Infinity and Applied Mathematics

David Silvester
School of Mathematics
University of Manchester

Outline

- What is Infinity?
- What is Applied Mathematics?

Outline

- What is Infinity?
- What is Applied Mathematics? $\Longleftrightarrow$
  The study of differential equations
  - A diffusion equation
  - The heat equation
  - The wave equation

To Infinity . . .

- Infinity . . . if we ask, most people think of something unimaginably large.
- Often they have seen the symbol ∞.

To Infinity . . .

- Infinity . . . if we ask, most people think of something unimaginably large.
- Often they have seen the symbol ∞.
- Usually “infinity” is a vague notion.
- Attempts to make it precise have led to a range of fundamental mathematical ideas with many, many applications (for example, think of all the ways that limits can come up).
- These ideas can be subtle and counter-intuitive.
- They can lead to paradoxes . . . and can resolve them!

To Infinity . . . and Beyond

The “Point at Infinity” — I

- If we say “*there are an infinite number of real numbers*”, we could mean:
  - ∗ There is no largest real number, or
  - ∗ If somehow we could list and then count all the real numbers, we would get an “infinite” number.
- The former occurs with the integers: you think of any integer, no matter how big a one you come up with, I can always provide a bigger one by adding $+1$.

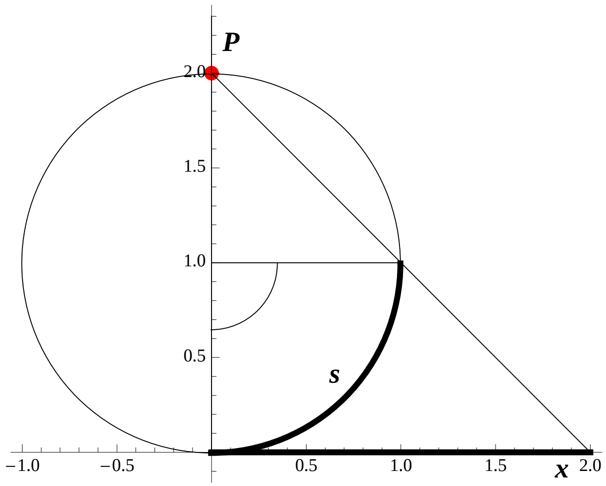
The “Point at Infinity” — II

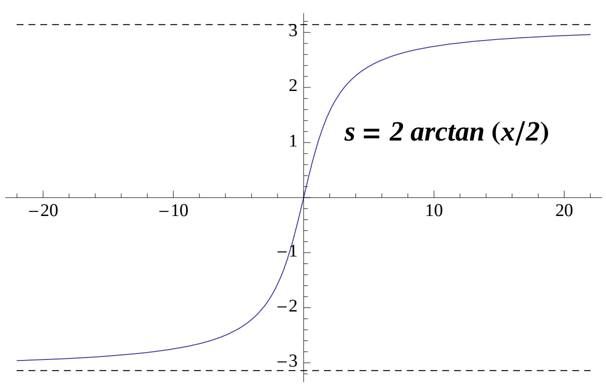
- It is useful to include in the integers and the real numbers something called infinity, ∞, to represent the limits of increasing sequences.
- For example:

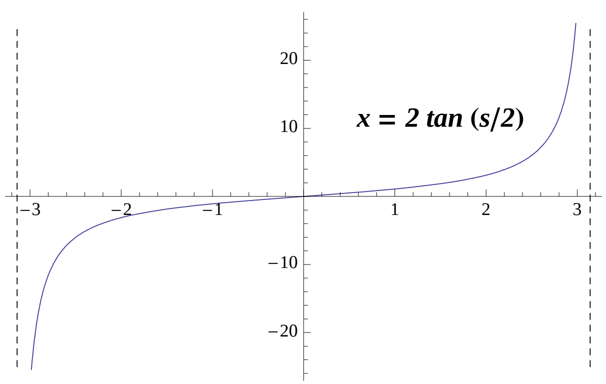
$$\lim_{x \to \infty} 2 \arctan \frac{x}{2} = \pi$$

$$\lim_{s \to \pi} 2 \tan \frac{s}{2} = \infty$$

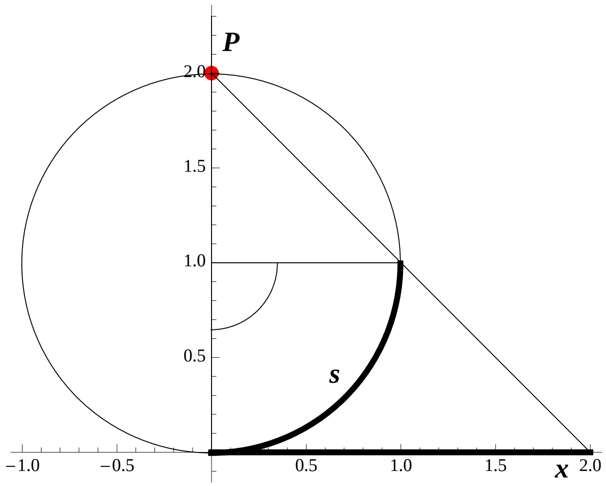
- This process is called “completion”.

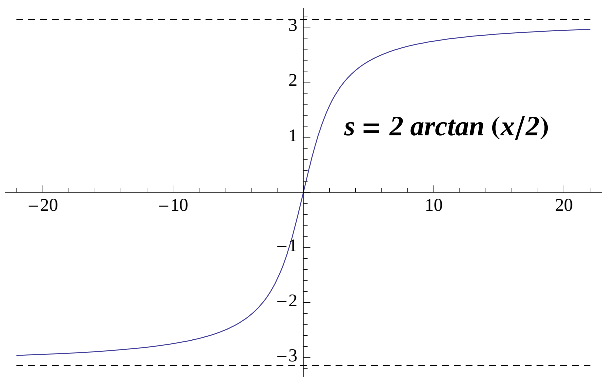


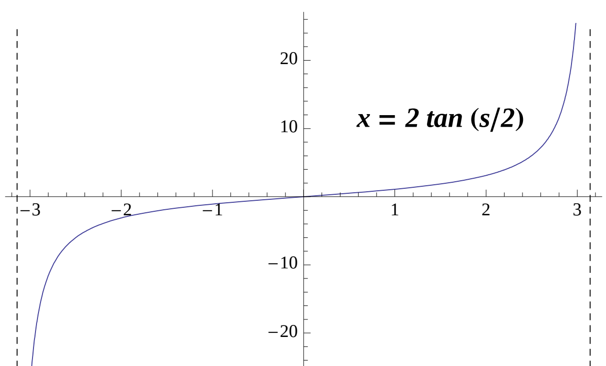




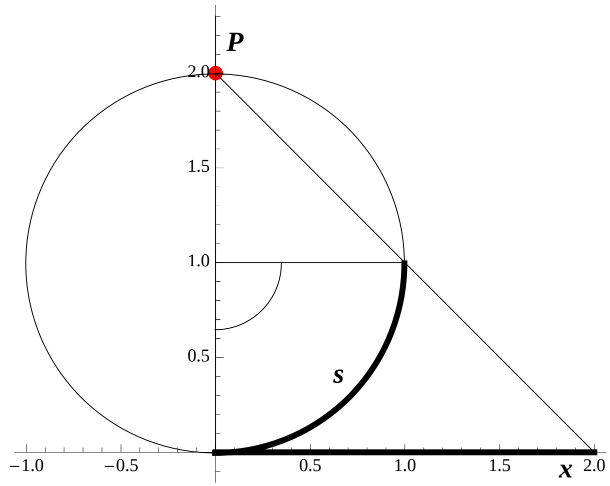
- Think of the projection of the real line onto a circle by drawing a line from any point $x \in \mathbb{R}$ to the “north pole” of the circle, P.
- For any real x the corresponding point on the circle is the angle s where $-\pi < s < \pi$.

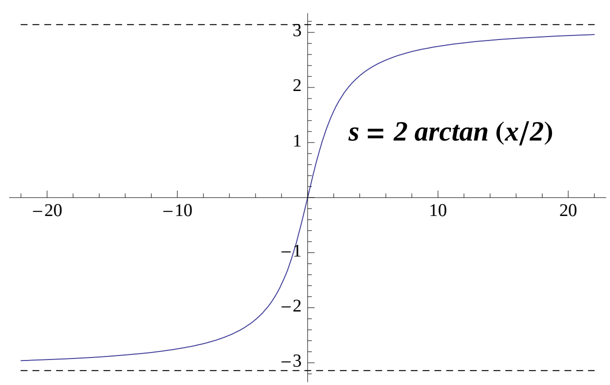


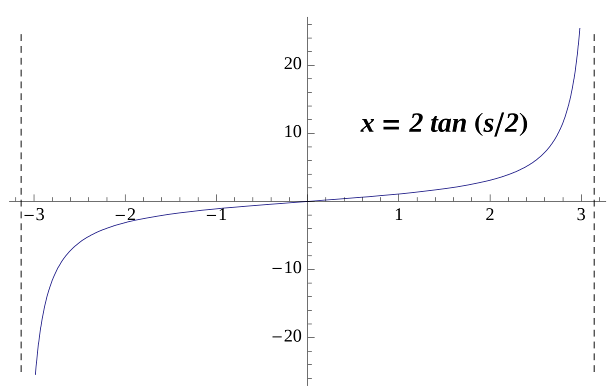




- Think of the projection of the real line onto a circle by drawing a line from any point $x \in \mathbb{R}$ to the "north pole" of the circle, P.
- For any real x the corresponding point on the circle is the angle s where $-\pi < s < \pi$.
- If we include the point P ($s = \pi$ or $-\pi$) then we complete the circle.







- Think of the projection of the real line onto a circle by drawing a line from any point $x \in \mathbb{R}$ to the "north pole" of the circle, P.
- For any real x the corresponding point on the circle is the angle s where $-\pi < s < \pi$.
- If we include the point P ($s = \pi$ or $-\pi$) then we complete the circle.
- Corresponding to this we include ∞ in the real line so that we can say

$$\lim_{x \to \infty} 2 \arctan \frac{x}{2} = \pi.$$

Achilles and the Tortoise — I

- Zeno of Elea (490 BC – 430 BC) gave four paradoxes which were treated as proofs by contradiction.
- If Achilles runs at speed v_A, the tortoise at v_T with $v_A > v_T$, and the tortoise gets a head start of d, when does Achilles catch the tortoise?

Achilles and the Tortoise — I

- Zeno of Elea (490 BC – 430 BC) gave four paradoxes which were treated as proofs by contradiction.
- If Achilles runs at speed v_A, the tortoise at v_T with $v_A > v_T$, and the tortoise gets a head start of d, when does Achilles catch the tortoise? $t_\infty = \frac{d}{v_A - v_T}$?

Achilles and the Tortoise — I

- Zeno of Elea (490 BC – 430 BC) gave four paradoxes which were treated as proofs by contradiction.
- If Achilles runs at speed v_A, the tortoise at v_T with $v_A > v_T$, and the tortoise gets a head start of d, when does Achilles catch the tortoise? $t_\infty = \frac{d}{v_A - v_T}$?
- Zeno argues that he never does
  - Achilles runs to where the tortoise starts.
  - When he arrives there the tortoise has advanced further.
  - Achilles runs there, but when he arrives the tortoise has advanced further $\ldots$
  - $\ldots$ and so on for ever $\ldots$

Achilles and the Tortoise — I

- Zeno of Elea (490 BC – 430 BC) gave four paradoxes which were treated as proofs by contradiction.
- If Achilles runs at speed v_A, the tortoise at v_T with $v_A > v_T$, and the tortoise gets a head start of d, when does Achilles catch the tortoise? $t_\infty = \frac{d}{v_A - v_T}$?
- Zeno argues that he never does
  - Achilles runs to where the tortoise starts.
  - When he arrives there the tortoise has advanced further.
  - Achilles runs there, but when he arrives the tortoise has advanced further $\ldots$
  - $\ldots$ and so on for ever $\ldots$
- This can be resolved by a better understanding of infinity.

Achilles and the Tortoise — II

- At $t = 0$, Achilles is at $x = 0$ and the tortoise is at $x = d$.

Achilles and the Tortoise — II

- At $t = 0$, Achilles is at $x = 0$ and the tortoise is at $x = d$.
- It takes Achilles $t = t_1$ to reach d, where $t_1 = d/v_A$.
  By then the tortoise is at $x = d + v_T t_1 = d(1 + v_T/v_A)$.

Achilles and the Tortoise — II

- At $t = 0$, Achilles is at $x = 0$ and the tortoise is at $x = d$.
- It takes Achilles $t = t_1$ to reach d, where $t_1 = d/v_A$.
  By then the tortoise is at $x = d + v_T t_1 = d(1 + v_T/v_A)$.
- It takes Achilles t_2 to reach here, $t_2 = d v_T / v_A^2$.
  By then the tortoise is has reached
  $x = d + v_T t_1 + v_T t_2 = d\left(1 + v_T/v_A + (v_T/v_A)^2\right)$

Achilles and the Tortoise — II

- At $t = 0$, Achilles is at $x = 0$ and the tortoise is at $x = d$.
- It takes Achilles $t = t_1$ to reach d, where $t_1 = d/v_A$.
  By then the tortoise is at $x = d + v_T t_1 = d(1 + v_T/v_A)$.
- It takes Achilles t_2 to reach here, $t_2 = d v_T / v_A^2$.
  By then the tortoise is has reached
  $x = d + v_T t_1 + v_T t_2 = d \left(1 + v_T/v_A + (v_T/v_A)^2\right)$
- Keep doing this ... at time $t = t_1 + t_2 + \ldots + t_n$
  - Achilles and the tortoise are $d \, (v_T/v_A)^n$ apart.
  - Since $v_A > v_T$, Achilles catches the tortoise as $n \to \infty$

Achilles and the Tortoise — II

- At $t = 0$, Achilles is at $x = 0$ and the tortoise is at $x = d$.
- It takes Achilles $t = t_1$ to reach d, where $t_1 = d/v_A$.
  By then the tortoise is at $x = d + v_T t_1 = d(1 + v_T/v_A)$.
- It takes Achilles t_2 to reach here, $t_2 = d v_T/v_A^2$.
  By then the tortoise is has reached
  $x = d + v_T t_1 + v_T t_2 = d\left(1 + v_T/v_A + (v_T/v_A)^2\right)$
- Keep doing this . . . at time $t = t_1 + t_2 + \ldots + t_n$
  - Achilles and the tortoise are $d\,(v_T/v_A)^n$ apart.
  - Since $v_A > v_T$, Achilles catches the tortoise as $n \to \infty$
- There is no paradox because the time taken is finite :

$$\lim_{n\to\infty} \sum_{k=1}^{n} t_k = \frac{d}{v_T} \sum_{k=1}^{\infty} \left(\frac{v_T}{v_A}\right)^k = \frac{d}{v_A - v_T}.$$

Conclusion (part I)

- Integers and real numbers are totally ordered, and our notion of infinity represent the limit of taking larger and larger integers or reals.
- The Greeks had a big problem with dynamics. . . but Zeno's paradox can be resolved if we observe that it is possible for an infinite series to have a finite limit.
- In part II we shall see that this fact underlies many problems in Applied Mathematics.

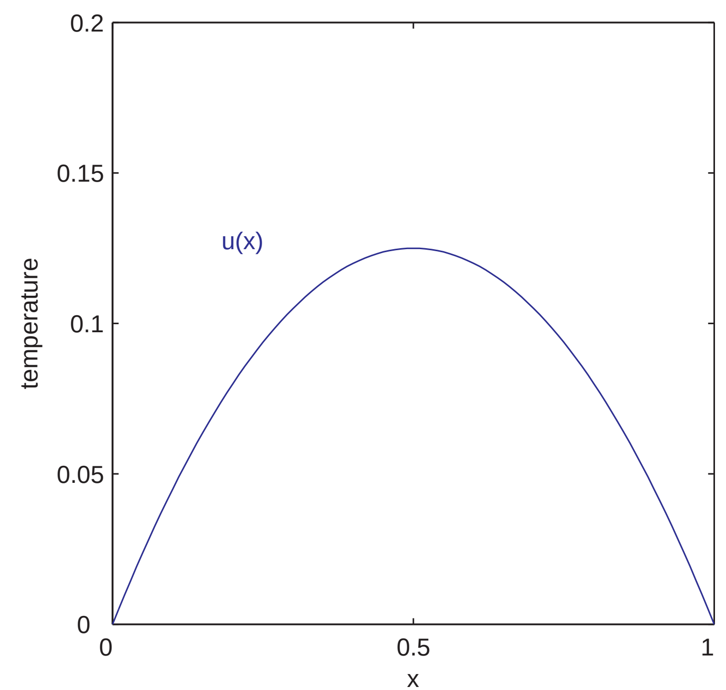
The Diffusion Problem

A general statement is: given a continuous “source” function f, we seek a function $u(x)$ satisfying

$$\left.\begin{array}{c} -\dfrac{d^2u}{dx^2} = f \quad \text{for} \quad 0 < x < 1 \\ u(0) = 0; \quad u(1) = 0. \end{array}\right\} \qquad (D)$$

Example : $f = 1$

$$\left.\begin{array}{l} -\dfrac{d^2u}{dx^2} = f \quad \text{for} \quad 0 < x < 1 \\ u(0) = 0; \quad u(1) = 0. \end{array}\right\}$$



with solution

$$u(x) = \frac{1}{2}(x - x^2).$$

- This is a model for the steady-state temperature in a wire with the ends kept in ice. There is a current flowing in the wire which generates heat.

The heat equation

A general statement is: given a continuous "source" function f, and an "initial condition" function $u_0(x)$, we seek a function $u(x,t)$ satisfying

$$\begin{aligned} u_t - u_{xx} &= f(x) && \text{in } (0,1)\times(0,\tau] \\ u(0,t)=0 \quad &; \quad u(1,t)=0 && \text{for all } t>0 \\ u(x,0) &= u_0(x) && \text{for all } x\in[0,1] \end{aligned}$$

The heat equation

A general statement is: given a continuous "source" function f, and an "initial condition" function $u_0(x)$, we seek a function $u(x,t)$ satisfying

$$\begin{aligned} u_t - u_{xx} &= f(x) && \text{in } (0,1)\times(0,\tau] \\ u(0,t)=0 \quad ; \quad u(1,t) &= 0 && \text{for all } t>0 \\ u(x,0) &= u_0(x) && \text{for all } x\in[0,1] \end{aligned}$$

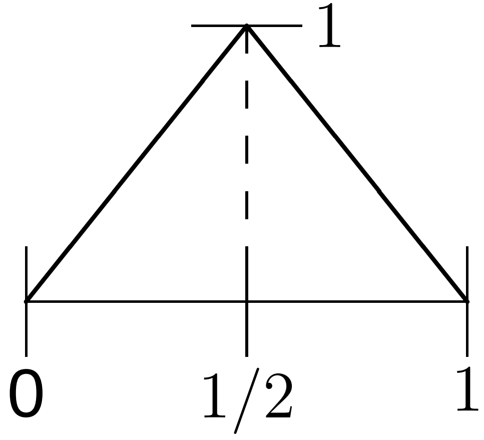
- Idea: find the steady-state solution $u_*(x)$ satisfying the diffusion equation (D) and combine it with u the solution of the homogeneous heat equation:

$$\left.\begin{aligned} u_t - u_{xx} &= 0 && (x,t)\in(0,1)\times(0,\tau] \\ u(0,t)=0, \quad u(1,t) &= 0 && t\in(0,\tau] \\ u(x,0) &= u_0(x) && x\in[0,1]. \end{aligned}\right\} (H)$$

Example I

$$\left.\begin{array}{rll} u_t - u_{xx} & = 0 & (x,t) \in (0,1) \times (0,\tau] \\ u(0,t) = 0, \quad u(1,t) & = 0 & t \in (0,\tau] \\ u(x,0) & = u_0(x) & x \in [0,1]. \end{array}\right\}$$

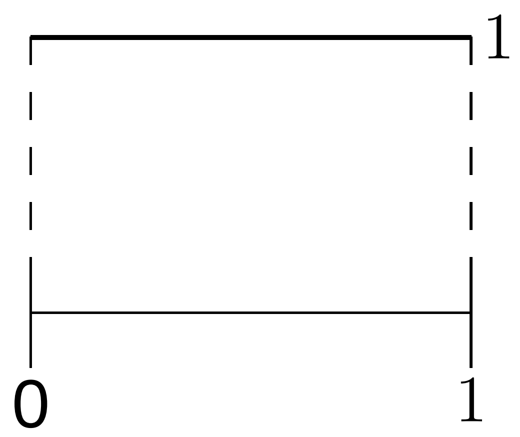
$$u_0(x) = \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ 2 - 2x & \frac{1}{2} \le x \le \ell \end{cases}$$



Example II

$$\left.\begin{array}{rll} u_t - u_{xx} & = 0 & (x,t) \in (0,1) \times (0,\tau] \\ u(0,t) = 0, \quad u(1,t) & = 0 & t \in (0,\tau] \\ u(x,0) & = u_0(x) & x \in [0,1]. \end{array}\right\}$$

$$u_0(x) = 1 \quad 0 \le x \le 1$$



A solution to the heat equation

$$\left.\begin{aligned} u_t - u_{xx} &= 0 && (x,t) \in (0,1) \times (0,\tau] \\ u(0,t) = 0, \quad u(1,t) &= 0 && t \in (0,\tau] \\ u(x,0) &= u_0(x) && x \in [0,1]. \end{aligned}\right\} (H)$$

A solution to the heat equation

$$\left.\begin{aligned} u_t - u_{xx} &= 0 && (x,t) \in (0,1) \times (0,\tau] \\ u(0,t) = 0, \quad u(1,t) &= 0 && t \in (0,\tau] \\ u(x,0) &= u_0(x) && x \in [0,1]. \end{aligned}\right\} (H)$$

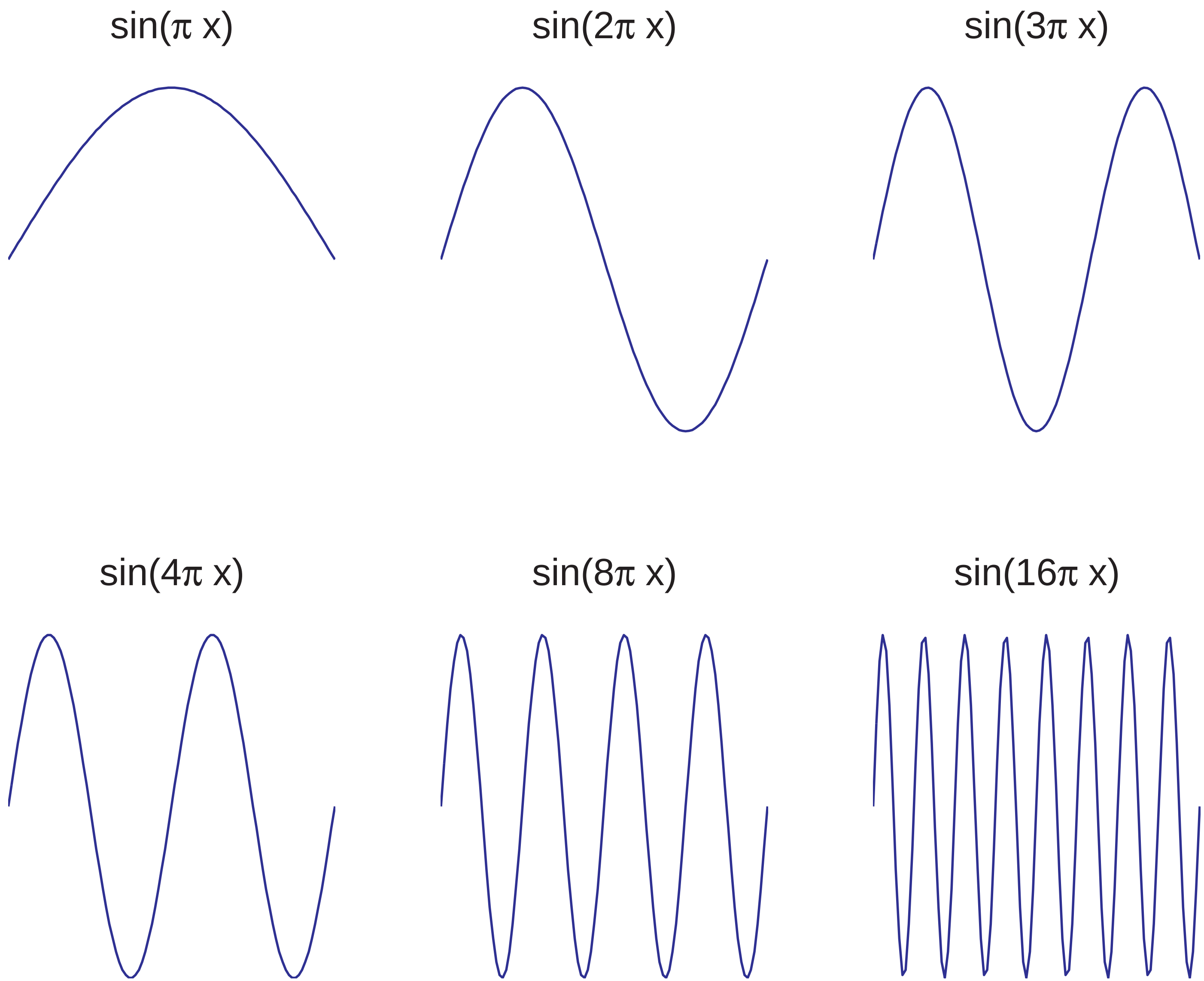
- Joseph Fourier (1768–1830) showed that if the initial condition is given by a combination of waves

$$u_0(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x),$$

then the general solution to (H) is also an infinite series of waves :

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t} \sin(n\pi x).$$

Fourier Modes



$$\left.\begin{aligned} u_t - u_{xx} &= 0 & (x,t) &\in (0,1)\times(0,\tau] \\ u(0,t) = 0, \quad u(1,t) &= 0 & t &\in (0,\tau] \\ u(x,0) &= u_0(x) & x &\in [0,1]. \end{aligned}\right\} (H)$$

- Fourier showed that the general solution to (H) is the infinite series of waves :

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t} \sin(n\pi x).$$

$$\left.\begin{array}{rll} u_t - u_{xx} & = 0 & (x,t) \in (0,1) \times (0,\tau] \\ u(0,t) = 0, \quad u(1,t) & = 0 & t \in (0,\tau] \\ u(x,0) & = u_0(x) & x \in [0,1]. \end{array}\right\} (H)$$

- Fourier showed that the general solution to (H) is the infinite series of waves :

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t} \sin(n\pi x).$$

- Fourier also showed that the coefficients are readily computed: $c_n = 2\int_0^1 u_0(x) \sin(n\pi x)\, dx.$

$$\left.\begin{array}{rll} u_t - u_{xx} & = 0 & (x,t) \in (0,1) \times (0,\tau] \\ u(0,t) = 0, \quad u(1,t) & = 0 & t \in (0,\tau] \\ u(x,0) & = u_0(x) & x \in [0,1]. \end{array}\right\} (H)$$

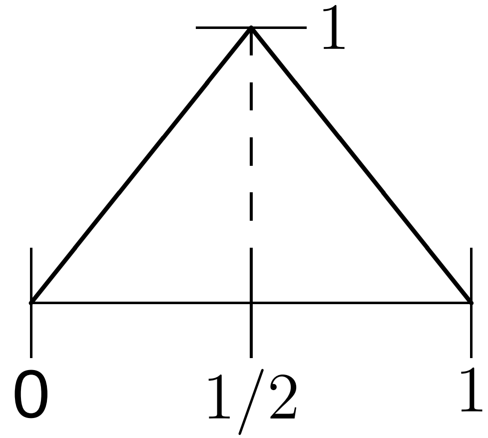
- Fourier showed that the general solution to (H) is the infinite series of waves :

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t} \sin(n\pi x).$$

- Fourier also showed that the coefficients are readily computed: $c_n = 2\int_0^1 u_0(x) \sin(n\pi x)\, dx.$
- Fourier's theory was extremely controversial; the story can be found in The Mathematical Experience, by P. Davis & R. Hersch (1980).

Example I

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t} \sin(n\pi x)$$



$$c_n = \frac{8}{\pi^2}\left\{\frac{\mathbf{1}}{\mathbf{1^2}}, \frac{0}{2^2}, \frac{\mathbf{-1}}{\mathbf{3^2}}, \frac{0}{4^2}, \frac{\mathbf{1}}{\mathbf{5^2}}, \frac{0}{6^2}, \frac{\mathbf{-1}}{\mathbf{7^2}}, \ldots\right\} \quad \sim \frac{1}{n^2}.$$

Hence

$$u(x,t) = \frac{8}{\pi^2}\left\{\frac{1}{\mathbf{1}} e^{-\pi^2 t}\sin(\pi x) - \frac{1}{\mathbf{9}} e^{-9\pi^2 t}\sin(3\pi x) + \frac{1}{\mathbf{25}} \ldots\right\}$$

Example II

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t} \sin(n\pi x)$$

$$c_n = \frac{4}{\pi}\left\{\frac{1}{\mathbf{1}}, \frac{0}{2}, \frac{1}{\mathbf{3}}, \frac{0}{4}, \frac{1}{\mathbf{5}}, \frac{0}{6}, \ldots\right\} \quad \sim \frac{1}{n}.$$

$$u(x,t) = \frac{4}{\pi}\left\{\frac{1}{\mathbf{1}} e^{-\pi^2 t}\sin(\pi x) + \frac{1}{\mathbf{3}} e^{-9\pi^2 t}\sin(3\pi x) + \frac{1}{\mathbf{5}}\ldots\right\}$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t} \sin(n\pi x)$$

What are the general properties of the solution?

- All Fourier modes decay exponentially in time; thus $u \to 0$ as $t \to \infty$.
- High frequency waves decay faster than low frequency ones.

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t} \sin(n\pi x)$$

What are the general properties of the solution?

- All Fourier modes decay exponentially in time; thus $u \to 0$ as $t \to \infty$.
- High frequency waves decay faster than low frequency ones.

"Difficult" questions include:

- Does the series converge to a well defined function u_∞?
- Does the limit u_∞ satisfy (H)?

The wave equation

Given an “initial displacement” $u_0(x)$ and an “initial velocity’ $u_1(x)$, we seek a function $u(x,t)$ satisfying

$$\begin{aligned} u_{tt} - u_{xx} &= 0 \qquad \text{in } (0,1)\times(0,\tau] \\ u(0,t) = 0 \quad &; \quad u(1,t) = 0 \qquad \text{for all } t>0 \\ u(x,0) = u_0(x); &\qquad u_t(x,0) = u_1(x) \qquad x \in [0,1]. \end{aligned}$$

The wave equation

Given an “initial displacement” $u_0(x)$ and an “initial velocity’ $u_1(x)$, we seek a function $u(x,t)$ satisfying

$$\begin{aligned} u_{tt} - u_{xx} &= 0 \qquad \text{in } (0,1)\times(0,\tau] \\ u(0,t) = 0 \quad &; \quad u(1,t) = 0 \quad \text{for all } t > 0 \\ u(x,0) = u_0(x); \quad & \quad u_t(x,0) = u_1(x) \qquad x \in [0,1]. \end{aligned}$$

- The Fourier solution is the infinite series of waves :

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos(n\pi t) + b_n \sin(n\pi t)) \sin(n\pi x)$$

with coefficients

$$\begin{aligned} a_n &= 2\int_0^1 u_0(x) \sin(n\pi x)\, dx; \\ b_n &= \frac{2}{n\pi}\int_0^1 u_1(x) \sin(n\pi x)\, dx. \end{aligned}$$

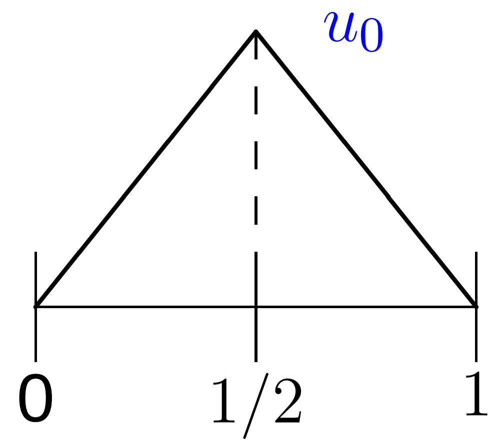
$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos(n\pi t) + b_n \sin(n\pi t)) \sin(n\pi x)$$

What are the properties of the solution?

- All Fourier modes oscillate in time forever.
- High frequency waves oscillate faster than low frequency ones.

Example

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos(n\pi t) + b_n \sin(n\pi t)\right) \sin(n\pi x)$$



$$a_n = \frac{8}{\pi^2} \left\{ \frac{\mathbf{1}}{\mathbf{1^2}}, \frac{0}{2^2}, \frac{\mathbf{-1}}{\mathbf{3^2}}, \frac{0}{4^2}, \frac{\mathbf{1}}{\mathbf{5^2}}, \frac{0}{6^2}, \frac{\mathbf{-1}}{\mathbf{7^2}}, \ldots \right\}.$$

Also $u_1 = 0 \iff b_n = 0$ for all n.
Thus,

$$u(x,t) = \frac{8}{\pi^2} \left\{ \cos(\pi t) \sin(\pi x) - \frac{1}{\mathbf{9}} \cos(3\pi t) \sin(3\pi x) + \frac{1}{\mathbf{25}} \ldots \right\}$$

Conclusion (part II)

- Even the simplest problems in Applied Mathematics have infinite series solutions.

Conclusion (part II)

- Even the simplest problems in Applied Mathematics have infinite series solutions.
- An understanding of the importance of ∞ is the “mark” of a mathematics graduate.