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Perturbation theory for homogeneous polynomial eigenvalue problems

Jean-Pierre Dedieu [a], Françoise Tisseur [b,*,1]

[a] *MIP, Département de Mathématique, Université Paul Sabatier, 31062 Toulouse, Cedex 04, France*
[b] *Department of Mathematics, University of Manchester, Manchester M13 9PL, UK*

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Abstract

We consider polynomial eigenvalue problems $P(A, \alpha, \beta)x = 0$ in which the matrix polynomial is homogeneous in the eigenvalue $(\alpha, \beta) \in \mathbb{C}^2$. In this framework infinite eigenvalues are on the same footing as finite eigenvalues. We view the problem in projective spaces to avoid normalization of the eigenpairs. We show that a polynomial eigenvalue problem is well-posed when its eigenvalues are simple. We define the condition numbers of a simple eigenvalue (α, β) and a corresponding eigenvector x and show that the distance to the nearest ill-posed problem is equal to the reciprocal of the condition number of the eigenvector x. We describe a bihomogeneous Newton method for the solution of the homogeneous polynomial eigenvalue problem (homogeneous PEP).
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* Corresponding author. Tel.: +44-0161 275 5823; fax: +44-0161 275 5819.
E-mail addresses: dedieu@mip.ups-tlse.fr (J.-P. Dedieu), ftisseur@ma.man.ac.uk (F. Tisseur).
URL: http://www.ma.man.ac.uk/~ftisseur/ (F. Tisseur).
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1. Introduction

Let $A = (A_0, A_1, \ldots, A_m)$ be an $(m+1)$-tuple of $n \times n$ complex matrices. We consider homogeneous matrix polynomials $P(A, \alpha, \beta)$ defined by

$$P(A, \alpha, \beta) = \sum_{k=0}^{m} \alpha^k \beta^{m-k} A_k, \tag{1}$$

that is, $P(A, \alpha, \beta)$ is homogeneous of degree m in $(\alpha, \beta) \in \mathbb{C}^2$. The homogeneous polynomial eigenvalue problem (homogeneous PEP) is to find pairs of scalars $(\alpha, \beta) \neq (0, 0)$ and nonzero vectors $x, y \in \mathbb{C}^n$ satisfying

$$P(A, \alpha, \beta)x = 0, \quad y^* P(A, \alpha, \beta) = 0.$$

The vectors x, y are called right and left eigenvectors corresponding to the eigenvalue (α, β). For instance, the homogeneous generalized eigenvalue problem

$$(\beta A - \alpha B)x = 0$$

and the homogeneous quadratic eigenvalue problem

$$(\alpha^2 A + \alpha\beta B + \beta^2 C)x = 0$$

are major cases of homogeneous PEPs whose importance lies in the diverse roles they play in the solution of problems in science and engineering. In particular the QEP is currently receiving much attention because of its extensive applications in areas such as the dynamic analysis of mechanical systems in acoustics and linear stability of flows in fluid mechanics. A comprehensive survey on QEPs can be found in [26].

Most of the theory concerning matrix polynomials [10,11,15] is developed for nonhomogeneous polynomials of the form

$$Q(A, \lambda) = P(A, \lambda, 1) = \lambda^m A_m + \cdots + \lambda A_1 + A_0.$$

In this context, the nonhomogeneous PEP is to find a complex scalar λ and nonzero vectors $x, y \in \mathbb{C}^n$ such that

$$Q(A, \lambda)x = 0, \quad y^* Q(A, \lambda) = 0.$$

When (λ, x, y) is a solution of the nonhomogeneous PEP, $((\lambda, 1), x, y)$ is a solution of the homogeneous PEP. Conversely, if $((\alpha, \beta), x, y)$ is a solution of the homogeneous PEP and if $\beta \neq 0$, then $(\alpha/\beta, x, y)$ is a solution of the corresponding nonhomogeneous PEP. As for the linear eigenvalue problem, the eigenvalues are solutions of the scalar equation $\det P(A, \alpha, \beta) = 0$ in the homogeneous case and $\det Q(A, \lambda) = 0$ in the nonhomogeneous case. These two polynomials may be simultaneously identically zero or not. We say that an $(m+1)$-tuple A is regular when $\det P(A, \alpha, \beta) \not\equiv 0$. When A is regular, the polynomial $\det P(A, \alpha, \beta)$ has degree mn. Its zeros consist of mn complex lines in $\mathbb{C}^2$ through the origin, counting multiplicities. The nonhomogeneous problem has a different behaviour: the polynomial

$\det Q(A, \lambda)$ has degree less than or equal to mn and $\deg Q(A, \lambda) < mn$ if and only if A_m is singular. In such a case, $Q(A, \lambda)$ has $r = \deg Q(A, \lambda)$ eigenvalues λ_i, $i = 1{:}r$ corresponding to r homogeneous eigenvalues $(\lambda_i, 1)$, $i = 1{:}r$. The other homogeneous eigenvalue is $(1, 0)$ with multiplicity $mn - r$. This eigenvalue is called an infinite eigenvalue. For example, with

$$A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A_0 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

the homogeneous polynomial $\det P(A, \alpha, \beta) = \beta^4(\beta + \alpha)(2\beta - \alpha)$ has six zeros corresponding to the values $(\alpha, \beta) = (-1, 1)$ and $(2, 1)$ with multiplicity 1 and $(1, 0)$ and with multiplicity 4. On the other hand, the nonhomogeneous polynomial $\det Q(A, \lambda) = \det P(A, \lambda, 1) = (1 + \lambda)(2 - \lambda)$ has only two roots.

In the nonhomogeneous case, infinite eigenvalues require a special treatment [11] but not in the homogeneous case. For this reason we prefer to deal with homogeneous polynomials. In [13] it is shown that pseudospectra for matrix polynomials with infinite eigenvalues and their applications in control theory can be elegantly treated in the homogeneous framework.

When (α, β) is an eigenvalue of the homogeneous PEP, the (right) eigenvector $x \in \mathbb{C}^n$ is a solution of $P(A, \alpha, \beta)x = 0$. This equation is bihomogeneous: it is homogeneous of degree 1 in x and homogeneous of degree m in (α, β). For these reasons, it is natural to consider the eigenvalues and eigenvectors in projective spaces $\mathbb{P}(\mathbb{C}^2)$ and $\mathbb{P}(\mathbb{C}^n)$, respectively, where $\mathbb{P}(\mathbb{C}^k)$ is the set of vector lines in $\mathbb{C}^k$. When working in complex vector spaces instead of projective spaces, the common strategy is to use a normalization of x and (α, β). Taking unit vectors for x and (α, β) works well in the real case because the unit sphere cuts any real vector line in only two antipodal points. In the complex case, this strategy causes some difficulties as a complex line in $\mathbb{C}^n$ cuts the unit sphere in a circle. Andrew et al. [2, Section 4] discuss various normalization strategies for the eigenvector. The advantage of working in the way we do is that we do not need to introduce a normalization, that is, a local chart.

In this paper we study the condition number of the homogeneous PEP. The condition number of a problem measures the sensitivity of the output to small changes in the input. We study the sensitivity of the eigenvalue (α, β) and eigenvector x to perturbations to the matrix $(m + 1)$-tuple A. Previous condition number analyses of PEPs [24] were done in complex vector spaces for nonhomogeneous polynomials and with the assumption that λ is a simple and finite eigenvalue. Our analysis covers the case of infinite eigenvalues. We use the homogeneous character of the equation $P(A, \alpha, \beta)x = 0$ in both (α, β) and x to view the problem in a product of two projective spaces. This approach has been used by Stewart and Sun [23] and Dedieu [5] for the generalized eigenvalue problem $\beta Ax = \alpha Bx$. Stewart and Sun define the condition number of a simple eigenvalue in terms of the chordal distance between the exact eigenvalue and the perturbed eigenvalue. This paper adopts Shub and Smale's [22] and Dedieu's [5] approaches.

In Section 2, we define the map from the matrix $(m+1)$-tuple A to an eigenvalue and eigenvector and define the condition operator as the differential of this map. The condition number is the norm of this differential. All these computations are made in projective spaces, that is, we measure the angular variations between the exact eigenvalue and the perturbed eigenvalue and between the exact eigenvector and the perturbed eigenvector. In Section 3 we characterize well-posed problems, that is, problems for which the condition operator is defined. We compute the condition numbers in Section 4. We also relate the condition number of the eigenvector to the distance to ill-posed problems in Section 7. Our results are illustrated by numerical examples in Section 8.

In Section 9 we describe a bihomogeneous Newton method for the solution of the homogeneous PEP. The projective version of Newton method's has been studied extensively by Shub and Smale [20–22] for finding the zeros of homogeneous systems of n equations and $n+1$ unknowns. Here, as our problem is bihomogeneous, we use the multihomogeneous Newton method introduced by Dedieu and Shub [6] for finding the zeros of multihomogeneous functions.

2. Background and definitions

Let $A=(A_0, A_1, \ldots, A_m) \in \mathscr{M}_n(\mathbb{C})^{m+1}$, where $\mathscr{M}_n(\mathbb{C})^{m+1}$ denotes the set of $(m+1)$-tuples of $n \times n$ complex matrices. Throughout this paper we assume that A is regular, that is, $\det P(A, \alpha, \beta) \not\equiv 0$.

We denote by $\mathbb{P}_{n-1}(\mathbb{C}) = \mathbb{P}(\mathbb{C}^n)$ the set of vector lines in $\mathbb{C}^n$. More precisely, $\mathbb{P}(\mathbb{C}^n)$ is the quotient of $\mathbb{C}^n \setminus \{0\}$ for the equivalence relation "$x \sim y$ if and only if $x = \rho y$ for some $\rho \in \mathbb{C} \setminus \{0\}$". The set $\mathbb{P}(\mathbb{C}^n)$ can equally be viewed as the quotient of the unit sphere in $\mathbb{C}^n$:

$$\mathbb{S}^{2n-1} = \left\{x \in \mathbb{C}^n \colon \|x\|^2 = 1\right\}$$

for the equivalence relation "$x \sim y$ if and only if $x = \mathrm{e}^{\mathrm{i}\theta} y$ for some $\theta \in \mathbb{R}$", that is, $\mathbb{P}(\mathbb{C}^n)$ is the set of great circles in $\mathbb{S}^{2n-1}$. This is a smooth complex algebraic variety of dimension $n-1$ and also a complex Riemannian manifold.

Let $T_x\mathbb{P}_{n-1}$ be the tangent space at x to $\mathbb{P}_{n-1}$. This tangent space is usually identified with

$$x^{\perp} = \left\{\dot{x} \in \mathbb{C}^n \colon \langle \dot{x}, x\rangle = 0\right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the usual Hermitian inner product over $\mathbb{C}^n$. The Hermitian structure on $T_x\mathbb{P}_{n-1}$ is defined to be

$$\langle v, w\rangle_{x^{\perp}} = \frac{\langle v, w\rangle}{\langle x, x\rangle} \tag{2}$$

for $v, w \in x^{\perp}$. If we take another representative in the same complex line, say $y = \lambda x$, $\lambda \in \mathbb{C} \setminus \{0\}$, the tangent vectors corresponding to v and w become λv and λw

so that the Hermitian structure $\langle\cdot,\cdot\rangle_{x^\perp}$ is invariant under the identifications defining $\mathbb{P}_{n-1}$ (see [9, Sections 2.29 and 2.30]). The corresponding Riemannian distance in $\mathbb{P}_{n-1}(\mathbb{C})$ is given by

$$d_{\mathrm{R}}(x, y) = \arccos\left(\frac{|\langle x, y\rangle|}{\|x\|\,\|y\|}\right)$$

for any $x, y \in \mathbb{P}_{n-1}(\mathbb{C})$. Here we identify $x \in \mathbb{P}_{n-1}(\mathbb{C})$ with any nonzero representative $x \in \mathbb{C}^n$. Notice that the Riemannian distance $d_{\mathrm{R}}(x, y)$ is just the angle between the vector lines defined by x and y. Another popular distance is the "chordal distance" between two vector lines. This distance is the sine of the angle between these two lines: $d_{\mathrm{chordal}}(x, y) = \sin d_{\mathrm{R}}(x, y)$.

We want to study the first-order variations of the eigenvalue (α, β) and eigenvector x in terms of the variations of the $(m+1)$-tuple A. The relation between these three quantities is given implicitly via the equation $P(A, \alpha, \beta)x = 0$. For this reason we define the set of polynomial eigenvalue problems by the following algebraic variety:

$$\mathscr{V}_P = \left\{(A, x, \alpha, \beta) \in \mathscr{M}_n(\mathbb{C})^{m+1} \times \mathbb{P}_{n-1} \times \mathbb{P}_1 \colon P(A, \alpha, \beta)x = 0\right\}.$$

$\mathscr{V}_P$ is a smooth algebraic variety. $\mathscr{V}_P$ is a smooth manifold because the derivative of the equation defining $\mathscr{V}_P$ is onto (see Eq. (3) below) and $\mathscr{V}_P$ is homogeneous in x and in (α, β). The tangent space at (A, x, α, β) to $\mathscr{V}_P$ is given by (just differentiate the equation defining $\mathscr{V}_P$):

$$\begin{aligned} T_{(A,x,\alpha,\beta)}\mathscr{V}_P = \Big\{&(\dot A, \dot x, \dot\alpha, \dot\beta) \in \mathscr{M}_n(\mathbb{C})^{m+1} \times \mathbb{C}^n \times \mathbb{C}^2\colon \\ &P(\dot A, \alpha, \beta)x + P(A, \alpha, \beta)\dot x + \dot\alpha \mathscr{D}_\alpha P(A, \alpha, \beta)x \\ &+ \dot\beta \mathscr{D}_\beta P(A, \alpha, \beta)x = 0,\ \langle \dot x, x\rangle = 0,\ \dot\alpha\bar\alpha + \dot\beta\bar\beta = 0\Big\}. \end{aligned} \tag{3}$$

To avoid heavy notation we will often write P for $P(A, \alpha, \beta)$ and $\mathscr{D}_\alpha P$ and $\mathscr{D}_\beta P$ for $\mathscr{D}_\alpha P(A, \alpha, \beta)(\alpha, \beta)$ and $\mathscr{D}_\beta P(A, \alpha, \beta)(\alpha, \beta)$, respectively. We will use the first projection $\Pi_1\colon \mathscr{V}_P \to \mathscr{M}_n(\mathbb{C})^{m+1}$ given by $\Pi_1(A, x, \alpha, \beta) = A$ and the second projection $\Pi_2\colon \mathscr{V}_P \to \mathbb{P}_{n-1} \times \mathbb{P}_1$ given by $\Pi_2(A, x, \alpha, \beta) = (x, \alpha, \beta)$.

Definition 2.1. We say that (A, x, α, β) is *well-posed* when the derivative of the first projection Π_1 at (A, x, α, β) is an isomorphism. Otherwise, (A, x, α, β) is said to be *ill-posed*.

Note that this derivative is itself a projection: $D\Pi_1(A, x, \alpha, \beta)(\dot A, \dot x, \dot\alpha, \dot\beta) = \dot A$. When (A, x, α, β) is well-posed, by the inverse function theorem there exists a neighbourhood $U(x, \alpha, \beta) \subset \mathbb{P}_{n-1} \times \mathbb{P}_1$ of (x, α, β) and a neighbourhood $U(A) \subset \mathscr{M}_n(\mathbb{C})^{m+1}$ of A such that

$$\Pi_1\colon \mathscr{V}_P \cap \Big(U(A) \times U(x, \alpha, \beta)\Big) \to U(A)$$

is invertible (see [1, Section 3.5.1]). Its inverse gives rise to a smooth map

$$G = (G_1, G_2) = \Pi_2 \circ \Pi_1^{-1}\colon U(A) \to U(x, \alpha, \beta)$$

such that

$$\text{Graph}(G) = \mathscr{V}_P \cap (U(A) \times U(x, \alpha, \beta)).$$

For any $(m+1)$-tuple A with eigenvalue (α, β) and eigenvector x and any $(m+1)$-tuple A' close to A, G associates an eigenvalue (α', β') and eigenvector x' close to (α, β) and x, respectively. The map G is inaccessible but its derivative $DG(A)$ can be computed: the graph of $DG(A)$ is equal to the tangent space $T_{(A,x,\alpha,\beta)}\mathscr{V}_P$. We denote by

$$K(A, x, \alpha, \beta) = DG(A)\colon \mathscr{M}_n(\mathbb{C})^{m+1} \to T_x\mathbb{P}_{n-1} \times T_{(\alpha,\beta)}\mathbb{P}_1, \tag{4}$$

and

$$K_1(A, x, \alpha, \beta) = DG_1(A)\colon \mathscr{M}_n(\mathbb{C})^{m+1} \to T_x\mathbb{P}_{n-1},$$
$$K_2(A, x, \alpha, \beta) = DG_2(A)\colon \mathscr{M}_n(\mathbb{C})^{m+1} \to T_{(\alpha,\beta)}\mathbb{P}_1$$

the two components of this derivative. K_1 is the condition operator of the eigenvector and K_2 is the condition operator of the eigenvalue. These condition operators describe the first-order variations of x and (α, β) in terms of the first-order variations of A.

Definition 2.2. When (A, x, α, β) is well-posed, we define the *condition numbers* of (A, x, α, β) to be the norms of these linear operators:

$$C_1(A, x) = \max_{\|\dot{A}\| \leqslant 1} \|K_1(A, x, \alpha, \beta)(\dot{A})\|_{x^\perp},$$
$$C_2(A, \alpha, \beta) = \max_{\|\dot{A}\| \leqslant 1} \|K_2(A, x, \alpha, \beta)(\dot{A})\|_{(\alpha,\beta)^\perp}.$$

Here and throughout, $\|\cdot\|$ is the norm on $\mathscr{M}_n(\mathbb{C})^{m+1}$ associated with the Frobenius scalar product

$$\langle(\dot{A}_1, \ldots, \dot{A}_m), (\dot{B}_1, \ldots, \dot{B}_m)\rangle = \text{trace}\left(\sum_{k=1}^{m} \dot{B}_k^* \dot{A}_k\right).$$

3. Well-posed problems

In this section we characterize well-posed problems. Let $(A, x, \alpha_0, \beta_0) \in \mathscr{V}_P$ and define $P_0 = P(A, \alpha_0, \beta_0)$, $\mathscr{D}_\alpha P_0 = \mathscr{D}_\alpha P(A, \alpha, \beta)(\alpha_0, \beta_0)$, $\mathscr{D}_\beta P_0 = \mathscr{D}_\beta P(A, \alpha, \beta)$ (α_0, β_0). We also write

$$\text{Ker}(P_0) = \{x \in \mathbb{C}^n,\ P_0 x = 0\},$$
$$\text{Im}(P_0) = \{u \in \mathbb{C}^n,\ u = P_0 x \text{ for some } x \in \mathbb{C}^n\}.$$

The following two lemmas characterize some properties related to simple eigenvalues. These results are needed to prove Theorem 3.3 characterizing well-posed problems.

Lemma 3.1. *If* (α_0, β_0) *is a simple eigenvalue of* $P(A, \alpha, \beta)$ *and* x *is a corresponding right eigenvector, then* $v_0 = (\bar{\beta}_0 \mathscr{D}_\alpha P_0 - \bar{\alpha}_0 \mathscr{D}_\beta P_0)x \notin \mathrm{Im} P_0$.

Proof. We consider the local Smith form of $P(A, \alpha, \beta)$ (cf. [11, Theorem S1.10] and [15]). For any (α, β) in a neighbourhood of (α_0, β_0), we have

$$P(A, \alpha, \beta) = E(\alpha, \beta)D(\alpha, \beta)F(\alpha, \beta), \tag{5}$$

where

$$D(\alpha, \beta) = \mathrm{diag}\left((\alpha\beta_0 - \beta\alpha_0)^{\mu_1}, \dots, (\alpha\beta_0 - \beta\alpha_0)^{\mu_n}\right),$$

$\mu_1 \geqslant \cdots \geqslant \mu_n \geqslant 0$ are the partial multiplicities of (α_0, β_0) and the matrices $E(\alpha, \beta)$, $F(\alpha, \beta)$ are nonsingular.

If (α_0, β_0) is a simple eigenvalue, then $\mu_1 = 1$, $\mu_2 = \cdots = \mu_n = 0$ and $\dim \mathrm{Im} P_0 = n - 1$. Let $E_0 = E(\alpha_0, \beta_0)$, $F_0 = F(\alpha_0, \beta_0)$ and

$$D_1 + D_2 = [e_1, 0, \dots, 0] + [0, e_2, \dots, e_n] = I_n,$$

where e_k is the kth column of the identity matrix. Using the local Smith form of $P(A, \alpha, \beta)$ in (5), we have $P_0 = E_0 D_2 F_0$. As $P_0 x = 0$ it follows that $D_2 F_0 x = 0$. We also have

$$\begin{aligned}
\mathscr{D}_\alpha P_0 &= \mathscr{D}_\alpha E_0 D_2 F_0 + \beta_0 E_0 D_1 F_0 + E_0 D_2 \mathscr{D}_\alpha F_0, \\
\mathscr{D}_\beta P_0 &= \mathscr{D}_\beta E_0 D_2 F_0 - \alpha_0 E_0 D_1 F_0 + E_0 D_2 \mathscr{D}_\beta F_0
\end{aligned}$$

and therefore

$$\begin{aligned}
v_0 &= \bar{\beta}_0 \mathscr{D}_\alpha P_0 x - \bar{\alpha}_0 \mathscr{D}_\beta P_0 x \\
&= \left(|\alpha_0|^2 + |\beta_0|^2\right) E_0 D_1 F_0 x + E_0 D_2 (\bar{\beta}_0 \mathscr{D}_\alpha F_0 - \bar{\alpha}_0 \mathscr{D}_\beta F_0)x.
\end{aligned} \tag{6}$$

If $v_0 \in \mathrm{Im} P_0$, then $v_0 = P_0 u$ for some $u \in \mathbb{C}^n$. Multiplying (6) on the left by E_0^{-1} yields

$$(|\alpha_0|^2 + |\beta_0|^2) D_1 F_0 x + D_2 (\bar{\beta}_0 \mathscr{D}_\alpha F_0 - \bar{\alpha}_0 \mathscr{D}_\beta F_0)x = D_2 F_0 u.$$

This implies that $D_1 F_0 x \in \mathrm{Im} D_2$ so that $D_1 F_0 x = 0$. As $D_2 F_0 x = 0$ and $D_1 + D_2 = I_n$ we obtain $F_0 x = 0$. Since F_0 is nonsingular, we conclude that $x = 0$. Thus, $v_0 \notin \mathrm{Im} P_0$. □

In the following, $\Pi_{v^\perp}$ denotes the projection in $\mathbb{C}^n$ onto $v^\perp$ and $P_{|_{v^\perp}}$ denotes the restriction of P to $v^\perp$.

Lemma 3.2. *If* (α_0, β_0) *is a simple eigenvalue of* $P(A, \alpha, \beta)$ *with corresponding right eigenvector x and* $v_0 = (\bar{\beta}_0 \mathscr{D}_\alpha P_0 - \bar{\alpha}_0 \mathscr{D}_\beta P_0)x$, *then* $\Pi_{v_0^\perp} P_0|_{x^\perp}$ *is nonsingular.*

Proof. Suppose that $\Pi_{v_0^\perp} P_0|_{x^\perp}$ is singular. Then, there exists a vector $u \in x^\perp$, $u \neq 0$ such that $\Pi_{v_0^\perp} P_0|_{x^\perp} u = 0$. Thus, we have $P_0 u \in \mathrm{Im} P_0$ and $\Pi_{v_0^\perp} P_0 u = 0$ so that $v_0 \in \mathrm{Im} P_0$. But if (α_0, β_0) is a simple eigenvalue, then from Lemma 3.1 $v_0 \notin \mathrm{Im} P_0$ and therefore $\Pi_{v_0^\perp} P_0|_{x^\perp}$ is nonsingular. □

From (3), $(\dot{A}, \dot{x}, \dot{\alpha}, \dot{\beta}) \in T_{(A,x,\alpha_0,\beta_0)} \mathscr{V}_P$ if and only if

$$(\dot{A}, \dot{x}, \dot{\alpha}, \dot{\beta}) \in \mathscr{M}_n(\mathbb{C})^{m+1} \times \mathbb{C}^n \times \mathbb{C}^2$$

and

$$\mathscr{J}_P(x, \alpha_0, \beta_0) \begin{bmatrix} \dot{x} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} := \begin{bmatrix} P_0 & \mathscr{D}_\alpha P_0 x & \mathscr{D}_\beta P_0 x \\ x^* & 0 & 0 \\ 0 & \bar{\alpha}_0 & \bar{\beta}_0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = -\begin{bmatrix} P(\dot{A}, \alpha_0, \beta_0)x \\ 0 \\ 0 \end{bmatrix}. \tag{7}$$

Theorem 3.3. *Let* (α_0, β_0) *be an eigenvalue of* $P(A, \alpha, \beta)$ *with corresponding left and right eigenvectors y and x, respectively, and let* $v_0 = (\bar{\beta}_0 \mathscr{D}_\alpha P_0 - \bar{\alpha}_0 \mathscr{D}_\beta P_0)x$. *The following conditions are equivalent*:

(i) $D\Pi_1(A)$: $T_{(A,x,\alpha_0,\beta_0)} \mathscr{V}_P \to \mathscr{M}_n(\mathbb{C})^{m+1}$ *is an isomorphism, that is,* $(A, x, \alpha_0, \beta_0)$ *is well-posed.*
(ii) *The matrix* $\mathscr{J}_P(x, \alpha_0, \beta_0)$ *is nonsingular.*
(iii) $\mathrm{rank} P_0 = n - 1$ *and* $y^* v_0 \neq 0$.
(iv) (α_0, β_0) *is a simple eigenvalue.*

Proof. (i) ⇒ (ii): If

$$\mathscr{J}_P(x, \alpha_0, \beta_0) \begin{bmatrix} \dot{x} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = 0,$$

then $(0, \dot{x}, \dot{\alpha}, \dot{\beta}) \in T_{(A,x,\alpha_0,\beta_0)} \mathscr{V}_P$ and since $D\Pi_1$ is an isomorphism, $\dot{x} = 0$, $(\dot{\alpha}, \dot{\beta}) = (0, 0)$. Hence, $\mathscr{J}_P(x, \alpha_0, \beta_0)$ is nonsingular.

(ii) ⇒ (i): If $D\Pi_1(\dot{A}, \dot{x}, \dot{\alpha}, \dot{\beta}) = 0$, then $\dot{A} = 0$ and $P(\dot{A}, \alpha_0, \beta_0)x = 0$. Since $\mathscr{J}_P(x, \alpha_0, \beta_0)$ is nonsingular, $\dot{x} = 0$ and $(\dot{\alpha}, \dot{\beta}) = (0, 0)$ thus $D\Pi_1$ is an isomorphism.

(iii) ⇒ (ii): If

$$\mathscr{J}_P(x, \alpha_0, \beta_0) \begin{bmatrix} \dot{x} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = 0,$$

then

$$P_0\dot{x} + \dot{\alpha}\mathscr{D}_\alpha P_0 x + \dot{\beta}\mathscr{D}_\beta P_0 x = -P(\dot{A}, \alpha, \beta)x = 0, \tag{8}$$

$$\langle \dot{x}, x\rangle = 0, \tag{9}$$

$$\dot{\alpha}\bar{\alpha_0} + \dot{\beta}\bar{\beta_0} = 0. \tag{10}$$

Eq. (8) multiplied on the left by y^* and Eq. (10) gives

$$\begin{bmatrix} y^*\mathscr{D}_\alpha P_0 x & y^*\mathscr{D}_\beta P_0 x \\ \bar{\alpha_0} & \bar{\beta_0} \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = 0. \tag{11}$$

If $y^*v_0 \neq 0$, then (11) implies $\dot{\alpha} = \dot{\beta} = 0$. In this case (8) and (9) give

$$P_0\dot{x} = 0, \quad \langle \dot{x}, x\rangle = 0,$$

that is, x and $\dot{x} \in \operatorname{Ker} P_0$. As $x \neq 0$ and $\langle \dot{x}, x\rangle = 0$, if $\dim \operatorname{Ker} P_0 = 1$, then $\dot{x} = 0$.

(iv) ⇒ (iii): If (α_0, β_0) is a simple eigenvalue, then from Lemma 3.1 $v_0 \notin \operatorname{Im} P_0$. Let y be a left eigenvector corresponding to the eigenvalue (α_0, β_0). As $y^*P_0 = 0$ we have $y \perp \operatorname{Im} P_0$ but y is not orthogonal to v_0. Hence, $y^*v_0 \neq 0$.

(iii) ⇒ (iv): This is an easy consequence of the Smith form.

(ii) ⇒ (iii): Suppose that $\mathscr{J}_P(x, \alpha_0, \beta_0)$ is nonsingular. Without loss of generality we can assume that P_0 is diagonal (if not, just take the SVD of $P_0 = U\Sigma V^*$). If (α_0, β_0) is an eigenvalue, P_0 is singular and has at least one zero on its diagonal, so $\operatorname{rank} P_0 \leqslant n - 1$. $\mathscr{J}_P(x, \alpha_0, \beta_0)$ takes the form (for $n = 3$ for instance),

$$\begin{bmatrix} \times & 0 & 0 & \times & \times \\ 0 & d & 0 & \times & \times \\ 0 & 0 & 0 & \times & \times \\ \times & \times & \times & 0 & 0 \\ 0 & 0 & 0 & \times & \times \end{bmatrix}.$$

If $\operatorname{rank} P_0 < n - 1$, then $d = 0$ and clearly $\mathscr{J}_P(x, \alpha_0, \beta_0)$ is singular. Hence rank $P_0 = n - 1$.

We now prove that $\mathscr{J}_P(x, \alpha_0, \beta_0)$ nonsingular implies $y^*v_0 \neq 0$. We have

$$\begin{aligned} \det(\mathscr{J}_P(x, \alpha_0, \beta_0)) &= -\bar{\alpha}_0 \det \begin{bmatrix} P_0 & \mathscr{D}_\beta P_0 x \\ x^* & 0 \end{bmatrix} + \bar{\beta}_0 \det \begin{bmatrix} P_0 & \mathscr{D}_\alpha P_0 x \\ x^* & 0 \end{bmatrix} \\ &= -\bar{\alpha}_0 x^* P_0^A \mathscr{D}_\beta P_0 x - \bar{\beta}_0 x^* P_0^A \mathscr{D}_\alpha P_0 x \\ &= -x^* P_0^A v_0 \neq 0, \end{aligned} \tag{12}$$

where P_0^A is the adjoint of P_0. The adjoint has the property that $P_0^A P_0 = \det(P_0)I$. Let $y^* = x^*P_0^A$. Then $y^*P_0 = 0$ so that y is a left eigenvector of P_0 corresponding to (α_0, β_0). Then from (12), $y^*v_0 \neq 0$. □

Let us highlight some differences between the linear case ($m = 1$), and the polynomial case ($m > 1$). In the first case, that is, for the generalized eigenvalue problem,

$$(\alpha A_1 + \beta A_0)x = 0,$$

we have the following characterization of $(A, x, \alpha_0, \beta_0)$ being well-posed [5]:
(i) The matrix $\mathscr{J}_P(x, \alpha_0, \beta_0)$ is nonsingular.
(ii) $\operatorname{rank} P_0 = n - 1$.
(iii) (α_0, β_0) is a simple eigenvalue.

The main difference between the linear and the polynomial case is the possibility for a matrix polynomial to satisfy $\operatorname{rank} P_0 = n - 1$ even if (α_0, β_0) is not a simple eigenvalue. This is clearly shown by the following 2×2 example:

$$P(A, \alpha, \beta) = \operatorname{diag}((\alpha - \beta)^2, \alpha^2)$$

at $(\alpha_0, \beta_0) = (1, 1)$. In this case $\operatorname{rank} P_0 = 1$ and (α_0, β_0) is an eigenvalue with multiplicity 2.

When the matrix pair (A_0, A_1) is regular, there exist two unitary matrices U and V such that UA_0V and UA_1V are upper triangular. This is called the generalized Schur decomposition. Another major difference between the linear and the polynomial case is that the generalized Schur decomposition for pairs (A_0, A_1) does not extend to an $(m + 1)$-tuple of matrices with $m > 1$. To avoid this difficulty we use here the local Smith form of $P(A, \alpha, \beta)$.

4. Computation of the condition numbers

By definition, the condition operator of the eigenvector x and the condition operator of the eigenvalue (α, β) are

$$K_1(A, x, \alpha, \beta)(\dot{A}) = \dot{x}, \quad K_2(A, x, \alpha, \beta)(\dot{A}) = (\dot{\alpha}, \dot{\beta}).$$

The corresponding condition numbers are given by

$$C_1(A, x) = \max_{\|\dot{A}\| \leqslant 1} \|\dot{x}\|_{x^\perp} = \max_{\|\dot{A}\| \leqslant 1} \frac{\|\dot{x}\|}{\|x\|},$$

and

$$C_2(A, \alpha, \beta) = \max_{\|\dot{A}\| \leqslant 1} \|(\dot{\alpha}, \dot{\beta})\|_{(\alpha,\beta)^\perp} = \max_{\|\dot{A}\| \leqslant 1} \sqrt{\frac{|\dot{\alpha}|^2 + |\dot{\beta}|^2}{|\alpha|^2 + |\beta|^2}}.$$

Theorem 4.1. *Assume that (α, β) is a simple eigenvalue of $P(A, \alpha, \beta)$ with corresponding right eigenvector x and let $v = \bar{\beta}\mathscr{D}_\alpha P(A, \alpha, \beta)x - \bar{\alpha}\mathscr{D}_\beta P(A, \alpha, \beta)x$. Then*

$$K_1(A, x, \alpha, \beta)(\dot{A}) = -(\Pi_{v^\perp} P(A, \alpha, \beta)_{|_{x^\perp}})^{-1} \Pi_{v^\perp} P(\dot{A}, \alpha, \beta)x,$$

and

$$C_1(A, x) = \left(\sum_{k=0}^{m} |\alpha|^{2k}|\beta|^{2(m-k)}\right)^{1/2} \|(\Pi_{v^\perp} P(A, \alpha, \beta)_{|_{x^\perp}})^{-1}\|,$$

where $P(A, \alpha, \beta)_{|_{x^\perp}}$ *is the restriction of* $P(A, \alpha, \beta)$ *to* $x^\perp$ *and* $\Pi_{v^\perp}$ *is the projection over* $v^\perp$.

Proof. Eq. (10) is equivalent to $\dot\alpha = \dot\lambda\bar\beta$ and $\dot\beta = -\dot\lambda\bar\alpha$ for some $\dot\lambda \in \mathbb{C}$. As $\langle \dot x, x\rangle = 0$, $\dot x \in x^\perp$ and from (8) we obtain

$$P(A, \alpha, \beta)_{|_{x^\perp}} \dot x + \dot\lambda v = -P(\dot A, \alpha, \beta)x. \tag{13}$$

Projecting Eq. (13) over $v^\perp$ yields

$$\Pi_{v^\perp} P(A, \alpha, \beta)_{|_{x^\perp}} \dot x = -\Pi_{v^\perp} P(\dot A, \alpha, \beta)x.$$

By Lemma 3.2, the operator $\Pi_{v^\perp} P(A, \alpha, \beta)_{|_{x^\perp}}$ is nonsingular and the expression for K_1 in the theorem follows. For the condition number,

$$\|\dot x\| = \|(\Pi_{v^\perp} P(A, \alpha, \beta)_{|_{x^\perp}})^{-1} \Pi_{v^\perp} P(\dot A, \alpha, \beta)x\|.$$

Suppose that $\left(\sum_{k=0}^{m} |\alpha|^{2k}|\beta|^{2(m-k)}\right) = 1$. In this case, when $\dot A$ describes the unit ball in $\mathscr{M}_n(\mathbb{C})^{m+1}$, the vector $P(\dot A, \alpha, \beta)x$ describes the unit ball in $\mathbb{C}^n$ with projection over $v^\perp$ equal to the unit ball. Thus,

$$\begin{aligned} C_1(A, x) &= \max_{\|\dot A\| \leqslant 1} \frac{\|\dot x\|}{\|x\|} \\ &= \left(\sum_{k=0}^{m} |\alpha|^{2k}|\beta|^{2(m-k)}\right)^{1/2} \left\| \left(\Pi_{v^\perp} P(A, \alpha, \beta)_{|_{x^\perp}}\right)^{-1} \right\|. \qquad \square \end{aligned}$$

Theorem 4.2. *Assume that* (α, β) *is a simple eigenvalue of* $P(A, \alpha, \beta)$ *with corresponding right and left eigenvectors x and y, respectively, and let* $v = \bar\beta \mathscr{D}_\alpha P(A, \alpha, \beta)x - \bar\alpha \mathscr{D}_\beta P(A, \alpha, \beta)x$. *Then*

$$K_2(A, x, \alpha, \beta)(\dot A) = \frac{y^* P(\dot A, \alpha, \beta)x}{y^* v}(-\bar\beta, \bar\alpha),$$

and

$$C_2(A, \alpha, \beta) = \left(\sum_{k=0}^{m} |\alpha|^{2k}|\beta|^{2(m-k)}\right)^{1/2} \frac{\|x\|\,\|y\|}{|y^* v|}.$$

Proof. Eq. (10) is equivalent to $\dot\alpha = \dot\lambda\bar\beta$ and $\dot\beta = -\dot\lambda\bar\alpha$ for some $\dot\lambda \in \mathbb{C}$. Multiplying (8) on the left by y^* gives

$$\dot\lambda y^* v = -y^* P(\dot A, \alpha, \beta)x.$$

From Theorem 3.3, if (α, β) is a simple eigenvalue, then $y^*v \neq 0$ so that

$$(\dot{\alpha}, \dot{\beta}) = \dot{\lambda}(\bar{\beta}, -\bar{\alpha}) = -\frac{y^* P(\dot{A}, \alpha, \beta)x}{y^*v}(\bar{\beta}, -\bar{\alpha}).$$

For the condition number,

$$\frac{|\dot{\alpha}|^2 + |\dot{\beta}|^2}{|\alpha|^2 + |\beta|^2} = |\dot{\lambda}|^2 = \frac{|y^* P(\dot{A}, \alpha, \beta)x|^2}{|y^*v|^2}.$$

As

$$\|P(\dot{A}, \alpha, \beta)\| \leqslant \left(\sum_{k=0}^{m} |\alpha|^{2k}|\beta|^{2(m-k)}\right)^{1/2} \|\dot{A}\|,$$

we obtain

$$C_2(A, \alpha, \beta) \leqslant \left(\sum_{k=0}^{m} |\alpha|^{2k}|\beta|^{2(m-k)}\right)^{1/2} \frac{\|x\|\|y\|}{|y^*v|}.$$

Suppose that $x = e_1$, where e_1 the unit vector. We take

$$S = \left[\frac{y}{\|y\|}, 0\right], \quad \dot{A}_k = \frac{\bar{\alpha}^k \bar{\beta}^{m-k}}{\left(\sum_{k=0}^{m} |\alpha|^{2k}|\beta|^{2(m-k)}\right)^{1/2}} S, \quad 0 \leqslant k \leqslant m.$$

It is easy to check that $\|\dot{A}\| = 1$ and

$$C_2(A, \alpha, \beta) \geqslant \frac{|y^* P(\dot{A}, \alpha, \beta)x|}{|y^*v|} = \left(\sum_{k=0}^{m} |\alpha|^{2k}|\beta|^{2(m-k)}\right)^{1/2} \frac{\|x\|\|y\|}{|y^*v|}. \qquad \square$$

5. Invariance properties

Both condition operators and condition numbers are invariant under scaling of the eigenvector and eigenvalue because they are constructed projectively:

$$K_i(A, \rho x, \mu(\alpha, \beta)) = K_i(A, x, (\alpha, \beta)), \quad i = 1, 2,$$
$$C_i(A, \rho x, \mu(\alpha, \beta)) = C_i(A, x, (\alpha, \beta)), \quad i = 1, 2,$$

for any ρ and $\mu \in \mathbb{C} \setminus \{0\}$. However, under scaling of A we have

$$C_i(\rho A, x, (\alpha, \beta)) = |\rho|^{-1} C_i(A, x, (\alpha, \beta))$$

for any $\rho \in \mathbb{C} \setminus \{0\}$ and $i = 1, 2$. In the following we prove another invariance property with respect to unitary transformations in $\mathbb{C}^n$.

Lemma 5.1. *For any unitary matrices U and V, if* $P(A, \alpha, \beta)x = 0$, *then we also have* $P(V^*AU, \alpha, \beta)U^*x = 0$ *and*

$$\begin{aligned}
&K_1(A, x, \alpha, \beta)(\dot{A}) = UK_1(V^*AU, U^*x, \alpha, \beta)(V^*\dot{A}U),\\
&K_2(A, x, \alpha, \beta)(\dot{A}) = K_2(V^*AU, U^*x, \alpha, \beta)(V^*\dot{A}U),\\
&C_1(A, x) = C_1(V^*AU, U^*x),\\
&C_2(A, \alpha, \beta) = C_2(V^*AU, \alpha, \beta).
\end{aligned}$$

Proof. First, we have that if $(A, x, \alpha, \beta) \in \mathscr{V}_P$, then $(V^*AU, U^*x, \alpha, \beta) \in \mathscr{V}_P$ and if $(\dot{A}, \dot{x}, \dot{\alpha}, \dot{\beta}) \in T_{(A,x,\alpha,\beta)}\mathscr{V}_P$, then $(V^*\dot{A}U, U^*\dot{x}, \dot{\alpha}, \dot{\beta}) \in T_{(V^*AU,U^*x,\alpha,\beta)}\mathscr{V}_P$. Moreover,

$$\begin{aligned}
&\mathscr{D}\Pi_1(V^*AU, U^*x, \alpha, \beta)(V^*\dot{A}U, U^*\dot{x}, \dot{\alpha}, \dot{\beta})\\
&\quad = V^*\dot{A}U = V^*\mathscr{D}\Pi_1(A, x, \alpha, \beta)(\dot{A}, \dot{x}, \dot{\alpha}, \dot{\beta})U
\end{aligned}$$

for any $(\dot{A}, \dot{x}, \dot{\alpha}, \dot{\beta}) \in T_{(A,x,\alpha,\beta)}\mathscr{V}_P$. Hence if (A, x, α, β) is well-posed, then $(V^*AU, U^*x, \alpha, \beta)$ is well-posed. From the definition of the condition operator, if $(\dot{A}, \dot{x}, \dot{\alpha}, \dot{\beta}) \in T_{(A,x,\alpha,\beta)}\mathscr{V}_P$, then $K(A, x, \alpha, \beta)\dot{A} = (\dot{x}, \dot{\alpha}, \dot{\beta})$. But if $(\dot{A}, \dot{x}, \dot{\alpha}, \dot{\beta}) \in T_{(A,x,\alpha,\beta)}\mathscr{V}_P$, then $(V^*\dot{A}U, U^*\dot{x}, \dot{\alpha}, \dot{\beta}) \in T_{(V^*AU,U^*x,\alpha,\beta)}\mathscr{V}_P$. Thus

$$K(V^*AU, U^*x, \alpha, \beta)(V^*\dot{A}U) = (U^*\dot{x}, \dot{\alpha}, \dot{\beta}).$$

and

$$\begin{aligned}
&K_1(V^*AU, U^*x, \alpha, \beta)(V^*\dot{A}U) = U^*K_1(A, x, \alpha, \beta)(\dot{A}),\\
&K_2(V^*AU, U^*x, \alpha, \beta)(V^*\dot{A}U) = K_2(A, x, \alpha, \beta)(\dot{A}).
\end{aligned}$$

Since both the Hermitian norm in $\mathbb{C}^n$ and the Frobenius norm in $\mathscr{M}_n(\mathbb{C})$ are unitarily invariant, the last two equalities in the lemma follow. □

6. Special cases

In the case of the generalized eigenvalue problem $\beta Ax = \alpha Bx$, $\Pi_{v^\perp} = \Pi_{(Ax)^\perp}$ and for the condition number of the eigenvector we have

$$C_1(x) = (|\alpha|^2 + |\beta|^2)^{1/2} \left\| (\Pi_{(Ax)^\perp}(\beta Ax - \alpha Bx)_{|_{x^\perp}})^{-1} \right\|,$$

which is the expression obtained in [5]. For the condition number of the eigenvalue we obtain

$$C_2(\alpha, \beta) = \frac{(|\alpha|^2 + |\beta|^2)^{1/2}\|x\|\|y\|}{|\bar{\alpha}y^*Ax + \bar{\beta}y^*Bx|}.$$

We now use a result proved in [23, Chapter 6, Corollary 1.10]:

Lemma 6.1. *Let* (α, β) *be a simple eigenvalue of the regular pair* (A, B) *with right and left eigenvectors x and y. Then* $(\alpha, \beta) = \rho(y^*Ax, y^*Bx)$ *for some nonzero constant* $\rho \in \mathbb{C}$.

We take the representatives $\alpha = y^*Ax$, $\beta = y^*Bx$. Then

$$C_2(\alpha, \beta) = \frac{\|x\|\,\|y\|}{(|\alpha|^2 + |\beta|^2)^{1/2}},$$

which is the condition number derived by Stewart and Sun [23, p. 294].

Condition numbers of eigenvalues are used in the state-feedback pole assignment problem in control system design [14,16]. For instance, consider the second-order system

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = -Bu(t), \tag{14}$$

where M, C and K are the $n \times n$ matrices and $q(t) \in \mathbb{C}^n$, $B \in \mathbb{C}^{m\times m}$ and $u(t) \in \mathbb{C}^m$ with $m \leqslant n$. The control problem is to design a state-feedback controller $u(t)$ of the form

$$u(t) = F_C^{\mathrm{T}}\dot{q}(t) + F_K^{\mathrm{T}}q(t) + r(t),$$

where $F_C, F_K \in \mathbb{C}^{m\times n}$ and $r(t) \in \mathbb{C}^m$ are such that the closed-loop system

$$M\ddot{q}(t) + (C + BF_C^{\mathrm{T}})\dot{q}(t) + (K + BF_K^{\mathrm{T}})q(t) = -Br(t) \tag{15}$$

has the desired properties. The solution is, in general, underdetermined, with many degrees of freedom. The behaviour of the closed-loop system is governed by the eigenstructure of the corresponding quadratic eigenvalue problem

$$\lambda^2 A_2 x + \lambda A_1 x + A_0 x = 0, \tag{16}$$

where $A_2 = M$, $A_1 = C + BF_C^{\mathrm{T}}$ and $A_0 = K + BF_K^{\mathrm{T}}$. A desirable property is that the eigenvalues should be insensitive to perturbations in the coefficient matrices. This criterion is used to restrict the degrees of freedom in the assignment problem and to produce a well-conditioned or robust solution to the problem. To measure the robustness of the system we can take as a global measure

$$\nu^2 = \sum_{k=1}^{2n} \omega_k^2 \kappa(\lambda_k)^2,$$

where the ω_k are the positive weights and $\kappa(\lambda_k)$ is the condition number of the simple and finite eigenvalue λ_k. The control design problem is then to select the feedback matrices F_C and F_K to assign a given set of $2n$ nondefective eigenvalues to the second-order closed loop system and to minimize its robustness measure ν^2.

Several expressions for $\kappa(\lambda)$ have already been obtained. In [16], the condition number of λ is defined to be

$$\kappa(\lambda) = \lim_{\epsilon\to 0} \sup \left\{ \frac{|\dot{\lambda}|}{\epsilon}\colon Q(A+\dot{A}, \lambda+\dot{\lambda})(x+\dot{x}) = 0, \quad \|A_2^{-1}\dot{A}\|_2 \leqslant \epsilon \right\},$$

where $A, \dot{A}$ are the 3-tuples and $Q(A, \lambda)$ is the nonhomogeneous polynomial of degree 2 in λ. This is an absolute condition number and an explicit expression is given by

$$\kappa(\lambda) = \frac{(|\lambda|^4 + |\lambda|^2 + 1)^{1/2}\|x\|_2\|y^*A_2\|_2}{|y^*(2\lambda A_2 + A_1)x|}. \tag{17}$$

In [24] the condition number of λ is defined to be

$$\kappa(\lambda) = \lim_{\epsilon\to 0} \sup \left\{ \frac{|\dot{\lambda}|}{\epsilon|\lambda|}\colon Q(A+\dot{A}, \lambda+\dot{\lambda})(x+\dot{x}) = 0, \ \|\dot{A}_i\| \leqslant \alpha_i\epsilon, i = 0\colon 2 \right\},$$

where the α_k are nonnegative parameters that allow freedom in how perturbations are measured—for example, in an absolute sense ($\alpha_i \equiv 1$) or a relative sense ($\alpha_i = \|A_i\|$) and $\|\cdot\|$ denotes any subordinate matrix norm. This is a relative condition number and an explicit expression is given by

$$\kappa(\lambda) = \frac{(|\lambda|^2\alpha_2 + |\lambda|\alpha_1 + \alpha_0)}{|\lambda|\ |y^*(2\lambda A_2 + A_1)x|}\|y\|\|x\|. \tag{18}$$

Both condition numbers (17) and (18) assume that A_2 is nonsingular, which eliminates the case of infinite eigenvalues. This might be a problem in control design problems as singular matrices A_2 are frequent and a singular A_2 yields infinite eigenvalues. The advantage of working with the homogeneous form

$$\alpha^2 A_2 x + \alpha\beta A_1 x + \beta^2 A_0 x = 0$$

is that infinite eigenvalues are not a special case. For this problem our condition number is

$$C_2(\alpha, \beta) = \frac{\sqrt{|\alpha|^4 + |\alpha|^2|\beta|^2 + |\beta|^4}\|x\|\|y\|}{|y^*(2\bar{\beta}\alpha A_2 + (|\beta|^2 - |\alpha|^2)A_1 - 2\beta\bar{\alpha}A_0)x|}.$$

If (α, β) is an infinite eigenvalue, i.e., $\beta = 0$, then $C_2(\alpha, 0) = \|x\|\|y\|/|y^*A_1x|$.

An expression for the condition number of the eigenvector $\kappa(x)$ has been obtained in [26] in the nonhomogeneous case. As an eigenvector corresponding to a simple λ is unique only up to a scalar multiple, a linear normalization based on a constant vector g is used. The normwise condition number for the eigenvector x corresponding to the simple eigenvalue λ can be defined by

$$\begin{aligned} \kappa_\lambda(x) = \lim_{\epsilon\to 0} \sup \Bigg\{ \frac{\|\dot{x}\|}{\epsilon\|x\|}\colon & \big(Q(A+\dot{A}, \lambda+\dot{\lambda})\big)(x+\dot{x}) = 0, \\ & g^*(2\lambda A_2 + A_1)x = g^*(2\lambda A_2 + A_1)(x+\dot{x}) \equiv 1, \\ & \|\dot{A}_i\| \leqslant \alpha_i\epsilon,\ i = 0\colon 2 \Bigg\}, \end{aligned}$$

and an explict expression is given by (see [26, Theorem 2.7])

$$\kappa_\lambda(x) = \|V(W^*(\lambda^2 A_2 + \lambda A_1 + A_0)V)^{-1}W^*\|(|\lambda|^2\alpha_2 + |\lambda|\alpha_1 + \alpha_0), \quad (19)$$

where the full rank matrices $V, W \in \mathbb{C}^{n\times(n-1)}$ are chosen so that $g^*(2\lambda A_2 + A_1)V = 0$ and $W^*(2\lambda A_2 + \lambda A_1)x = 0$, the α_k are nonnegative parameters that allow freedom in how perturbations are measured and $\|\cdot\|$ denotes a subordinate matrix norm. This expression extends in an obvious way to nonhomogeneous polynomials of degree higher than 2.

The matrix $\Pi_{v^\perp}P(A,\alpha,\beta)_{|_{x^\perp}}$ in the expression for $C_1(A,x)$ in Theorem 4.1 can be explicitly constructed as follows. Let $Q_x^{\mathrm{T}}x = R_x = \pm\|x\|e_1$ and $Q_v^{\mathrm{T}}v = R_v = \pm\|v\|e_1$ be QR factorizations and let $\widetilde{U} = Q_x(:,2:n)$ and $\widetilde{V} = Q_v(:,2:n)$. Then for $\Pi_{v^\perp}P(A,\alpha,\beta)_{|_{x^\perp}}$ we take $\widetilde{V}^*P(A,\alpha,\beta)\widetilde{U}$. Hence, for $A = (A_0, A_1, A_2)$ we have

$$C_1(A,x) = \|(\widetilde{V}^*(\alpha^2 A_2 + \alpha\beta A_1 + \beta^2 A_0)\widetilde{U})^{-1}\|_{\mathrm{F}}\sqrt{|\alpha|^4 + |\alpha\beta|^2 + |\beta|^4}.$$

7. Distance to the nearest ill-posed problem

A problem is *ill-posed* when it is not well-posed. For ill-posed problems we take the condition number as equal to infinity. A problem is *ill-conditioned* when its condition number is large. The condition number of a problem is necessarily related to the distance from ill-posed instances because the inverse of the condition number is zero on ill-posed instances. The first example of such a result is due to Eckart and Young [8]: the Frobenius norm of the inverse of a nonsingular matrix is equal to the inverse of the Frobenius norm distance to the singular matrices. The case of homogeneous polynomial systems is studied by Shub and Smale in [20], and the eigenvalue problem by the same authors in [21]. For the generalized eigenvalue problem see [5]. See also [3] for polynomial equations of a single variable. All these results are put in a more general setting by Dedieu in [4] and by Demmel in [7] using differential inequalities.

Let $\Pi_2 : \mathscr{V}_P \to \mathbb{P}_{n-1}\times\mathbb{P}_1$ be the second projection and Σ be the set of ill-posed problems. We denote by $\Sigma_{(x,\alpha,\beta)} = \Sigma \cap \Pi_2^{-1}(x,\alpha,\beta)$ the set of ill-posed problems at (x,α,β) and $\Sigma'_{(x,\alpha,\beta)} \subset \Sigma_{(x,\alpha,\beta)}$ the set of ill-posed problems that are such that if $(B,x,\alpha,\beta) \in \Sigma'_{(x,\alpha,\beta)}$, then $v_B = \bar{\beta}\mathscr{D}_\alpha P(B,\alpha,\beta)x - \bar{\alpha}\mathscr{D}_\beta P(B,\alpha,\beta)x \neq 0$. We equip $\Sigma'_{(x,\alpha,\beta)}$ with the distance d_{F} deduced from the Frobenius norm over $\mathscr{M}_n(\mathbb{C})^{m+1}$.

Theorem 7.1. *Let (A,x,α,β) be a well-posed problem. Then*

$$d_{\mathrm{F}}\left((A,x,\alpha,\beta),\Sigma'_{(x,\alpha,\beta)}\right) = \frac{1}{C_1(A,x)}.$$

Proof. Let U, V be two unitary matrices such that $U = [\,\|x\|^{-1}x, \widetilde{U}\,]$ and $V = [\,\|v\|^{-1}v, \widetilde{V}\,]$. Then

$$V^*P(A,\alpha,\beta)U = \begin{bmatrix} 0 & \tilde{p}^* \\ 0 & P(\widetilde{A},\alpha,\beta) \end{bmatrix}, \tag{20}$$

where $\tilde{p}^* = \|v\|^{-1}v^*P(A,\alpha,\beta)\widetilde{U}$ and $\widetilde{A} = \widetilde{V}^*A\widetilde{U}$, so that $\widetilde{V}^*P(A,\alpha,\beta)\widetilde{U} = P(\widetilde{A},\alpha,\beta)$. The matrix $P(\widetilde{A},\alpha,\beta)$ is the matrix representation of the operator $\Pi_{v^\perp}P(A,\alpha,\beta)_{|_{x^\perp}}$ in two bases orthogonal to v and x, respectively. As $\|(\Pi_{v^\perp} \times P(A,\alpha,\beta)_{|_{x^\perp}})^{-1}\|$ is invariant under unitary transformations we take $x = e_1$ in what follows.

Let $B = (B_0, B_1, \ldots, B_m) \in \Sigma'_{(x,\alpha,\beta)}$ so that $P(B,\alpha,\beta)x = 0, v_B \neq 0$ and (α,β) is a multiple eigenvalue. If (α,β) is a semi-simple eigenvalue, then rank $P(B,\alpha,\beta) < n-1$ and therefore $\Pi_{v_B\perp}P(B,\alpha,\beta)|_{x^\perp}$ is singular. If (α,β) is a defective eigenvalue, then there is a generalized eigenvector x_1 such that $v_B + P(B,\alpha,\beta)x_1 = 0$. As $v_B \neq 0$, x_1 is not collinear to x. There exists $u \neq 0$ such that $x_1 = \rho x + u$ and $u^*x = 0$. Using $v_B = -P(B,\alpha,\beta)x_1 = -P(B,\alpha,\beta)u$ we have

$$\Pi_{v_B\perp}P(B,\alpha,\beta)|_{x^\perp}x_1 = 0$$

so that $\Pi_{v_B\perp}P(B,\alpha,\beta)|_{x^\perp}$ is singular. Let $V_B = [v_B, \widetilde{V}_B]$ be unitary. Then $V_B^*P(B,\alpha,\beta)U$ is in the form (20) and $P(\widetilde{B},\alpha,\beta) = \widetilde{V}_B^*P(B,\alpha,\beta)\widetilde{U}$ is a matrix representation of the operator $\Pi_{v_B\perp}P(B,\alpha,\beta)|_{x^\perp}$. We denote by $\mathscr{S}_{n-1}$ the set of $(n-1)\times(n-1)$ singular matrices. We have

$$\|P(\widetilde{B},\alpha,\beta)\| \leqslant \left(\sum_{k=0}^{m}|\alpha|^{2k}|\beta|^{2(m-k)}\right)\|B\|,$$

so that

$$d_{\mathrm{F}}(P(\widetilde{A},\alpha,\beta),\mathscr{S}_{n-1}) \leqslant \left(\sum_{k=0}^{m}|\alpha|^{2k}|\beta|^{2(m-k)}\right)d_{\mathrm{F}}(A,\Sigma'_{x,\alpha,\beta}). \tag{21}$$

Let $S \in \mathscr{S}_{n-1}$ be such that $d_{\mathrm{F}}(P(\widetilde{A},\alpha,\beta),S) = d_{\mathrm{F}}(P(\widetilde{A},\alpha,\beta),\mathscr{S}_{n-1})$. We define $\widetilde{B}$ by

$$\begin{aligned}\widetilde{B}_k &= \widetilde{A}_k - \bar{\alpha}^k\bar{\beta}^{(m-k)}L,\\ L &= \big(P(\widetilde{A},\alpha,\beta) - S\big)\Big/\left(\sum_{k=0}^{m}|\alpha|^{2k}|\beta|^{2(m-k)}\right).\end{aligned}$$

It is easy to check that $P(\widetilde{B},\alpha,\beta) = S$. Now, we consider $B \in \mathscr{M}_n(\mathbb{C})^{m+1}$ defined such that

$$\begin{aligned}(B_k)_{ij} &= (\widetilde{B}_k)_{i-1,j-1}, \quad 2 \leqslant i,j \leqslant n,\\ (B_k)_{1i} &= (A_k)_{1i}, \quad (B_k)_{i1} = (A_k)_{i1}, \quad 1 \leqslant i \leqslant n.\end{aligned}$$

As $P(B, \alpha, \beta)e_1 = 0$ and $P(\widetilde{B}, \alpha, \beta)$ is singular, $B \in \Sigma_{x,\alpha,\beta}$. We also have

$$\begin{aligned}
d_{\mathrm{F}}(P(\widetilde{A}, \alpha, \beta), S)^2 &= d_{\mathrm{F}}(P(\widetilde{A}, \alpha, \beta), P(\widetilde{B}, \alpha, \beta))^2 \\
&= \sum_{2\leqslant i,j\leqslant n} \left| \sum_{k=0}^{m} \alpha^k \beta^{m-k} (a_{ij} - b_{ij}) \right|^2 \\
&= \sum_{2\leqslant i,j\leqslant n} \left(\sum_{k=0}^{m} |\alpha|^{2k} |\beta|^{2(m-k)} \right)^2 |L_{ij}|^2 \\
&= \left(\sum_{k=0}^{m} |\alpha|^{2k} |\beta|^{2(m-k)} \right) \sum_{2\leqslant i,j\leqslant n} \sum_{k=0}^{m} |(A_k)_{i,j} - (B_k)_{ij}|^2 \\
&= \left(\sum_{k=0}^{m} |\alpha|^{2k} |\beta|^{2(m-k)} \right) \sum_{1\leqslant i,j\leqslant n} \sum_{k=0}^{m} |(A_k)_{i,j} - (B_k)_{ij}|^2 \\
&= \left(\sum_{k=0}^{m} |\alpha|^{2k} |\beta|^{2(m-k)} \right) d_{\mathrm{F}}(A, B)^2 \\
&\geqslant \left(\sum_{k=0}^{m} |\alpha|^{2k} |\beta|^{2(m-k)} \right) d_{\mathrm{F}}(A, \Sigma_{x,\alpha,\beta})^2,
\end{aligned}$$

so that

$$d_{\mathrm{F}}(P(\widetilde{A}, \alpha, \beta), \mathscr{S}_{n-1}) \geqslant \left(\sum_{k=0}^{m} |\alpha|^{2k} |\beta|^{2(m-k)} \right)^{1/2} d_{\mathrm{F}}(A, \Sigma_{x,\alpha,\beta}),$$

and from (21), equality holds. We now use a result due to Eckart and Young [8]:

$$d_{\mathrm{F}}(M, \mathscr{S}_n) = \|M^{-1}\|^{-1}.$$

We refer to Golub and Van Loan [12] or Stewart and Sun [23] for a proof. Therefore,

$$d_{\mathrm{F}}(P(\widetilde{A}, \alpha, \beta), \mathscr{S}_{n-1}) = \|P(\widetilde{A}, \alpha, \beta)^{-1}\|_{\mathrm{F}}^{-1} = \left\| \left(\Pi_{v^\perp} P(A, \alpha, \beta)_{|_{x^\perp}} \right)^{-1} \right\|.$$

Hence,

$$\begin{aligned}
C_1(A, x) &= \left(\sum_{k=0}^{m} |\alpha|^{2k} |\beta|^{2(m-k)} \right)^{1/2} \left\| \left(\Pi_{v^\perp} P(A, \alpha, \beta)_{|_{x^\perp}} \right)^{-1} \right\| \\
&= \left(\sum_{k=0}^{m} |\alpha|^{2k} |\beta|^{2(m-k)} \right)^{1/2} \Big/ d_{\mathrm{F}}(P(\widetilde{A}, \alpha, \beta, \mathscr{S}_{n-1})) \\
&\leqslant d_{\mathrm{F}}(A, \Sigma_{x,\alpha,\beta})^{-1}. \qquad \square
\end{aligned}$$

8. Numerical examples

Our tests have been performed with MATLAB, for which the working precision is $u = 2^{-53} \approx 1.1 \times 10^{-16}$.

As a first example we consider the problem with the homogeneous polynomial

$$P(A(\epsilon), \alpha, \beta) = \begin{bmatrix} \alpha^2 - 3\alpha\beta + 2\beta^2 & -\alpha^2 + \alpha\beta & -\alpha^2 + 9\beta^2 \\ 0 & \alpha^2 - \alpha\beta(1+\epsilon) & 0 \\ 0 & 0 & \alpha\beta - 3\beta^2 \end{bmatrix}, \quad (22)$$

or equivalently $A(\epsilon) = (A_0, A_1(\epsilon), A_2) \in \mathscr{M}_3(\mathbb{C})^3$ with

$$A_0 = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_1(\epsilon) = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -1-\epsilon & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 2 & 0 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix},$$

where ϵ is a positive constant. This problem is regular because

$$\det P = \alpha^5\beta - (7+\epsilon)\alpha^4\beta^2 + (17+6\epsilon)\alpha^3\beta^3 - (17+11\epsilon)\alpha^2\beta^4 + 6(1+\epsilon)\alpha\beta^5 \not\equiv 0.$$

There are six zero-lines (α_k, β_k), $1 \leqslant k \leqslant 6$, whose representatives in $\mathbb{C}^2$, chosen such that $|\alpha|^2 + |\beta|^2 = 1$, are given in Table 1, where $\nu = ((1+\epsilon)^2 + 1)^{-1}$. The two last lines of the table display the right and left eigenvectors normalized so that $\|x\|_\infty = 1$. We see that while $(\alpha_2, \beta_2) \neq (\alpha_4, \beta_4)$ and $(\alpha_5, \beta_5) \neq (\alpha_6, \beta_6)$ we have $x_2 = x_4$ and $y_5 = y_6$: the corresponding eigenvectors are the same.

For $\epsilon = \sqrt{u}$, the condition numbers are shown in Table 2. The eigenvalues $\alpha/\beta = 0, 2, 3$ and ∞ are simple and well-conditioned. Note that the condition number (18)

Table 1
Eigenvalues and right and left eigenvectors for $(A(\epsilon), x, \alpha, \beta)$ defined in (22)

k	1	2	3	4	5	6
(α_k, β_k)	$(0, 1)$	$\frac{1}{\sqrt{2}}(1, 1)$	$\nu(1+\epsilon, 1)$	$\frac{1}{\sqrt{5}}(2, 1)$	$\frac{1}{\sqrt{10}}(3, 1)$	$(1, 0)$
α_k/β_k	0	1	$1+\epsilon$	2	3	∞
x_k	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ \frac{\epsilon-1}{\epsilon+1} \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$
y_k	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{4} \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{5} \\ \frac{1}{5(1-\epsilon)} \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Table 2
Condition numbers for $(A(\epsilon), x, \alpha, \beta)$ as defined in (22) with $\epsilon = \sqrt{u}$

α_k/β_k	0	1	$1+\epsilon$	2	3	∞
$C_1(A(\epsilon), x)$	1.7	1×10^8	3×10^8	5.1	29.4	2.5
$C_2(A(\epsilon), \alpha, \beta)$	1.0	3.6	1.2	4.8	0.9	1.4

Table 3
Condition numbers for $(A(\epsilon), x, \alpha, \beta)$ as defined in (22) with $\epsilon = -1 + \sqrt{u}$

α/β	0	$\sqrt{u}$
$C_1(A(\epsilon), x)$	3×10^7	3×10^7
$C_2(A(\epsilon), \alpha, \beta)$	8×10^7	8×10^7

in [24] for nonhomogeneous PEPs is not defined for zero eigenvalues and infinite eigenvalues. As ϵ is small, α_2/β_2 and α_3/β_3 are close to an eigenvalue of multiplicity 2 which is semi-simple (x_2 and x_3 are not collinear). As predicted by the theory, the condition numbers of the corresponding eigenvectors are large; their inverses measure the distance to the nearest ill-posed problem. On the other hand, the condition number for the eigenvalue is small because

$$|y_i(\bar{\beta}_i \mathscr{D}_\alpha P(A(\epsilon), \alpha_i, \beta_i) - \bar{\alpha}_i \mathscr{D}_\beta P(A(\epsilon), \alpha_i, \beta_i))x_i| \geqslant 0.2, \quad i = 2, 3.$$

For $\epsilon = -1 + \sqrt{u}$, the two eigenvalues (α_1, β_1) and (α_3, β_3) are close to a defective eigenvalue. Their condition numbers are shown in Table 3. In this case, $v_i = (\bar{\beta}_i \mathscr{D}_\alpha P(A(\epsilon), \alpha_i, \beta_i) - \bar{\alpha}_i \mathscr{D}_\beta P(A(\epsilon), \alpha_i, \beta_i))x_i \approx e_1$ for $i = 1, 3$, $y_i \perp e_1$ for $i = 1, 3$ and therefore $C_2(A(\epsilon), \alpha, \beta)$ is large. Also, as $v_i \neq 0$ the problem is expected to be close to an ill-posed problem. This is confirmed by the large condition number for the eigenvector $C_1(A(\epsilon), x)$.

We now compare our condition number for the eigenvector with the condition number $\kappa_\lambda(x)$ in (19) that was obtained in the nonhomogeneous case. We take $\epsilon = 1 + 10^{-6}$ so that the two eigenvalues 2 and $1 + \epsilon$ are clustered and consider two different normalizations in the nonhomogeneous case. We measure the perturbations in an absolute sense so that $\alpha_0 = \alpha_1 = \alpha_2 = 1$ in (19). We find that for the eigenvector corresponding to $(\alpha, \beta) = (2, 1)$ or $\lambda = \alpha/\beta = 2$

$$\begin{aligned} & C_1(A(\epsilon), x) = 2 \times 10^6, \\ \text{for } g = x: \quad & \kappa_\lambda(x) = 1 \times 10^7, \\ \text{for } g = y: \quad & \kappa_\lambda(x) = 7 \times 10^{12}. \end{aligned}$$

This example shows that in the homogeneous case, the sensitivity of an eigenvector can depend strongly on how it is normalized.

As a second illustration, we consider the problem

$$P(A(\epsilon), \alpha, \beta) = \beta^2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \alpha\beta \begin{bmatrix} -2-\epsilon & 1 \\ 0 & -2 \end{bmatrix} + \alpha^2 \begin{bmatrix} 1+\epsilon & 1 \\ 0 & 0 \end{bmatrix}, \qquad (23)$$

with $\epsilon > 0$ and with eigenvalues

$$\{(0, 1), (1, 1), (1 + \epsilon, 1), (2, 1)\}.$$

The sets of right eigenvectors X and left eigenvectors Y are given by

$$X = \begin{bmatrix} 1 & 1 & 1 & -1 \\ -(1+\epsilon) & 0 & 0 & \frac{1-\epsilon}{7} \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 3 & a & 1 \end{bmatrix},$$

where $a = (\epsilon^2 + 3\epsilon + 3)/(\epsilon^2 - 1)$. For $\epsilon = \sqrt{u}$, $A(\epsilon)$ is close to a tuple that has a double eigenvalue which is defective. Surprisingly, the condition number for the eigenvector corresponding to the eigenvalues $\alpha_2/\beta_2 \approx \alpha_3/\beta_3 \approx 1$ is small; we found that $C_1(A(\epsilon), x) \approx 2$. In the proof of Theorem 7.1, we showed that $\Pi_{v^\perp} P(A(\epsilon), \alpha, \beta)_{|_{x^\perp}}$ is singular when (α, β) is a defective eigenvalue and $v \neq 0$. In this case, for $\epsilon = 0$ and $(\alpha, \beta) = (1, 1)$, $v = 0$.

9. Bihomogeneous Newton method

The idea of using Newton's method to solve the eigenvalue problem $Ax = \lambda x$ goes back to Unger [27] and has been investigated further by Peters and Wilkinson [17] and most recently by Tisseur [25]. Newton's method for the nonhomogeneous polynomial eigenvalue problem has been studied by Ruhe [18].

In this section, we apply the multihomogeneous Newton method of Dedieu and Shub [6] to the homogeneous PEP. The multihomogeneous Newton method generalizes to multihomogeneous systems the projective version of Newton method introduced by Shub in [19] and studied by Shub and Smale [20–22].

Given a regular $(m + 1)$-tuple $A = (A_0, A_1, \dots, A_m)$, our aim is to compute an eigenvector x and an eigenvalue (α, β) associated with this $(m + 1)$-tuple. Hence, we want to find the zeros of the bihomogeneous function

$$F\colon \mathbb{C}^n \times \mathbb{C}^2 \to \mathbb{C}^n, \quad F(x, \alpha, \beta) = P(A, \alpha, \beta)x.$$

From [6] we have that the bihomogeneous Newton map $N_F\colon \mathbb{P}_{n-1} \times \mathbb{P}_1 \to \mathbb{P}_{n-1} \times \mathbb{P}_1$ associated with F is given by

$$N_F(x, \alpha, \beta) = (x, \alpha, \beta) - \left(\mathscr{D}F(x, \alpha, \beta)_{|_{x^\perp \times (\alpha,\beta)^\perp}}\right)^{-1} F(x, \alpha, \beta).$$

The bihomogeneous Newton method has the usual properties of a Newton method: the fixed points for N_F correspond to the zeros for F and the convergence of Newton sequence to these zeros is quadratic.

Let $(x', \alpha', \beta') = N_{\mathrm{F}}(x, \alpha, \beta)$. Then

$$\mathscr{D}F(x, \alpha, \beta)(x' - x, \alpha' - \alpha, \beta' - \beta) = -F(x, \alpha, \beta),$$

with $x' - x \in x^{\perp}$ and $(\alpha' - \alpha, \beta' - \beta) \in (\alpha, \beta)^{\perp}$. Thus, x' and (α', β') are given by

$$\begin{aligned}
&P(x' - x') + (\alpha' - \alpha)\mathscr{D}_\alpha P x + (\beta' - \beta)\mathscr{D}_\beta P x = -Px,\\
&\langle x', x\rangle = \langle x, x\rangle,\\
&\langle(\alpha', \beta'), (\alpha, \beta)\rangle = \langle(\alpha, \beta), (\alpha, \beta)\rangle,
\end{aligned}$$

or equivalently

$$\begin{bmatrix} P & \mathscr{D}_\alpha P x & \mathscr{D}_\beta P x \\ x^* & 0 & 0 \\ 0 & \bar{\alpha} & \bar{\beta} \end{bmatrix} \begin{bmatrix} x' - x \\ \alpha' - \alpha \\ \beta' - \beta \end{bmatrix} = \begin{bmatrix} -Px \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, the bihomogeneous Newton iteration for the homogeneous PEP is given by

$$\mathscr{J}_P(x_k, \alpha_k, \beta_k) \begin{bmatrix} x_{k+1} - x_k \\ \alpha_{k+1} - \alpha_k \\ \beta_{k+1} - \beta_k \end{bmatrix} = - \begin{bmatrix} P(A, \alpha_k, \beta_k)x_k \\ 0 \\ 0 \end{bmatrix}, \quad k \geqslant 0, \tag{24}$$

where (x_0, α_0, β_0) is a given starting guess and $\mathscr{J}_P$ is as in (7). The condition number $\kappa(\mathscr{J}_P)$ of the matrix involved in the bihomogeneous Newton iteration is related to the condition number of the eigenvalue the method is trying to approximate. Let

$$P(A, \alpha, \beta) = U\Sigma V^* = [\widetilde{U}, y] \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} [\widetilde{V}, x]^*$$

be the singular value decomposition of $P(A, \alpha, \beta)$ and let

$$\begin{aligned}
\widetilde{\mathscr{J}} &:= \begin{bmatrix} U^* & \\ & I \end{bmatrix} \mathscr{J}_P(x, \alpha, \beta) \begin{bmatrix} V & \\ & I \end{bmatrix} \\
&= \begin{bmatrix} \Sigma & 0 & U^*\mathscr{D}_\alpha P x & U^*\mathscr{D}_\beta P x \\ 0 & 0 & y^*\mathscr{D}_\alpha P x & y^*\mathscr{D}_\beta P x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \bar{\alpha} & \bar{\beta} \end{bmatrix}.
\end{aligned}$$

Let

$$Q = \begin{bmatrix} I & 0 & 0 \\ 0 & 1 & \bar{\beta} \\ 0 & 0 & -\bar{\alpha} \end{bmatrix}$$

so that

$$\widetilde{\mathscr{J}} Q = \begin{bmatrix} \Sigma & 0 & U^*\mathscr{D}_\alpha P x & U^* v \\ 0 & 0 & y^*\mathscr{D}_\alpha P x & y^* v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \bar{\alpha} & 0 \end{bmatrix},$$

where $v = (\bar{\beta}\mathscr{D}_\alpha P - \bar{\alpha}\mathscr{D}_\beta P)x$. We denote by $\widetilde{\mathscr{J}}_0$ the matrix $\widetilde{\mathscr{J}}$ with y^*v in the last column replaced by 0. $\widetilde{\mathscr{J}}_0$ is a singular matrix so that its smallest singular value $\sigma_{n+2}^0 = 0$. Let σ_{n+2} be the smallest singular value of $\widetilde{\mathscr{J}}$. Then from [12, Theorem 2.5.3],

Table 4
Condition numbers for $(A(\epsilon), x, \alpha, \beta)$ defined in (22) with $\epsilon = 10^{-1}$

α_j/b_j	0	1	1.1	2	3	∞
$E_0(x,\alpha,\beta)$	1.7e − 2	1.7e − 2	2.0e − 2	1.5e − 2	1.5e − 2	1.4e − 2
$E_1(x,\alpha,\beta)$	1.0e − 4	2.7e − 3	4.0e − 3	6.1e − 4	4.2e − 4	3.1e − 4
$E_2(x,\alpha,\beta)$	9.5e − 9	2.1e − 5	7.1e − 6	8.3e − 7	5.6e − 8	2.6e − 7
$E_3(x,\alpha,\beta)$	9.2e − 17	1.3e − 10	9.8e − 11	2.3e − 14	6.4e − 15	1.6e − 13
$E_4(x,\alpha,\beta)$		1.1e − 16	6.4e − 16	1.2e − 16	8.9e − 16	1.1e − 16

$$\sigma_{n+2} = |\sigma_{n+2} - \sigma^0_{n+2}| \leqslant \|\widetilde{\mathscr{J}} - \widetilde{\mathscr{J}}_0\|_2 = |y^* v|.$$

Hence,

$$\begin{aligned}\kappa(\mathscr{J}_P(x,\alpha,\beta)) &\geqslant \frac{\kappa(\widetilde{\mathscr{J}}Q)}{\kappa(Q)} \geqslant \frac{\kappa(Q)^{-1}\|\widetilde{\mathscr{J}}Q\|_2}{|y^*v|} \\ &\geqslant \frac{\left(\sum_{k=0}^m |\alpha|^{2k}|\beta|^{2(m-k)}\right)^{-1/2}}{\kappa(Q)\|x\|\|y\|} C_2(A,\alpha,\beta).\end{aligned}$$

Therefore, as long as Q is not too ill-conditioned, the condition number of the matrix involved in the bihomogeneous Newton iteration will be large when (α, β) is an ill-conditioned eigenvalue. A similar conclusion holds for the Jacobian matrix in the Newton method for nonhomogeneous PEPs with normalization condition $x^*Q'(A, \lambda)x = 1$. Thus from the numerical point of view, the bihomogeneous Newton method will not perform better than Newton's method for nonhomogeneous PEPs when approximating eigenvalues with large condition numbers. However, an advantage of the bihomogeneous Newton method is that infinite eigenvalues can be computed in the same way as finite eigenvalues.

We implemented the bihomogeneous Newton method in MATLAB. To illustrate the method we consider the matrix polynomial $P(A(\epsilon), \alpha, \beta)$ defined in (22), for which the eigenvalues and eigenvectors are known exactly (see Table 1). We chose $\epsilon = 0.1$ so that the eigenvalues are well separated.

As a starting guess, we use a random perturbation of the order 10^{-2} of the true eigenpairs. We measure the error in the approximate eigenpair by

$$\begin{aligned}E_k(x,\alpha,\beta) &= \|(x,\alpha,\beta) - (x_k,\alpha_k,\beta_k)\|_{x^\perp \times (\alpha,\beta)^\perp} \\ &= \left(\frac{\|x - x_k\|^2}{\|x\|^2} + \frac{|\alpha-\alpha_k|^2 + |\beta-\beta_k|^2}{|\alpha|^2+|\beta|^2}\right)^{1/2},\end{aligned}$$

where (x, α, β) is the true eigenpair we are seeking and (x_k, α_k, β_k) the kth iterate. The errors $E_k(x, \alpha, \beta)$ are given in Table 4 and illustrate the quadratic convergence of the iterates. We see that the infinite eigenvalue is found with no more difficulty than the finite ones.

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