K}^n$ *s.t.* $q(x) = q(b) = 0$, *determine the "slightly" modified set*

$$\widetilde{\mathcal{G}} = \{A \in \mathbb{G} : \ Ax = b, \ A(M^{-1}\bar{x}) = \left(\frac{x^*x}{b^*b}\right)(M^{-1}\bar{b})\}.$$

We need rank-2 matrices:

$$\widetilde{P}_x := \frac{xx^* + (xx^*)^\star}{x^*x}, \qquad \widetilde{S}_{b,x} := \frac{bx^*}{x^*x} + \frac{(xb^*)^\star}{b^*b}.$$

Can show that

$$\widetilde{\mathcal{G}} = \{\widetilde{S}_{b,x} + Q(I - \widetilde{P}_x) : Q \in \mathbb{G}, Q\widetilde{P}_x = \widetilde{P}_b Q\}.$$

Isotropic Case

Let $0 \neq x, b \in \mathbb{K}^n$ be given vectors s.t. $q(x) = q(b) = 0$ and define

$$\mathcal{G} := \{A \in \mathbb{G} : \ Ax = b\}.$$

Can show that for any $A \in \mathcal{G}$, there exist two $\mathbb{G}$-reflectors G_1,G_2 s. t. $\widetilde{A} = G_2 G_1 A$ satisfies

$$\widetilde{A}x = b \quad \text{and} \quad \widetilde{A}(M^{-1}\bar{x}) = \left(\frac{x^*x}{b^*b}\right)(M^{-1}\bar{b}).$$

Then

$$\mathcal{G} = \{{G_1}^{-1}{G_2}^{-1}\left(\widetilde{S}_{b,x} + Q(I - \widetilde{P}_x)\right) : Q \in \mathbb{G}\ , \quad Q\widetilde{P}_x = \widetilde{P}_b Q\},$$

where
$$\widetilde{P}_x := \frac{xx^* + (xx^*)^\star}{x^*x}, \qquad \widetilde{S}_{b,x} := \frac{bx^*}{x^*x} + \frac{(xb^*)^\star}{b^*b}.$$

Summary

Given a class of structured matrices $\mathbb{S}$,

(1): For which nonzero vectors x, b is it possible to find some $A \in \mathbb{S}$ such that $Ax = b$?

(2): Determine the set $\mathcal{S} = \{A \in \mathbb{S} : \; Ax = b\}$.

- Consider $\mathbb{S} = \mathbb{G}$, $\mathbb{J}$ or $\mathbb{L}$ of any orthosymmetric $\langle \cdot, \cdot \rangle_{\mathrm{M}}$.
- Solved (1) and (2) for any $\mathbb{L}$ and $\mathbb{J}$ (linear structures).
- Solved
  - (1) for all $\mathbb{G}$ (nonlinear structures).
  - (2) for all $\mathbb{G}$ when x, b are non-isotropic.
- For isotropic x, b, solved (2) for a particular class of orthosymmetric $\langle \cdot, \cdot \rangle_{\mathrm{M}}$.