Flutter in a quasi-one-dimensional model of a collapsible channel

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The effects of wall inertia on instabilities in a collapsible channel with a long finite-length flexible wall containing a high Reynolds number flow of incompressible fluid are studied. Using the ideas of interactive boundary layer theory, the system is described by a one-dimensional model that couples inviscid flow outside the boundary layers formed on the channel walls with the deformation of the flexible wall. The observed instability is a form of flutter, which is superposed on the behaviour of the system when the wall mass is neglected. We show that the flutter has a positive growth rate because the fluid loading acts as a negative damping in the system. We discuss these findings in relation to other work on self-excited oscillations in collapsible channels.

Key words: collapsible channel, flutter, self-excited oscillations, interactive boundary layer theory.

1. Introduction

Self-excited oscillations in a collapsible channel have been studied widely in the fluid-mechanical literature because of their relevance to various physiological phenomena, such as wheezing in lung airways and Korotkov sounds in blood-pressure measurement. Similar instabilities are readily observed in experimental collapsible-tube systems, such as the Starling resistor (Bertram and Elliott, 2003), but are yet to be fully understood in all flow regimes (Carpenter and Pedley, 2003). Moreover, although collapsible channels are inherently global systems with self-excited oscillations often being driven or considerably modified by the presence of finite boundaries (Stewart et al., 2009), they are undoubtedly linked to the local instabilities in homogeneous domains with flexible walls (Carpenter and Garrad, 1985, 1986; Davies and Carpenter, 1997; Stewart et al., 2010a).

In the simplest collapsible channel model a finite-Reynolds-number fluid flow is driven through a two-dimensional channel that has a finite segment of one of the walls replaced with a prestretched elastic membrane under external pressure. Depending on the system parameters and the controlled boundary conditions, different types of oscillations that correspond to different modes, or number of local extrema in the wall shape, are observed. When the pressure drop is controlled in the system, it typically loses stability through mode-1 ‘sloshing’ oscillations, described by Jensen and Heil (2003) and Stewart et al. (2010b). The mechanism for sustaining these oscillations relies on the kinetic energy flux into the system.
being greater than the energy loss due to viscous dissipation. Hence ‘sloshing’ is only observed in channels whose downstream rigid segments are longer than the upstream rigid segments. At lower values of the wall tension, these oscillations are replaced by large-amplitude ‘slamming’ oscillations, during which the membrane briefly almost touches the opposite wall (Stewart et al., 2010b). On the other hand, when the flow rate is the controlled boundary condition, the self-excited oscillations appear as a consequence of instability of the steady solution due to small-amplitude mode-2 perturbations (Luo and Pedley, 1996; Luo et al., 2008). If the flexible wall of the collapsible channel is long, but similar in length to the downstream rigid section, then an oscillatory mode arises through an interaction between two static eigenmodes (Xu et al., 2013). Crucially however neither of the classical scenarios includes significant wall inertia, which is believed to complicate the dynamical picture even further. For example, for a non-zero wall mass Luo and Pedley (1998) observed a higher-frequency flutter instability superimposed on the original large-amplitude, low-frequency oscillations. Unlike the oscillations of the massless wall, the flutter grew with time from a small amplitude until it completely dominated the original slower mode. Therefore, one anticipates a whole new range of mechanisms operating when the structural inertia is accounted for as well.

One such mechanism was explored by Huang (2001) who studied a finite-length channel conveying a Poiseuille flow with a segment of one of the walls replaced by a heavy tensioned membrane. By focusing on the linear stability of the uniform Poiseuille flow, the author identified eigenmodes related to flutter, which were stabilised by the structural damping in the system, but destabilised by the fluid viscosity. The asymmetric loading created by the fluid on the membrane led to oscillations with different preferential modes in the membrane shape depending on the controlled boundary conditions.

Another flutter mechanism in collapsible channels was studied by Mandre and Mahadevan (2009), who explained it as a 1 : 1 resonance involving a coincidence of two frequencies in a dynamical system. However, this type of flutter was limited to systems, dominated by the normal mode oscillations of elastic components, but also susceptible to weak fluid forcing with time scale much smaller that the natural time scale of the elastic system oscillations.

Here we present a different form of flutter in a collapsible channel, which relies on coupling between the inviscid flow outside the boundary layers formed on the channel walls and the deformation of the flexible heavy membrane under longitudinal tension. We start by presenting our model based on interactive boundary layer theory in section 2. In section 3 we present our results, and discuss them in section 4.

2. The model

We focus on a high Reynolds number flow ($Re \gg 1$) of incompressible viscous fluid, with mean velocity $U_0$, density $\rho$ and kinematic viscosity $\nu$, in a two-dimensional channel of width $a$ with a long slender indentation in the lower wall of a finite length $\lambda a$, $\lambda \gg 1$, and amplitude $a\varepsilon F$, $\varepsilon \ll 1$ (see Fig. 1). The flow rate at the
Figure 1. Diagram of the collapsible channel. All quantities are given in dimensional variables. The line in the middle of the channel indicates the streamline displacement due to the elastic wall deformation.

entrance is taken to be fixed. Under the following non-dimensionalization

\[
\begin{align*}
[\hat{x}, \hat{y}] &= a[\lambda x, y], \quad [\hat{u}, \hat{v}] = U_0 \left[ u \frac{v}{\lambda} \right], \quad \hat{p} = \rho U_0^2 p, \quad \hat{t} = \frac{10\lambda^3 a}{U_0^2} t, \quad Re = \frac{a U_0}{\nu},
\end{align*}
\]

where \(\hat{x}\) indicates a dimensional variable, the flow field in the inviscid core, outside the boundary layers formed on the channel walls, is found to be:

\[
\begin{align*}
\hat{u} &= u_0(y) + \varepsilon A(x, t) u'_0(y) + \varepsilon^2 u_2(x, y, t) + O(\varepsilon^3), \\
\hat{v} &= -\varepsilon A_x(x, t) u_0(y) + \varepsilon^2 v_2(x, y, t) + O(\varepsilon^3), \\
\hat{p} &= -\frac{12\lambda_x}{Re} + \varepsilon^2 \left[ P(x, t) + A_{xx}(x, t) \frac{1}{\lambda^2 \varepsilon} \int_0^y u_0^2(\tilde{y})d\tilde{y} \right] + O(\varepsilon^3),
\end{align*}
\]

where \(Re^{1/7} \leq \lambda \ll Re\) and \(Re^{-2/7} \ll \varepsilon \ll 1\) (see Pedley (2000)). Here \(u_0(y) = 6y(1-y)\) is the Poiseuille velocity profile, \(-12\lambda_x/Re\) is a small term required to drive the Poiseuille flow, \(-\varepsilon A_x(x, t)\) is the lateral displacement of streamlines in the core flow, which would in the absence of the indentation stay straight, and \(P\) is the unknown contribution to the pressure \(p\). From (2.3) it can also be seen that

\[
\hat{p}_y = \varepsilon^2 \frac{5\sigma}{6} A_{xx} u_0^2(y) + O(\varepsilon^3),
\]

where the parameter \(\sigma = 6/5\lambda^2 \varepsilon\) represents the importance or otherwise of the cross-stream pressure gradient. Similar systems have been studied many times before, most notably by Smith (1976a,b); Bogdanova and Ryzhov (1983); Pedley and Stephanoff (1985); Pedley (2000); Pihler-Puzović and Pedley (2013). We follow Pedley and Stephanoff (1985) in assuming that the height of the indentation is much greater than the thickness of the boundary layers which form at the walls, but is still small compared to 1, and \(\sigma = O(1)\). This assumption allows us to proceed to \(O(\varepsilon^2)\) in the x-momentum Navier-Stokes equation to obtain the following equation in the inviscid core

\[
\begin{align*}
\sigma u_0 u_2x + u_0' v_2 + \frac{\sigma}{12} A_t u_0' + A A_x ((u_0')^2 - u_0 u_0'') = -P_x - \frac{5\sigma}{6} A_{xxx} \int_0^y u_0^2(\tilde{y})d\tilde{y}.
\end{align*}
\]
and using the no-penetration boundary conditions
\[ v = \varepsilon \left( uF_x + \frac{1}{10\lambda^2} F_t \right) \quad \text{on} \quad y = \varepsilon F(x, t) \quad \text{and} \quad v = 0 \quad \text{on} \quad y = 1, \]
find the unknown velocity terms from (2.4) in the vicinity of \( y = 0 \) and \( y = 1 \). It follows that,
\[ \begin{align*}
\text{at } y = 0: & \quad 36(FA_x + AF_x + FF_x) + \frac{\sigma}{2} F_t + \frac{\sigma}{2} A_t + 36AA_x = -P_x, \\
\text{at } y = 1: & \quad -\frac{\sigma}{2} A_t + 36AA_x = -P_x - \sigma A_{xxx}.
\end{align*} \]

If \( P \) is eliminated from equations (2.5) and (2.6), the equation for the streamline displacement becomes:
\[ A_t = A_{xxx} - \frac{1}{2} F_t - \frac{36}{\sigma} \left[ FA + \frac{1}{2} F^2 \right]_x, \]
which was solved previously by Pedley and Stephanoff (1985) for a prescribed \( F(x, t) \). They predicted and confirmed experimentally the formation of a wavetrain downstream from the indentation, when the wall was forced with a given frequency. Downstream and upstream from the flexible wall (2.7) reduces to the linearized Korteweg-de Vries (KdV) equation, which supports downstream propagating waves whose group velocity is three times larger than the phase velocity. A common feature of the flow in two-dimensional flexible channels (Luo and Pedley, 1996, 1998), these waves were named ‘vorticity waves’, because they owe their existence to the non-zero vorticity gradient in the main flow. For \( \sigma = O(1) \) the scalings of the model correspond to the lower branch Tollmien-Schlichting (T-S) neutral stability curve, suggesting further that the vorticity waves are a manifestation of the T-S instability. It is yet to be established if the vorticity waves are the cause or a consequence of the self-excited oscillations in collapsible channels (Stewart et al., 2010c; Xu et al., 2013).

In derivation of equation (2.7) it was implicitly assumed that the boundary layers at the channel walls stay attached at all times and there is no breakaway separation appearing in the flow. However, for some deformations of the wall a small region of reversed flow appears at the upstream corner of the hump as well as at its downstream end, and at a finite time \( t_{cr} \) the solution to the classical unsteady boundary-layer problem with an adverse pressure gradient breaks down due to the van Dommelen singularity (van Dommelen and Shen, 1980). As soon as breakaway separation of the boundary layer appears (or, in other words, for \( t > t_{cr} \)), (2.7) becomes invalid. Pedley and Stephanoff (1985) concluded that if \( F_x \) is \( O(1) \) and \( \sigma \) is \( O(1) \), then so is \( t_{cr} = O((\lambda^2 \varepsilon)^{-1}) \). As \( \lambda^2 \varepsilon \) becomes smaller, the inviscid theory remains valid for a longer time. These estimates were also confirmed by the experimental findings, albeit for the case of prescribed wall oscillations.

Unlike Pedley and Stephanoff (1985), we will be solving (2.7) for the elastic wall freely interacting with the fluid. Assuming that the wall is considerably prestretched, the membrane equation for the flexible wall movement is
\[ MF_{tt} - TF_{xx} = P_{ext} - P, \quad (2.8) \]
where $T$ is the initial longitudinal tension, which is constant up to the required order of accuracy, $p_{ext}$ is the pressure external to the channel and $M$ represents the mass of the flexible wall. These quantities are related to the dimensional tension $\hat{T}$, external pressure $\hat{p}_{ext}$ and membrane mass per unit area $\hat{M}$ via

$$
\hat{T} = \varepsilon \rho U_0^2 \lambda \alpha T,
\hat{p}_{ext} = \varepsilon^2 \rho U_0^2 P_{ext}
\text{ and } \hat{M} = 10^2 \varepsilon \rho \lambda \alpha M.
$$

With the membrane equation (2.8) we close the fluid-structure interaction problem (2.6)-(2.7). Note that all unknowns $A, P, F$ are functions of time $t$ and longitudinal coordinate $x$ only.

Pihler-Puzović and Pedley (2013) and Kudenatti et al. (2012) showed that if the pressure is fixed upstream from the membrane region, as well as the flow rate, the problem (2.6)-(2.8) is ill-posed. Therefore, we only study these equations when $P$ is fixed downstream. In its essence the model is inviscid, so if waves propagate in the rigid channels, boundary conditions will be needed to suppress their reflection from the ends. In the experiments, the waves would be attenuated by the presence of viscosity and would (subject to the rigid channels being sufficiently long) decay before the end of the domain. For the theoretical model formulated here this implies that the correct boundary conditions need to be given at infinity: far away from the membrane, both upstream and downstream, the flow should be unperturbed. However, since any numerical solution is only possible on a finite domain, we instead apply absorbing boundary conditions (ABCs) at the upstream end of the domain $x = b_1$ (for derivation of the boundary conditions, please see Appendix A)

$$
A(b_1, t) - I_t^{2/3} A_{xx}(b_1, t) = 0,
$$

$$
A_x(b_1, t) + I_t^{1/3} A_{xx}(b_1, t) = 0,
$$

where

$$
I_t^\delta f(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t - \tau)^{\delta - 1} f(\tau) d\tau, \quad t > 0.
$$

Here $\Gamma(*)$ is the Gamma function. Similarly, if $x = b_2$ is the downstream end of the system, then the ABC for the downstream end has the form

$$
A(b_2, t) - I_t^{1/3} A_x(b_2, t) + I_t^{2/3} A_{xx}(b_2, t) = 0.
$$

The equations (2.6)-(2.8) subject to the derived boundary conditions were solved numerically using a combination of Chebyshev collocation points for the spatial discretization with the Crank-Nicolson scheme for the time evolution of the system. The details of the numerical method are presented in Appendix B).

3. Results

(a) Solution of the steady problem

In the following we briefly study the stationary version of the temporal problem (2.6)-(2.8). Although the real interest lies in unsteady results, if the system is stable, in the long-time limit it should converge to a particular solution of the steady problem. Moreover, the model was derived using multiple assumptions,
Figure 2. (a) The membrane shape $F^s(x)$, (b) the streamline displacement $A^s(x)$ and (c) pressure distribution $P^s(x)$ plotted for $P_{ext} = 100$, $\sigma = 0.0001$ and $T = 100$. The solution obtained numerically is indicated with the crosses, whereas the analytical solution is represented with the solid line.

Figure 3. (a) Solutions of the inner (dots) and outer (crosses) problems in the vicinity of $x = 0$ for the streamline displacement $A^s(x)$. $P_{ext} = 100$, $\sigma = 0.0001$ and $T = 100$. The exact solution obtained numerically is represented by the solid line. (b) The boundaries for existence of the steady solution in $T - P_{ext}$ parameter space on the logarithmic scale for different values of $\sigma$. Crosses correspond to the computational data, whereas lines are generated by connecting data points by cubic spline. The direction of increasing $\sigma$ is indicated with an arrow.

which suggests that it is likely to break down in certain regions of parameter space. By studying the steady problem we will explore these regions. The steady equations are invariant with respect to the map $0 \rightarrow 1$ and $1 \rightarrow 0$, so we expect symmetry of solutions about $x = 0.5$.

(a.1) Large tension, large pressure and small cross-stream pressure gradient limit

The steady problem for $A = A^s(x)$, $F = F^s(x)$ and $P = P^s(x)$ can be solved easily in the asymptotic limit of large tension $T \gg 1$, large external pressure $P_{ext} = TP_e$ and small cross-stream pressure gradient $\sigma \ll 1$. The solution in $0 \leq x \leq 1$ is
ought to introduce a magnified inner coordinate perturbation problem. Therefore, in order to solve the problem completely, the steady version of the equation (2.7) is lost and we have to deal with a singular \( \sigma \) (Puzović, 2011), because in the limit of \( \sigma = 0 \) the function \( F(x) \) for the function varied, but all other parameters, such as the external pressure \( P_{\text{ext}} \) and the cross-stream pressure gradient parameter \( \sigma \), were kept constant (Guneratne and Pedley, 2006). We find a similar bifurcation structure of the steady solution (see Fig. 4) with one major qualitative difference: in our model the membrane takes only one form at each of the membrane ends and consider the appropriate transformed equation.

The direction of increasing tension is indicated with an arrow. Membrane shapes are given for \( T \) equal to 0.05, 0.055, 0.06, 0.065, 0.07, 0.075, 0.08, 0.085, 0.9, 0.095 and 1 respectively.

(see also Fig. 2)

\[
F^s(x) = F_0(x) + T^{-1} F_1(x) + O(T^{-2})
\]

\[
= \frac{1}{2} P_0 (1 - x) + T^{-1} \left( \frac{3}{160} P_0^2 (x - 5x^4 + 6x^5 - 2x^6) + O(T^{-2}) \right),
\]

\[
A^s(x) = A_0(x) + T^{-1} A_1(x) + O(T^{-2})
\]

\[
= -\frac{1}{2} F_0 + T^{-1} \left( -\frac{1}{2} F_1 + \frac{1}{16} \sigma F_0 \right) + O(T^{-2}),
\]

\[
P^s(x) = P_0(x) + T^{-1} P_1(x) + O(T^{-2})
\]

\[
= -\frac{9}{2} F_0^2 + T^{-1} (18 F_1 A_0 - \frac{9}{4} \sigma F_0 (A_0 + F_0)) + O(T^{-2}).
\]

This solution does not satisfy the steady version of the boundary conditions for the function \( A \) at the membrane ends (\( A^s_0 = 0 \) at \( x = 0 \) and \( x = 1 \), see Pihler-Puzović (2011)), because in the limit of \( \sigma \ll 1 \) the highest derivative in the steady version of the equation (2.7) is lost and we have to deal with a singular perturbation problem. Therefore, in order to solve the problem completely, we ought to introduce a magnified inner coordinate \( X = \frac{x}{\gamma} \) within an \( O(\gamma) \) vicinity of each of the membrane ends and consider the appropriate transformed equation. The inner problem does not have an analytical solution, but a relatively simple numerical solution is in good agreement with the steady state solver (see Fig. 3a).

(a.2) Steady simulations

Other studies of collapsible channels have previously reported a saddle-node bifurcation of the steady channel configurations if the membrane tension \( T \) was varied, but all other parameters, such as the external pressure \( P_{\text{ext}} \) and the cross-stream pressure gradient parameter \( \sigma \), were kept constant (Guneratne and Pedley, 2006). We find a similar bifurcation structure of the steady solution (see Fig. 4) with one major qualitative difference: in our model the membrane takes only

![Figure 4](image-url)
mode-one configurations. Moreover, for \( P_{ext} = 0 \) the only solution to our problem is an unperturbed Poiseuille flow in a parallel-sided channel (or \( A^* = F^* = P^* = 0 \)); the confirmation of this statement comes from solving a perturbation problem to the trivial solution.

Perhaps unsurprisingly, for each fixed cross-stream pressure gradient and non-zero external pressure, there exists a critical tension below which the steady problem has no solution and the model breaks down. On the other hand, for larger tensions there are two possible steady solutions. The lower branch solution in Fig. 4b corresponds to the less collapsed channel as the tension is increased. An infinite tension gives the limit of a rigid wall, so we anticipate (and confirm later in this paper) that if the system is stable, the unsteady simulations converge to this particular branch. On the other hand, the upper branch in Fig. 4c is never stable and it is clearly unphysical.

Of some interest is the bifurcation point. If \( \sigma \) is fixed at a small value, the boundary for existence of the steady solutions represents a parabola (see Fig. 3b). Although for higher values of \( \sigma \) this dependence is strictly speaking more complicated, the numerical results nevertheless suggest a weak dependence of the boundary position on the value of \( \sigma \) (see the different lines in Fig. 3b). When \( \sigma = 0 \), the steady equations simplify to \( A^*(x) = -\frac{1}{2} F^*(x) \) at the leading order. Therefore, the nature of the bifurcation point can be inferred from the problem

\[
\begin{align*}
\frac{d^2 f}{dx^2} + \frac{9}{2} f^2 + \tilde{p}_{ext} &= 0, \\
 f|_{x=0} = f|_{x=1} &= 0,
\end{align*}
\]

where \( P_{ext} = \tilde{p}_{ext} T^2 \) and \( F^*(x) = T f(x) \). From (3.1)-(3.2) it can be shown that there exists a constant \( C \) such that for \( \tilde{p}_{ext} > C \) there are no solutions to the steady problem (Pihler-Puzović, 2011).

(b) Unsteady problem

Temporal behaviour of the model was studied by perturbing the steady solution (i) by increasing the external pressure from zero to the constant value to mimic an experimental setup \( (P_{ext}(t) = P_{ext}^0 (1 - (1 + \exp(2(t - 10))))^{-1}) \) as in Fig. 6), (ii) by introducing a short-time spike-like perturbation to the external pressure \( (P_{ext}(t) = P_{ext}^0 (-100(t - 1/2)^2) \) as in Fig. 7 (a)), and (iii) by starting the simulations from the steady solution corresponding to a point in parameter space close to the parameter values of interest.

(b.1) \( M = 0 \).

In the regime of \( M = 0 \), no fluid-structure instabilities are observed and an ‘inviscid mechanism’, similar to the one reported by Kudenatti et al. (2012) and Pihler-Puzović and Pedley (2013), acts to stabilise the system. When \( \sigma \sim 0 \), the same asymptotic expansion as for the steady solution in section 3 (a.1) suggests that the problem (2.6)-(2.8) is quasi-stationary at all orders away from the membrane ends \( (x = 0 \) and \( x = 1) \). On the other hand, increasing the influence of the cross-stream pressure gradient by making \( \sigma \sim O(1) \) has little impact on the stability of the system: as the external pressure is increased from zero, the system adjusts to the changes in \( P_{ext} \) and in the long-time limit always tends to a lower
Figure 5. In (a)-(c), the behaviour of the pressure $P(0.5,t)$, membrane displacement $F(0.5,t)$ and streamline displacement $A(0.5,t)$ at the point $x = 0.5$ with time $t$ for $T = P_{\text{ext}} = \sigma = 1$. The corresponding steady solutions are indicated with solid lines. In (d), plot of the difference between instantaneous values of these functions and constants corresponding to the steady solution for $t > 20$ on a logarithmic scale. The linear fits to curves are shown as solid lines. The slope of the solid lines in (d) is $-\frac{1}{3}$.

Figure 6. (a) Time-evolution of the membrane displacement $F(0.5,t)$ at $x = 0.5$ obtained by solving (2.6)-(2.8) when $\sigma = T = P_{\text{ext}} = 1$, the external pressure is increased from zero using the sigmoid function and $M = 0.01$ (curve with oscillations) and $M = 0$ (curve without oscillations). (b) The instantaneous membrane shapes $F(x,t)$ during mode-2 oscillations in (a) at given times $t$.

branch stationary solution as in Figure 5 (a)-(c). Furthermore, for $M = 0$ the solution to the fully coupled nonlinear problem has a distinct algebraic decay to the steady solution (Figure 5 (d)). This follows from the solution to the linearised KdV equation in the rigid segments (Pihler-Puzović (2011)).

(b.2) $M \neq 0$.

Adding wall inertia to the model has a dramatic effect on the stability of the system - instabilities are now readily observed, independent of the way the system is perturbed.

If the simulations are started by increasing the external pressure to some finite value of interest, the elastic wall always initially assumes a mode-1 shape in accord with the steady solution in Fig. 4. However, even when the perturbation to the vector fields decays at first, the system starts oscillating later in time. A typical example is shown in Fig. 6 (a), where we follow the membrane displacement at $x = 0.5$. Notice that the solution to the problem when $M \neq 0$ is exactly superimposed on the decay predicted by the model when $M = 0$. By inspecting the corresponding
Figure 7. (a) Comparison between the two time-evolutions of the perturbation to the membrane displacement $F_1(0.5,t)$ at $x=0.5$ obtained using the full non-linear set of equations (dashed line) and by solving the linearized problem about the steady solution (solid line) when $M=0.01$, $\sigma = T = P_{ext} = 1$. The system was perturbed using the spike-like perturbation centered around $t=0.5$. (b) The same as in (a), except the dashed line was obtained by solving an eigenvalue problem (3.3) and the intial conditions for the nonlinear equations (2.6)-(2.8) corresponded to the most unstable eigenmode.

Figure 8. Dependence of (a) the growthrate $\Re(\alpha)$ and (b) the frequency $\Im(\alpha)$ of the fastest growing eigenvalue on the membrane mass $M$ when $P_{ext} = T = \sigma = 1$. Dots are drawn for clarity to relate real and imaginary parts of selected eigenvalues in (a) and (b). The change in $\Re(\alpha)$ with (c) the external pressure $P_{ext}$ and (d) cross-stream pressure gradient parameter $\sigma$ for selected eigenvalues when (c) $\sigma = T = M = 1$ and (d) $P_{ext} = T = M = 1$. In (b)-(d), $n$ is the number of extrema in the elastic wall deformation of the corresponding eigenvector.
Figure 9. Time evolution of \( \sigma \left( \frac{\partial^2 A_1}{\partial x^2} + \frac{1}{2} \int_1^x \frac{\partial F_1}{\partial t}(z,t)dz \right) \) and \( \frac{\partial F_1}{\partial t} \) for \( P_{ext} = 0, T = M = \sigma = 1 \) and the initial conditions that correspond to the most unstable eigenmode.

We conclude that the early-time mode-1 deformation is replaced by mode-2 deformations when the system starts oscillating. This instability (its growth rate and frequency) is very robust to the manner in which the simulations are started, so the behavior of the system can be understood by assuming

\[
[F(x,t), A(x,t), P(x,t)] = [F^s(x), A^s(x), P^s(x)] + \mu [F_1(x,t), A_1(x,t), P_1(x,t)] + O(\mu^2), \quad \text{where} \quad \mu \ll 1,
\]

and exploring the solutions to the linearized problem around the steady states computed in section 3 (a.2). A comparison between this problem and the full non-linear problem (2.6)-(2.8) is presented in Fig. 7 (a) when the spike-like perturbation is introduced to the external pressure. The agreement between the two solutions is excellent if the amplitude of oscillations is not too large. Furthermore, we can assume

\[
\mathcal{F}_1(x,t) = [F_1(x,t), A_1(x,t), P_1(x,t)]^T = \mathcal{F}(x) \exp(\alpha t), \quad \text{where} \quad \alpha = \Re(\alpha) + i\Im(\alpha),
\]

to obtain a generalized eigenvalue problem

\[
A \mathcal{F} = \alpha B \mathcal{F}. \quad (3.3)
\]

Here the matrix \( A \) is determined by the steady solutions \( A^s(x), F^s(x) \) and \( P^s(x) \), and the matrix \( B \) is necessarily singular because of the boundary conditions and the pressure field \( P \), which has no explicit time-dependence in the equations (2.6)-(2.8). Once again, the solutions to (3.3) are in good agreement with the nonlinear simulations (see Fig. 7 (b)), so the eigenvalue analysis can be used to explore the variation in the system’s behavior with the parameters.

The eigenvalue analysis suggests that the system is neutrally stable only when \( \sigma = 0 \) (see Fig. 8 and Fig. 8(d) in particular). Although the model formally breaks
down when $\sigma = 0$, we can still consider the limit when the cross-stream pressure gradient is negligible. In the absence of the pressure gradient, the flow feels the presence of the deformed wall only between $x = 0$ and $x = 1$, suggesting that the fluid load on the membrane is exactly $P = -\frac{9}{2} F^2$. Thus, when $\sigma = 0$ the flexible wall behaves in accordance with the equation

$$M \frac{\partial^2 F}{\partial t^2} - T \frac{\partial^2 F}{\partial x^2} = P_{\text{ext}} + \frac{9}{2} F^2,$$

and any perturbation results in the system oscillating around the steady state. Therefore, in order for the instabilities to grow in amplitude, $\sigma$ must be non-zero and vorticity waves must be generated in the downstream rigid channel.

Clearly, the wall inertia is crucial for the observed instability. The plots of the fastest growing eigenvalue in Fig. 8 (a), (b) suggest a complicated dependence on $M$, but, interestingly, $M \to 0$ is also a singular limit of the model. The reason for this lies in the mechanism of the observed instability. The same mechanism also explains why for $\sigma \neq 0$ the frequency of the fastest growing eigenvalues is very close but not equal to a normal mode of the elastic wall $\pi n \sqrt{T/M}$, with the corresponding $n$, indicated in Fig. 8 (b), matching the number of extrema in the wall deformation.

From Fig. 8 (c) it follows that the system stability does not change dramatically if $P_{\text{ext}}$ is small, and that the instability growth rate stays approximately constant if $P_{\text{ext}} \lesssim 0.01$. In order to infer the nature of the fluid load on the flexible wall and thus explain the mechanism of the instability captured by the model, we consider the limit $P_{\text{ext}} = 0$. This simplifies the linearised problem significantly, because the corresponding steady solution is unique for all values of $T$ and $\sigma$: it represents an unperturbed Poiseuille flow in the parallel-sided channel with $A^s = F^s = 0$. Therefore, the linearised problem becomes

$$M \frac{\partial^2 F}{\partial t^2} - T \frac{\partial^2 F}{\partial x^2} = \sigma \left( \frac{\partial^2 A}{\partial x^2} + \frac{1}{2} \int_1^x \frac{\partial F}{\partial t} (z,t) dz \right).$$

The last term in (3.6) can be interpreted as negative damping. Indeed, in Fig. 9 we compare the time-evolution of this term against $\partial F_1 / \partial t$ for one set of parameters, and to a good approximation it is equal to $D \partial F_1 / \partial t$ for constant $D$. Therefore, the equation (3.6) essentially behaves like the telegraph equation, for which the solution grows as $\exp(Dt/2M)$ and the frequency of the $n$th mode oscillation is $\sqrt{\pi^2 n^2 T/M - D}$ (Polyanin (2001)). Equally important is the last term in (3.5) that excites the fluid, so that there is feedback between the fluid and the elastic wall.

Finally, although well-posed, the present model never leads to constant amplitude oscillations at large times. In other words, the nonlinear terms in the equations (2.6)-(2.8) are insufficient to saturate the growth rate of the predicted oscillations. Instead, the instabilities grow until the second branch of steady solutions is encountered and the numerics break down, because the new equilibrium state is exponentially unstable to small amplitude perturbations.
Similar behaviour was reported by Luo and Pedley (1998) from their numerical solution to the full Navier-Stokes equations.

4. Discussion of results and conclusions

In this paper we have studied a rationally derived quasi-one-dimensional model for a collapsible channel that uses the ideas from interactive boundary layer theory. In agreement with the previous findings (Stewart et al. (2010a), Kudenatti et al. (2012), Pihler-Puzović and Pedley (2013)), the model predicts oscillations, but only if the wall mass is accounted for as well. This result is in qualitative agreement with other studies of flutter in collapsible channels, which predict that the flutter is initially superimposed precisely upon the overall temporal behaviour of the massless system, until the relevant eigenmode is excited (Luo and Pedley (1998)).

The model covers instabilities in a long, but finite collapsible channel in the limit of asymptotically high Reynolds number, for which the flow is described by the inviscid equations, but must have some shear. Therefore, the principal features of the resulting flow are independent of the large Reynolds number. The fluid flow creates a loading on the flexible wall, which acts as a negative damping and excites the wall oscillations. The oscillating wall in turn represents a source of the downstream propagating vorticity waves.

Despite the evident simplicity of the present model, the ideas of the interactive boundary layer theory can only take us so far. For example, ‘sloshing’-like instabilities predicted in other theoretical work would only come in our model at higher asymptotic orders than the one considered here (see Jensen and Heil (2003), Stewart et al. (2010b), Pihler-Puzović and Pedley (2013)). Moreover, the scaling appropriate for T-S instabilities and used in this paper is insufficient to capture travelling-wave flutter, which is a structural mode destabilized in the absence of membrane inertia (Stewart et al. (2010a)). Similarly, the numerical simulations by Luo and Pedley (1996) and Luo et al. (2008) describe regimes in which the boundary layer thickness $\varepsilon$ and cross-stream pressure gradient parameter $\sigma$ are of the same order, which is in contrast with the assumptions made here. Nevertheless, many of the assumptions made in the present model are essentially the same as in the recent work by Xu et al. (2013), except that their membrane was even longer than ours, i.e. $\lambda = O(Re)$, so that they could reduce the spatial dimension of the system by assuming a parabolic velocity profile everywhere in the channel. This led to a rational choice for both the fluid inertia and the viscous terms. Xu et al. (2013) found an oscillatory instability even in the absence of wall inertia. As the presence of the viscous terms is the essential difference between the two models, we believe that including suitable ad hoc terms in our equations would also lead to self-excited oscillations when the wall mass is neglected. However, there are no rational ways of improving the present model, while remaining in the domain of validity of interactive boundary layer theory, so we proceed no further.

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Appendix A. The derivation of the boundary conditions

Here we derive the absorbing boundary conditions (ABCs), used for the numerical solution of the equations (2.6)-(2.8).

In the rigid channels away from the membrane the equation (2.7) is exactly the linearized KdV equation of the form

\[ A_t = A_{xxx}. \] (4.1)

Following Zheng et al. (2007), we perform the Laplace transformation on (4.1):

\[ s \bar{A} - \bar{A}_{xxx} = 0, \] (4.2)

where \( s \in \mathbb{C} \) with \( \Re(s) > 0 \) is the argument in the Laplace-transformed space and \( \bar{A} \) is the transform of \( A \). The general solution to (4.2) looks like:

\[ \bar{A}(x,s) = c_1(s) \exp(\lambda_1(s)x) + c_2(s) \exp(\lambda_2(s)x) + c_3(s) \exp(\lambda_3(s)x), \]

with \( \lambda_1(s) = s^{1/3}, \lambda_2(s) = \omega s^{1/3}, \lambda_3(s) = \omega^2 s^{1/3}, \omega = \exp(2\pi i/3) \). Therefore,

\[ \Re(\lambda_1(s)) > 0, \quad \Re(\lambda_2(s)) < 0, \quad \Re(\lambda_3(s)) < 0. \]

For the flow field to be unperturbed upstream (at \( x = -\infty \)), \( c_2(s) \) and \( c_3(s) \) must be equal to zero. If \( x = b_1 \) corresponds to the upstream end of the shortened domain, it follows that

\[ \bar{A}(b_1,s) - \frac{1}{\lambda_1^2} \bar{A}_{xx}(b_1,s) = 0, \quad \bar{A}_x(b_1,s) - \frac{1}{\lambda_1} \bar{A}_{xx}(b_1,s) = 0. \] (4.3)

Finally, the inverse Laplace transformation on (4.3) reveals upstream ABCs (2.9)-(2.10). If \( x = b_2 \) is the downstream end of the shortened domain, then a similar procedure leads to the ABC (2.12). A typical validation of the ABCs is shown in Fig. 10.

It is also useful to consider the leading asymptotic behaviour of (2.9), (2.10) and (2.12) in the long-time limit (Pihler-Puzović, 2011). This will enable us to compare results of the linear stability analysis with numerical solutions to (2.6)-(2.8), both of which are discussed in section 3 (b.2). Assuming that

\[ A(x,t) = \tilde{A}(x) \exp(-\alpha t), \]

where \( \alpha \) is without loss of generality real, then as \( t \to \infty \) the boundary conditions, if \( \alpha > 0 \), become

\[ \tilde{A}_x(b_1) = 0, \quad \tilde{A}_{xx}(b_1) = 0 \quad \text{and} \quad \tilde{A}_{xx}(b_2) = 0, \] (4.4)

or, if \( \alpha \leq 0 \), the boundary conditions are

\[ \tilde{A}(b_1) - \alpha^{-2/3} \tilde{A}_{xx}(b_1) = 0, \quad \tilde{A}_x(b_1) + \alpha^{-1/3} \tilde{A}_{xx}(b_1) = 0 \]

and

\[ \tilde{A}(b_2) - \alpha^{-1/3} \tilde{A}_x(b_2) + \alpha^{-2/3} \tilde{A}_{xx}(b_2) = 0. \] (4.5)

These boundary conditions have been validated by solving the generalized eigenvalue problem (3.3) for different values of \( b_1 \) and \( b_2 \). If we assume the simplified boundary conditions and consider dependence of the fastest growing eigenvalue on the length of the rigid domains, for sufficiently large values of \( b_1 \)
Figure 10. Solution for the stream-line displacement function $A(x, 200)$ at $t = 200$ for $\sigma = P_{ext} = T = 1$ and $M = 0$. Dots correspond to the simulation with $b_1 = 80$ and $b_2 = 81$, whereas the solid line represent the solution for $b_1 = -100$ and $b_2 = 101$.

Figure 11. The change in the real part of the biggest eigenvalue ($\Re(\alpha)$) with the length of the rigid domains ($|b_1| = b_2 - 1$). The parameter values are $M = 0.01$ and $\sigma = T = P_{ext} = 1$. 
Figure 12. Different domains in the quasi one-dimensional model and respective equations and boundary conditions.

and $b_2$ the growth rate saturates to a constant (see Fig. 11). Throughout section 3 the results of the eigenvalue analysis and numerical simulations were presented for $b_1 = -10$ and $b_2 = 11$.

Appendix B. Computational method

Strictly speaking, there are three subdomains with different equations and boundary conditions in problem (2.6)-(2.8): the region upstream from the membrane, the membrane region and the region downstream from the membrane. In the upstream and downstream rigid regions we have to solve (4.1) with (2.9)-(2.10) and (2.12) boundary conditions, respectively. At the junction of the domains we use conditions of continuity of the streamline displacement function and its derivatives ($A_x$, $A_{xx}$ and $A_{xxx}$) to match the solution in different domains, so that equivalence between the single and multidomain problems is guaranteed. By taking this approach we are effectively solving the pressure (2.6) and the membrane (2.8) equations only in the membrane domain (see Fig. 12).

For the spatial discretisation of the equations (2.6)-(2.8) we used the collocation method based on Chebyshev polynomial expansion (similar to Pavoni (1988)). The equations are therefore satisfied only at the location of nodal points and the grid is denser near the domain ends. Hence, the domain decomposition discussed previously is not only natural, but it also helps to resolve regions around the membrane ends. We map subdomains $[b_1, 0]$, $[0, 1]$ and $[1, b_2]$ into the $[-1, 1]$ interval, so that the interpolation points represent the extrema of the $N$th order Chebyshev polynomial and we use Chebyshev collocation differential matrices to approximate $x$-derivatives of functions at the collocation points. The number of collocation points is different in different domains and so are the lengths of the different segments. Therefore, differential matrices are in general different in the three segments.

There are three boundary conditions in each domain, but only two boundary points available. This problem is solved by enforcing one of the boundary conditions implicitly by incorporating it into the differential operators.

For the temporal discretisation the one-step, first order backward Euler/forward Euler method is used to validate the code when $M = 0$. The linear part of the differential operator is evaluated at the new time-level, whereas non-linearities are computed at the previous time level, so that the linear system
that is solved at each time-level has a fixed matrix. This approach speeds up computations significantly. However, as soon as the wall mass is added to the system, the simple implicit Euler is no longer adequate, since it is energy dissipative. Instead, we use the second order Crank-Nicolson scheme. Consequently, at each time step we apply a Newton-Raphson linearization and solve the linear system for the corrections. The problem is solved simultaneously in all three domains for both fluid and structure motions. The resulting matrix is dense since we are using a spectral method for the spatial discretization, but the number of collocation points is relatively small. Therefore, it is possible to invert the matrices by applying a straightforward LU-decomposition (Press et al., 2003).

References


