Structured Condition Numbers and Backward Errors in Scalar Product Spaces

Françoise Tisseur
Department of Mathematics
University of Manchester

ftisseur@ma.man.ac.uk
http://www.ma.man.ac.uk/~ftisseur/

Joint work with Stef Graillat (Univ. of Perpignan).

THE UNIVERSITY of MANCHESTER

Motivations

- Condition numbers and backward errors play an important role in numerical linear algebra.

$$\textit{forward error} \;\leq \textit{condition number} \times \; \textit{backward error}.$$

- Growing interest in structured perturbation analysis.
- Substantial development of algorithms for structured problems.
- Backward error analysis of structure preserving algorithms may be difficult.

Motivations Cont.

- For symmetric linear systems and for distances measured in the 2– or Frobenius norm: *It makes no difference whether perturbations are restricted to be symmetric or not*.
- Same holds for skew-symmetric and persymmetric structures. [S. Rump, 03].

Our contribution:

Extend and unify these results to

- Structured matrices in Lie and Jordan algebras,
- Several structured matrix problems.

Structured Problems

- Normwise *structured condition numbers* for
  - Matrix inversion,
  - Nearness to singularity,
  - Linear systems,
  - Eigenvalue problems.

- Normwise *structured backward errors* for
  - Linear systems,
  - Eigenvalue problems.

Scalar Products

A scalar product $\langle\cdot,\cdot\rangle_{\mathsf{M}}$ is a non degenerate (M nonsingular) bilinear or sesquilinear form on $\mathbb{K}^n$ ($\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$).

$$\langle x,y\rangle_{\mathsf{M}} = \begin{cases} x^T M y, & \text{real or complex bilinear forms,} \\ x^* M y, & \text{sesquilinear forms.} \end{cases}$$

Adjoint $A^\star$ of $A \in \mathbb{K}^{n\times n}$ wrt $\langle\cdot,\cdot\rangle_{\mathsf{M}}$:

$$A^\star = \begin{cases} M^{-1}A^T M, & \text{for bilinear forms,} \\ M^{-1}A^* M, & \text{for sesquilinear forms.} \end{cases}$$

$\langle\cdot,\cdot\rangle_{\mathsf{M}}$ orthosymmetric if $\begin{cases} M^T = \pm M, & \text{(bilinear),} \\ M^* = \alpha M, |\alpha| = 1, & \text{(sesquilinear).} \end{cases}$

$\langle\cdot,\cdot\rangle_{\mathsf{M}}$ is unitary if $M = \beta U$ for some unitary U and $\beta > 0$.

Jordan and Lie algebras

Two important classes of matrices associated with $\langle \cdot, \cdot \rangle_M$:

Lie algebra: $\mathbb{L} = \{A \in \mathbb{K}^{n \times n} : A^\star = -A\}$.

Jordan algebra: $\mathbb{J} = \{A \in \mathbb{K}^{n \times n} : A^\star = A\}$.

Recall that

$$A^\star = \begin{cases} M^{-1} A^T M, & \text{for bilinear forms,} \\ M^{-1} A^* M, & \text{for sesquilinear forms.} \end{cases}$$

Concentrate on Jordan and Lie algebras of orthosymmetric and unitary scalar products $\langle \cdot, \cdot \rangle_M$.

Some Structured Matrices

Space	M	Jordan Algebra	Lie Algebra
Bilinear forms			
$\mathbb{R}^n$	I	Symm.	Skew-symm.
$\mathbb{C}^n$	I	Complex symm.	Complex skew-symm.
$\mathbb{R}^n$	R	Persymmetric	Perskew-symm.
$\mathbb{R}^n$	$\Sigma_{p,q}$	Pseudo symm.	Pseudo skew-symm.
$\mathbb{R}^{2n}$	J	Skew-Hamiltonian.	Hamiltonian
Sesquilinear form			
$\mathbb{C}^n$	I	Hermitian	Skew-Herm.
$\mathbb{C}^n$	$\Sigma_{p,q}$	Pseudo Hermitian	Pseudo skew-Herm.
$\mathbb{C}^{2n}$	J	J-skew-Hermitian	J-Hermitian

$$R = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \qquad \Sigma_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$$

Matrix Inverse

Structured condition number for matrix inverse ($\nu = 2, F$):

$$\kappa_\nu(A;\mathbb{S}) := \lim_{\epsilon\to 0}\sup\left\{\frac{\|(A+\Delta A)^{-1} - A^{-1}\|_\nu}{\epsilon\|A^{-1}\|_\nu} : \frac{\|\Delta A\|_\nu}{\|A\|_\nu} \le \epsilon, \Delta A \in \mathbb{S}\right\}.$$

$\mathbb{S}$: Jordan or Lie algebra of orthosymm. and unitary $\langle\cdot,\cdot\rangle_{\mathrm{M}}$.

For nonsingular $A \in \mathbb{S}$,

$$\kappa_2(A;\mathbb{S}) = \kappa_2(A;\mathbb{C}^{n\times n}) = \|A\|_2\|A^{-1}\|_2$$
$$\kappa_F(A;\mathbb{S}) = \kappa_F(A;\mathbb{C}^{n\times n}) = \|A\|_F\|A^{-1}\|_2^2/\|A^{-1}\|_F.$$

Nearness to Singularity

Structured distance to singularity ($\nu = 2, F$):

$$\delta_\nu(A;\mathbb{S}) = \min\Big\{\epsilon : \frac{\|\Delta A\|_\nu}{\|A\|_\nu} \le \epsilon, A + \Delta A \text{ singular}, \Delta A \in \mathbb{S}\Big\}.$$

$\mathbb{S}$: Jordan or Lie algebra of $\langle\cdot,\cdot\rangle_{\mathsf{M}}$ orthosymm. and unitary.

For nonsingular $A \in \mathbb{S}$,

$$\delta_2(A;\mathbb{S}) = \delta_2(A;\mathbb{C}^{n\times n}) = \frac{1}{\|A\|_2\|A^{-1}\|_2},$$

$$\delta_F(A;\mathbb{C}^{n\times n}) \le \delta_F(A;\mathbb{S}) \le \sqrt{2}\,\delta_F(A;\mathbb{C}^{n\times n}).$$

Linear Systems

Structured condition number for linear system $Ax = b$, $x \neq 0$:

$$\operatorname{cond}_\nu(A, x; \mathbb{S}) = \lim_{\epsilon \to 0} \sup \left\{ \frac{\|\Delta x\|_2}{\epsilon \|x\|_2} : (A + \Delta A)(x + \Delta x) = b + \Delta b, \right.$$
$$\left. \frac{\|\Delta A\|_\nu}{\|A\|_\nu} \le \epsilon, \frac{\|\Delta b\|_2}{\|b\|_2} \le \epsilon, \Delta A \in \mathbb{S} \right\}, \ \nu = 2, F.$$

$\mathbb{S}$: Jordan or Lie algebra of $\langle \cdot, \cdot \rangle_{\mathsf{M}}$ orthosymm. and unitary.

For nonsingular $A \in \mathbb{S}$, $x \neq 0$ and $\nu = 2, F$,

$$\frac{\operatorname{cond}_\nu(A, x; \mathbb{C}^{n \times n})}{\sqrt{2}} \le \operatorname{cond}_\nu(A, x; \mathbb{S}) \le \operatorname{cond}_\nu(A, x; \mathbb{C}^{n \times n}).$$

Key Tools

Define $\mathrm{Sym}(\mathbb{K}) = \{A \in \mathbb{K}^{n\times n} : \; A^T = A\}, \; \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$,

$$\mathrm{Skew}(\mathbb{K}) = \{A \in \mathbb{K}^{n\times n} : \; A^T = -A\}.$$

$\mathbb{S}$: Lie algebra $\mathbb{L}$ or Jordan algebra $\mathbb{J}$ of *orthosymm.* $\langle\cdot,\cdot\rangle_{\mathrm{M}}$.

Orthosymmetry $\Rightarrow \mathbb{K}^{n\times n} = \mathbb{J} \oplus \mathbb{L}$ and,

$$M\cdot\mathbb{S} = \begin{cases} \mathrm{Sym}(\mathbb{K}) & \text{if } \begin{cases} M = M^T \text{ and } \mathbb{S} = \mathbb{J}, \\ M = -M^T \text{ and } \mathbb{S} = \mathbb{L}, \end{cases} \\ \mathrm{Skew}(\mathbb{K}) & \text{if } \begin{cases} M = M^T \text{ and } \mathbb{S} = \mathbb{L}, \\ M = -M^T \text{ and } \mathbb{S} = \mathbb{J}. \end{cases} \end{cases}$$

(bilinear forms)

Left multiplication of $\mathbb{S}$ by M is a bijection from $\mathbb{K}^{n\times n}$ to $\mathbb{K}^{n\times n}$ taking $\mathbb{J}$ and $\mathbb{L}$ to $\mathrm{Sym}(\mathbb{K})$ and $\mathrm{Skew}(\mathbb{K})$.

Key Tools Cont.

Define $\mathrm{Sym}(\mathbb{K}) = \{A \in \mathbb{K}^{n\times n} : \ A^T = A\}, \ \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$,

$$\mathrm{Skew}(\mathbb{K}) = \{A \in \mathbb{K}^{n\times n} : \ A^T = -A\},$$

$$\mathrm{Herm}(\mathbb{C}) = \{A \in \mathbb{C}^{n\times n} : \ A^* = A\}.$$

$\mathbb{S}$: Lie algebra $\mathbb{L}$ or Jordan algebra $\mathbb{J}$ of *orthosymm.* $\langle\cdot,\cdot\rangle_{\mathrm{M}}$.

$$M\cdot\mathbb{S} = \begin{cases} \mathrm{Sym}(\mathbb{K}) & \text{if } \begin{cases} M = M^T \text{ and } \mathbb{S} = \mathbb{J}, \\ M = -M^T \text{ and } \mathbb{S} = \mathbb{L}, \end{cases} \\ \mathrm{Skew}(\mathbb{K}) & \text{if } \begin{cases} M = M^T \text{ and } \mathbb{S} = \mathbb{L}, \\ M = -M^T \text{ and } \mathbb{S} = \mathbb{J}. \end{cases} \end{cases} \quad \text{(bilinear forms)}$$

$$M\cdot\mathbb{S} = \begin{cases} \mathrm{Herm}(\mathbb{C}) & \text{if } \ \mathbb{S} = \mathbb{J}, \\ i\,\mathrm{Herm}(\mathbb{C}) & \text{if } \ \mathbb{S} = \mathbb{L}. \end{cases} \quad \text{(sesquilinear forms)}$$

Distance to Singularity

Recall $\delta_2(A;\mathbb{S}) = \min\left\{\epsilon : \frac{\|\Delta A\|_2}{\|A\|_2} \leq \epsilon, A + \Delta A \text{ singular}, \Delta A \in \mathbb{S}\right\}$.

Want to show that $\delta_2(A;\mathbb{S}) = \delta_2(A;\mathbb{C}^{n\times n})$ $\quad (\star)$

$\langle\cdot,\cdot\rangle_{\mathsf{M}}$ unitary $\Rightarrow \begin{cases} \delta_2(A;\mathbb{S}) = \delta_2(MA; M\cdot\mathbb{S}), \\ \delta_2(MA;\mathbb{C}^{n\times n}) = \delta_2(A;\mathbb{C}^{n\times n}). \end{cases}$

$\Rightarrow$ Just need to prove $(\star)$ for $\mathbb{S} = \mathrm{Sym}(\mathbb{K}), \mathrm{Skew}(\mathbb{K})$, $\mathrm{Herm}(\mathbb{C})$.

Proof of $\delta_2(A;\mathbb{S}) = \delta_2(A;\mathbb{C}^{n\times n})$

Suppose $\mathbb{S} = \mathrm{Skew}(\mathbb{K}) = \{A \in \mathbb{K}^{n\times n} : \; A^T = -A\}$. Clearly,

$$\delta_2(A;\mathrm{Skew}(\mathbb{K})) \geq \delta_2(A;\mathbb{C}^{n\times n}) = 1/(\|A\|_2\|A^{-1}\|_2).$$

Assume $\|A\|_2 = 1$. Need to find $\Delta A \in \mathrm{Skew}(\mathbb{K})$ s.t.

- $\|\Delta A\|_2 = \sigma_{\min}(A) = 1/\|A^{-1}\|_2$
- and $A + \Delta A$ singular.

Let u, v s.t. $Av = \sigma_{\min}(A)u$. $\quad A \in \mathrm{Skew}(\mathbb{K}) \Rightarrow \bar{u}^* v = 0$.
Let Q unitary s.t. $Q[e_1, -e_2] = [v, \bar{u}]$. Then,

- $\Delta A = -\sigma_{\min}(A) Q(e_1 e_2^T - e_2 e_1^T)Q^T \in \mathrm{Skew}(\mathbb{K})$,
- $\|\Delta A\|_2 = \sigma_{\min}(A)$,
- $(A + \Delta A)v = 0$. $\quad \square$

Eigenvalue Condition Number

$$\kappa(A,\lambda;\mathbb{S}) = \lim_{\epsilon\to 0}\sup\Big\{\frac{|\Delta\lambda|}{\epsilon} : \lambda+\Delta\lambda \in Sp(A+\Delta A),$$
$$\|\Delta A\| \le \epsilon,\ \Delta A \in \mathbb{S}\Big\}.$$

$\mathbb{S}$: Jordan or Lie algebra of orthosymm. and unitary $\langle\cdot,\cdot\rangle_M$.

- For sesquilinear forms: $\kappa(A,\lambda;\mathbb{S}) = \kappa(A,\lambda,\mathbb{C}^{n\times n})$.
- For bilinear forms:
  - if $M\cdot\mathbb{S} = \mathrm{Sym}(\mathbb{C})$, $\kappa(A,\lambda;\mathbb{S}) = \kappa(A,\lambda,\mathbb{C}^{n\times n})$.
  - if $M\cdot\mathbb{L} = \mathrm{Skew}(\mathbb{C})$, $1 \le \kappa(A,\lambda;\mathbb{S}) \le \kappa(A,\lambda;\mathbb{C}^{n\times n})$.

Still incomplete.

Structured Backward Errors

$$\mu_\nu(y, r, \mathbb{S}) = \min\{\|\Delta A\|_\nu : \ \Delta A y = r, \ \Delta A \in \mathbb{S}\}, \qquad \nu = 2, F.$$

- For linear systems: $y \neq 0$ is the approx. sol. to $Ax = b$ and $r = b - Ay$.
- For eigenproblems: (y, λ) approx. eigenpair of A, $r = (\lambda I - A)y$.

$\mathbb{S}$: Jordan or Lie algebra of $\langle \cdot, \cdot \rangle_{\mathrm{M}}$ orthosymm. and unitary.

$\mu_\nu(y, r, \mathbb{S}) \neq \infty$ iff y, r satisfies the conditions:

$M \cdot \mathbb{S}$	Condition
$\mathrm{Sym}(\mathbb{K})$	none
$\mathrm{Skew}(\mathbb{K})$	$r^T y = 0$
$\mathrm{Herm}(\mathbb{C})$	$r^* y \in \mathbb{R}$.

Structured Backward Errors Cont.

$$\mu_\nu(y,r,\mathbb{S}) = \min\{\|\Delta A\|_\nu : \ \Delta A y = r,\ \Delta A \in \mathbb{S}\}, \qquad \nu = 2, F.$$

Recall $\mu_\nu(y,r;\mathbb{C}^{n\times n}) = \|r\|_2/\|y\|_2$.

$\mathbb{S}$: Jordan or Lie algebra of $\langle\cdot,\cdot\rangle_{\mathsf{M}}$ orthosymm. and unitary.

If $\mu_\nu(y,r,\mathbb{S}) \neq \infty$ ($\nu = 2, F$),

$$\mu_\nu(y,r;\mathbb{C}^{n\times n}) \leq \mu_\nu(y,r;\mathbb{S}) \leq \sqrt{2}\,\mu_\nu(y,r;\mathbb{C}^{n\times n}).$$

In particular for $\nu = F$,

$$\mu_F(y,r;\mathbb{S}) = \frac{1}{\|y\|_2}\sqrt{2\|r\|_2^2 - \frac{|\langle y,r\rangle_{\mathsf{M}}|^2}{\beta^2\|y\|_2^2}}.$$

Example

Take $\mathbb{S} = \text{Skew}(\mathbb{R}) = \{A \in \mathbb{R}^{n \times n} : \ A = A^T\}$.

Let $A = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix} \in \text{Skew}(\mathbb{R})$ and $b = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

True solution $x = [1, \ 1]^T$ satisfies $b^T x = 0$.

- Let $y = [1 + \epsilon, \ 1 - \epsilon]^T$ be an approximate solution. Then $r := b - Ay = \alpha \epsilon x$ and $r^T y = 2\alpha\epsilon \neq 0 \Rightarrow$ $\mu_F(y, r; \text{Skew}(\mathbb{R})) = \infty$.
- Using a structure preserving algorithm $\Rightarrow$ backward error matrix $\Delta A = \begin{bmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{bmatrix} \in \text{Skew}(\mathbb{R})$ and $y = (\alpha/(\epsilon + \alpha))x$. Hence, $r = b - Ay = (\epsilon/(\epsilon + \alpha))b$ satisfies $r^T y = 0$ and $\mu_F(y, r; \text{Skew}(\mathbb{R})) = \sqrt{2}\|r\|_2/\|y\|_2 \neq \infty$.

Conclusion

For matrices in Jordan or Lie algebras of orthosymmetric and unitary scalar products,

This includes symmetric, complex symmetric, skew-symmetric, pseudo symmetric, persymmetric, Hamiltonian, skew-Hamiltonian, Hermitian and J-Hermitian matrices.

Conclusion

For matrices in Jordan or Lie algebras of orthosymmetric and unitary scalar products,

- Usual unstructured perturbation analysis sufficient for
  - matrix inversion condition number,
  - distance to singularity,
  - linear system condition number.

Conclusion

For matrices in Jordan or Lie algebras of orthosymmetric and unitary scalar products,

- Usual unstructured perturbation analysis sufficient for
  - matrix inversion condition number,
  - distance to singularity,
  - linear system condition number.
- Partial answer for *eigenvalue condition numbers.*

Conclusion

For matrices in Jordan or Lie algebras of orthosymmetric and unitary scalar products,

- Usual unstructured perturbation analysis sufficient for
  - matrix inversion condition number,
  - distance to singularity,
  - linear system condition number.
- Partial answer for *eigenvalue condition numbers.*
- Structured backward error:
  - may be ∞ when using non structure preserving algorithm,
  - when finite, is within a small factor of the unstructured one.