

ON THE SOLUTION OF WIENER–HOPF PROBLEMS INVOLVING NONCOMMUTATIVE MATRIX KERNEL DECOMPOSITIONS*

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Abstract. Many problems in physics and engineering with semi-infinite boundaries or interfaces are exactly solvable by the Wiener–Hopf technique. It has been used successfully in a multitude of different disciplines when the Wiener–Hopf functional equation contains a single scalar kernel. For complex boundary value problems, however, the procedure often leads to coupled equations which therefore have a kernel of matrix form. The key step in the technique is to decompose the kernel into a product of two functions, one analytic in an upper region of a complex (transform) plane and the other analytic in an overlapping lower half-plane. This is straightforward for scalar kernels but no method has yet been proposed for general matrices.

In this article a new procedure is introduced whereby Padé approximants are employed to obtain an approximate but explicit noncommutative factorization of a matrix kernel. As well as being simple to apply, the use of approximants allows the accuracy of the factorization to be increased almost indefinitely. The method is demonstrated by way of example. Scattering of acoustic waves by a semi-infinite screen at the interface between two compressible media with different physical properties is examined in detail. Numerical evaluation of the approximate factorizations together with results on an upper bound of the absolute error reveal that convergence with increasing Padé number is extremely rapid. The procedure is computationally efficient and of simple analytic form and offers high accuracy (error $< 10^{-7}\%$ typically). The method is applicable to a wide range of initial or boundary value problems.

Key words. Wiener–Hopf technique, matrix Wiener–Hopf equations, scattering, acoustics, Padé approximants

AMS subject classifications. 30E10, 41A21, 45E10, 73D25, 76Q05, 78A45

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1. Introduction and difficulties in solving matrix Wiener–Hopf equations. The Wiener–Hopf technique has, since its invention in 1931 [34], proved an immensely important method in mathematical physics. It offers one of the very few approaches to obtaining an exact solution to a class of integral equations. These equations are of either first or second kind, are defined over a half-line, and have a difference kernel. An enormous variety of physically important problems can be cast into this form of equation, and all have the characteristic feature of boundary conditions defined on semi-infinite planes. Fields of application include diffraction of acoustic [7], elastic [30], and electromagnetic waves; fracture mechanics; flow problems; diffusion models; and geophysical applications [16], to name but a few. The solution method, although initially proposed to follow via construction of the integral equation, now usually proceeds directly from the boundary value problem to an equation in the complex plane defined in an infinite strip [23]. This is accomplished by Fourier transformation (or other appropriate transform) or Green’s theorem, and the essential physical details are manifested in the singularity structure of the Fourier transformed difference kernel often called the Wiener–Hopf kernel, $K(\alpha)$ (Noble [29]), where α is the transform parameter. This kernel is usually arranged to be singularity free in a finite-width strip of the complex α -plane which contains the real line as

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$|\alpha| \rightarrow \infty$. Further, this strip need not necessarily be straight (i.e., enclose the whole real line) and is denoted henceforth as \mathcal{D} . Solution of the Wiener–Hopf equation is straightforward once $K(\alpha)$ is decomposed into a product of two functions $K^+(\alpha)$, $K^-(\alpha)$, where $K^+(\alpha)$ is regular and zero free in the region above and including the strip \mathcal{D} , denoted \mathcal{D}^+ , and $K^-(\alpha)$ is regular and zero free in the region below and including \mathcal{D} , denoted \mathcal{D}^- . Further, $K^\pm(\alpha)$ must have at worst algebraic growth as $|\alpha| \rightarrow \infty$ in \mathcal{D}^\pm , respectively. Cauchy’s integral theorem provides a convenient method for obtaining the explicit sum factorization of a function, e.g., of the form $g(\alpha) = g^+(\alpha) + g^-(\alpha)$, where \pm denotes the above analyticity properties, and so by exponentiation we deduce (see Theorem C of Noble [29])

$$(1) \quad K^\pm(\alpha) = \exp[g^\pm(\alpha)] = \exp \left\{ \frac{\pm 1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log K(\zeta)}{\zeta - \alpha} d\zeta \right\},$$

where ζ is a contour in \mathcal{D} and α lies above (below) ζ for $K^+(\alpha)$ ($K^-(\alpha)$). Sometimes a limiting procedure is necessary to ensure convergence of these integrals.

For problems in which $K(\alpha)$ is a scalar function, then, apart from some technical difficulties associated with computing integrals of the above form (see Abrahams and Lawrie [5]), the Wiener–Hopf procedure goes through without a hitch. However, in more complex boundary value problems, it is not unusual to obtain a matrix functional Wiener–Hopf equation with a square matrix kernel $\mathbf{K}(\alpha)$. This presents two potential obstacles to allow the solution scheme to go through. The first, which it is the aim of this article to tackle, is the factorization procedure (1). For any matrix function, $\mathbf{K}(\alpha)$, the logarithm $\mathbf{G}(\alpha) = \log(\mathbf{K}(\alpha))$ can be defined, for example by its power series expansion $\log[\mathbf{I} + (\mathbf{K}(\alpha) - \mathbf{I})] = \mathbf{K}(\alpha) - \mathbf{I} - \frac{1}{2}(\mathbf{K}(\alpha) - \mathbf{I}) + \dots$, where \mathbf{I} is the identity matrix, or otherwise. Hence the sum split of $\mathbf{G}(\alpha)$ is accomplished by the Cauchy-type integrals acting on each element of the matrix. However, the final step to obtain the product factors, i.e.,

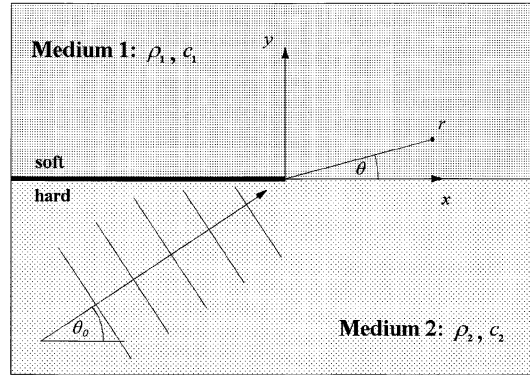
$$(2) \quad \exp\{\mathbf{G}^+(\alpha) + \mathbf{G}^-(\alpha)\} = \exp\{\mathbf{G}^+(\alpha)\} \exp\{\mathbf{G}^-(\alpha)\},$$

is true if and only if $\mathbf{G}^+(\alpha)$ and $\mathbf{G}^-(\alpha)$ commute (Heins [20]). For general matrices $\mathbf{K}(\alpha)$ this will not be the case, and although no procedure is presently available for tackling such arbitrary matrix Wiener–Hopf problems, Gohberg and Krein have proved the existence of product factors in all cases [17]. Fortunately, many physically interesting problems naturally give rise to kernels which, although intrinsically matrix in form, have a commutative factorization or can be reworked (by premultiplication and/or postmultiplication by suitable matrices) into such form. Khrapkov [26, 27], in articles concerned with the stresses in elastostatic wedges with notches, was the first author to express the commutative factorization in a form which indicates the subalgebra associated with this class of kernels. Many other authors have also examined the commutative cases, and, in particular, reference should be made to the ingenious approach to the simplest nontrivial commutative example by Rawlins [31], the direct scheme of Daniele [14], and the Hilbert problem technique of Hurd [21]. The latter approach, which transforms the factorization problem to a pair of uncoupled Hilbert problems defined on a semi-infinite branch cut, is generalizable to an interesting range of cases [32], one example of which is explored in [1]. It is generally accepted that the generalized range of kernels susceptible to Hurd’s method and Khrapkov’s class of commutative matrices suitably multiplied by entire matrices are in fact equivalent [24].

The second difficulty associated with matrix kernels is that, although decomposition into matrix factors with overlapping domains of regularity may be achieved, the growth of such factors could be exponential in the region of analyticity. This prohibits the application of Liouville’s theorem at a later stage of analysis, and so a solution to the matrix Wiener–Hopf equation cannot be obtained. Various techniques have been proposed in the literature to overcome this difficulty, including a general scheme by the author (Abrahams and Wickham [7]) and one specific to matrices of Khrapkov form (Daniele [15]). A full bibliography illustrating the relevant physical applications can be found in [6].

The aim of this article is to demonstrate a procedure for obtaining noncommutative matrix factors which have algebraic growth. Thus, we concentrate on the first rather than the second difficulty outlined above. As already mentioned, many useful and important problems have, or can be cast into, commutative factorization form, but the most important, and by far the more diverse, range of physical cases are intrinsically noncommutative. Matrix Wiener–Hopf equations arise in all areas of physics, including those mentioned at the start of this introduction. Usually the matrices yield noncommutative factorizations because they contain multiple branch cut functions, infinite sequences of poles or zeros, or a combination of both of these singularity structures. In certain special cases the factorization procedure can be completed by subtraction of an infinite family of poles from both sides of the Wiener–Hopf equation. The coefficients of each pole term are then found from the solution of an infinite algebraic system of equations (see Idemen [22], Abrahams [2], Abrahams and Wickham [8], and references quoted therein). The objective of this article is to illustrate a new procedure for obtaining explicit but approximate noncommutative matrix factors. The method should be applicable to many kernels of this previously unsolved matrix class and is based on the replacement of certain components of the matrix function by Padé approximants. This allows the level of approximation to be increased to very high accuracies while at the same time offering a surprisingly simple factorization form. An alternative technique has recently been proposed by Wickham [33] which, to the author’s knowledge, is the first to offer a computational route to factorize noncommutative kernels. This involves the derivation of certain exact coupled integral equations from which rigorous bounds on the convergence of such solutions can be derived. However, the method described herein is a good deal simpler to use in practice and appears much more direct than that suggested by Wickham. The relationship between the two methods is discussed further in the concluding remarks.

The clearest and most concise method for illustrating the Padé factorization procedure is to examine one specific example. We take a typical physical situation which gives rise to a noncommutative kernel, namely a semi-infinite barrier at an interface between two compressible media (Figure 1). This yields the simplest and therefore the canonical matrix kernel and so will omit any extraneous or obfuscating technical details. The plan of the paper is as follows. In section 2 the interfacial boundary value problem is stated mathematically, and in section 3 it is reduced to a matrix Wiener–Hopf equation. The explicit kernel factorization, employing Padé approximants for the approximate matrix kernel, is performed in section 4, and a discussion of the derivation of an ansatz used in that section can be found in Appendix B. The formal solution to the scattered field is given in section 5, and in section 6 the pressure in the far-field is derived asymptotically for the upper half-plane. Section 6 also presents numerical results and timings for different approximant numbers. A summary of the important properties of Padé approximants is written in Appendix A, and concluding remarks are made in section 7.

FIG. 1. *The physical configuration.*

2. The boundary value problem. As discussed in the introduction, we will study a two-dimensional problem involving a simple planar interfacial boundary because it offers the simplest form of a nontrivial matrix Wiener–Hopf kernel (with two branch cut pairs). Defining (x, y) as a two-dimensional Cartesian coordinate system, a semi-infinite screen lying along $x < 0, y = 0$ is placed between two different inviscid compressible media. The upper medium has sound speed c_1 , say, and lower c_2 . Along the extension of the screen ($x > 0, y = 0$) it is assumed that they have equal pressures and normal velocities, and if the continua are fluids, then they do not mix. Thus, it could be envisaged that on this line ($x > 0$) the two fluids are separated by a thin membrane of negligible mass and surface tension. On the physical screen ($x < 0$), the upper surface ($y = 0+$) is chosen to be perfectly soft (vanishing pressure p), whereas the underside ($y = 0-$) is taken to be perfectly rigid (vanishing normal velocity v). Figure 1 illustrates this geometry.

The governing equations for material motions can be expressed in terms of velocity potentials as

$$(3) \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{c_1^2} \frac{\partial^2}{\partial t^2} \right) \Phi_1 = 0, \quad y \geq 0,$$

$$(4) \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{c_2^2} \frac{\partial^2}{\partial t^2} \right) \Phi_2 = 0, \quad y \leq 0,$$

where the pressure and velocity are related to these potentials via

$$(5) \quad p = -\rho_1 \frac{\partial \Phi_1}{\partial t}, \quad y \geq 0; \quad p = -\rho_2 \frac{\partial \Phi_2}{\partial t}, \quad y \leq 0,$$

$$(6) \quad \mathbf{u} = (u, v) = \nabla \Phi_j, \quad j = 1, 2 \text{ for } y > 0, y < 0.$$

Here ρ_1, ρ_2 and c_1, c_2 are the density and sound speeds in media 1, 2, respectively.

For brevity, only the case of scattering by plane waves, incident from below, will be examined in this article, and the propagation speed c_2 is taken to be less than c_1 . All other cases can be dealt with in an identical fashion. It is the purpose of this paper to demonstrate the efficacy of the solution method, not to provide exhaustive details of the scattered field. Writing the incident waves (satisfying (4)) with angular frequency ω as

$$(7) \quad \Phi_{\text{inc}}(x, y; t) = \Re \{ e^{i\omega[(x \cos \theta_0 + y \sin \theta_0)/c_2 - t]} \},$$

then, for steady-state motions, the total velocity potential is given by

$$(8) \quad \Phi_1(x, y; t) = \Re\{\psi_1(x, y)e^{-i\omega t}\},$$

$$(9) \quad \Phi_2(x, y; t) = \Phi_{\text{inc}}(x, y; t) + \Phi_{\text{inc}}(x, -y; t) + \Re\{\psi_2(x, y)e^{-i\omega t}\}.$$

Note that the reflected wave $\Phi_{\text{inc}}(x, -y; t)$ has been written explicitly for convenience, and $\psi_1(x, y)$ and $\psi_2(x, y)$ are the scattered potentials which must be purely outgoing as $(x^2 + y^2)^{\frac{1}{2}} \rightarrow \infty$. Substituting (8) and (9) into the governing equations and boundary conditions gives

$$(10) \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c_1^2}\right) \psi_1 = 0, \quad y \geq 0,$$

$$(11) \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c_2^2}\right) \psi_2 = 0, \quad y \leq 0,$$

$$(12) \quad \psi_1(x, 0) = 0, \quad \frac{\partial \psi_2}{\partial y}(x, 0) = 0, \quad x \leq 0,$$

$$(13) \quad \frac{\rho_1}{\rho_2} \psi_1(x, 0) - \psi_2(x, 0) = 2e^{i\frac{\omega x}{c_2} \cos \theta_0}, \quad \frac{\partial \psi_1}{\partial y}(x, 0) - \frac{\partial \psi_2}{\partial y}(x, 0) = 0, \quad x \geq 0.$$

In order to exclude extraneous sound sources, it must also be insisted that

$$(14) \quad \psi_1, \psi_2 \text{ contain only outgoing waves as } \sqrt{x^2 + y^2} \rightarrow \infty;$$

$$(15) \quad \psi_1, \psi_2 \text{ are bounded everywhere, including the origin } (0, 0).$$

The boundary value problem is fully specified by statements (10)–(15).

3. Reduction to a Wiener-Hopf equation. To solve the boundary value problem defined in the previous section it is useful to introduce Fourier transforms. As there is no intrinsic length scale in the model, it is advantageous first to scale x and y on the length c_1/ω , viz.

$$(16) \quad x = Xc_1/\omega, \quad y = Yc_1/\omega,$$

and to write the sound speed ratio as

$$(17) \quad c_1/c_2 = k.$$

As stated earlier, and with no real loss of generality, c_1 is taken to be greater than c_2 , which means that $k > 1$. (If this is not the case, then x and y could be scaled on c_2/ω , k redefined as c_2/c_1 in order to force it to be larger than unity, and the following analysis repeated appropriately.) The Fourier transform of ψ_j can be written as

$$(18) \quad \Psi_j(\alpha, Y) = \int_{-\infty}^{\infty} e^{i\alpha X} \psi_j(Xc_1/\omega, Yc_1/\omega) dX, \quad j = 1, 2,$$

where α can at present be taken as a real parameter, and so, assuming convergence of transformed terms, (10) and (11) give

$$(19) \quad \frac{\partial^2 \Psi_1}{\partial Y^2} - (\alpha^2 - 1)\Psi_1 = 0, \quad Y \geq 0,$$

$$(20) \quad \frac{\partial^2 \Psi_2}{\partial Y^2} - (\alpha^2 - k^2)\Psi_2 = 0, \quad Y \leq 0.$$

The existence of all transformed expressions (i.e., integrals of the form (18)) can be confirmed a posteriori. These equations have general solutions

$$(21) \quad \Psi_1(\alpha, Y) = A(\alpha)e^{\gamma(\alpha)Y} + B(\alpha)e^{-\gamma(\alpha)Y}, \quad y \geq 0,$$

$$(22) \quad \Psi_2(\alpha, Y) = C(\alpha)e^{\delta(\alpha)Y} + D(\alpha)e^{-\delta(\alpha)Y}, \quad y \leq 0,$$

where $A(\alpha), \dots, D(\alpha)$ are as yet arbitrary functions of α and

$$(23) \quad \gamma(\alpha) = (\alpha^2 - 1)^{\frac{1}{2}}, \quad \delta(\alpha) = (\alpha^2 - k^2)^{\frac{1}{2}}.$$

The branch cut functions are *chosen*, for α real, to be

$$(24) \quad \gamma(\alpha) = |\alpha^2 - 1|^{\frac{1}{2}}, \quad |\alpha| \geq 1; \quad \gamma(\alpha) = -i|1 - \alpha^2|^{\frac{1}{2}}, \quad |\alpha| < 1,$$

and

$$(25) \quad \delta(\alpha) = |\alpha^2 - k^2|^{\frac{1}{2}}, \quad |\alpha| \geq k; \quad \delta(\alpha) = -i|k^2 - \alpha^2|^{\frac{1}{2}}, \quad |\alpha| < k.$$

Thus, $\Psi_1(\alpha, Y)$ will either grow exponentially or have incoming waves (of the form $e^{-i|\gamma(\alpha)|Y - i\omega t}$) as $Y \rightarrow \infty$ unless $A(\alpha) \equiv 0$. Similarly, as $Y \rightarrow -\infty$, it is clear that $\Psi_2(\alpha, Y)$ is bounded or has outgoing waves if and only if $D(\alpha) \equiv 0$. The two remaining unknowns $B(\alpha)$ and $C(\alpha)$ are specified by examining the boundary conditions on $Y = 0$.

First, from (12),

$$(26) \quad \Psi_1(\alpha, 0) = \int_0^\infty e^{i\alpha X} \psi_1(Xc_1/\omega, 0) dX \equiv \Psi_1^+(\alpha),$$

which is given the superscript + to denote that it must be a function analytic for all α above and on the inverse integration path (which at present is taken as the real line); see for example Chapter 1 of Noble [29]. Also from (12),

$$(27) \quad \frac{\partial \Psi_2}{\partial Y}(\alpha, 0) = \int_0^\infty e^{i\alpha X} \frac{\partial \psi_2}{\partial Y}(Xc_1/\omega, 0) dX \equiv \frac{\partial \Psi_2^+}{\partial Y}(\alpha),$$

and, from (12) and (13),

$$(28) \quad \begin{aligned} & \int_{-\infty}^\infty e^{i\alpha X} \left[\frac{\rho_1}{\rho_2} \psi_1(Xc_1/\omega, 0) - \psi_2(Xc_1/\omega, 0) \right] dX \\ &= - \int_{-\infty}^0 e^{i\alpha X} \psi_2(Xc_1/\omega, 0) dX + 2 \int_0^\infty e^{iX(\alpha + k \cos \theta_0)} dX \\ &= -\Psi_2^-(\alpha) + \frac{2i}{(\alpha + k \cos \theta_0)_+}, \end{aligned}$$

where the $-$ superscript denotes functions analytic below and on the contour of the inverse transform, and the simple pole at $\alpha = -k \cos \theta_0$ lies just below this path. Note that the integral of the exponential function converges in a generalized sense (Lighthill [28]) for real α and k . The remaining condition (13), which is continuity of normal velocity across $y = 0, x \geq 0$, yields

$$(29) \quad \frac{\partial \Psi_1^+}{\partial Y}(\alpha) - \frac{\partial \Psi_2^+}{\partial Y}(\alpha) = 0.$$

Hence, from (21) and (26),

$$(30) \quad \Psi_1^+(\alpha) = B(\alpha),$$

and, differentiating (22) and using (27),

$$(31) \quad \frac{\partial \Psi_2^+}{\partial Y}(\alpha) = \delta(\alpha)C(\alpha).$$

Further, differentiating (21) and employing (29) give

$$(32) \quad \frac{\partial \Psi_1^-}{\partial Y}(\alpha) + \frac{\partial \Psi_2^+}{\partial Y} = -\gamma(\alpha)B(\alpha),$$

and finally (21), (22), and (28) yield

$$(33) \quad \frac{\rho_1}{\rho_2}B - C = -\Psi_2^-(\alpha) + \frac{2i}{(\alpha + k \cos \theta_0)_+}.$$

The unknowns B and C can be eliminated from these four equations to give the vector Wiener-Hopf equation, valid at present only along the line $\Im(\alpha) = 0$:

$$(34) \quad \mathbf{K}(\alpha)\Psi^+(\alpha) = \Psi^-(\alpha) + \mathbf{F}^+(\alpha),$$

where the kernel is

$$(35) \quad \mathbf{K}(\alpha) = \begin{pmatrix} 1 & \mu\gamma(\alpha) \\ -\mu/\delta(\alpha) & 1 \end{pmatrix}$$

and the unknown column vectors are

$$(36) \quad \Psi^+(\alpha) = \left\{ \frac{\partial \Psi_2^+}{\partial Y}(\alpha), \frac{1}{\mu}\Psi_1^+(\alpha) \right\}^T,$$

$$(37) \quad \Psi^-(\alpha) = \left\{ -\frac{\partial \Psi_1^-}{\partial Y}(\alpha), -\mu\Psi_2^-(\alpha) \right\}^T.$$

Also,

$$(38) \quad \mathbf{F}^+(\alpha) = \frac{2i\mu}{(\alpha + k \cos \theta_0)_+} \{0, 1\}^T,$$

where μ is the constant

$$(39) \quad \mu = \sqrt{\frac{\rho_2}{\rho_1}}.$$

The inverse transform path has so far been taken along the real line in the α -plane. If this contour is to be deformed into the complex plane then it is necessary to define $\gamma(\alpha)$ and $\delta(\alpha)$ as complex valued functions of the complex variable α . In order to ensure the behavior defined in (24), (25) it is easy to show that $\gamma(\alpha)$ and $\delta(\alpha)$ have branch points at $\pm 1, \pm k$, respectively, and the integration path passes above the branch points at $-1, -k$, and below those at $+1, +k$. This path is illustrated as C in Figure 2, and the branch cuts are selected to run horizontally, as indicated. For convenience, and ultimately without loss of generality, θ_0 is confined for the moment

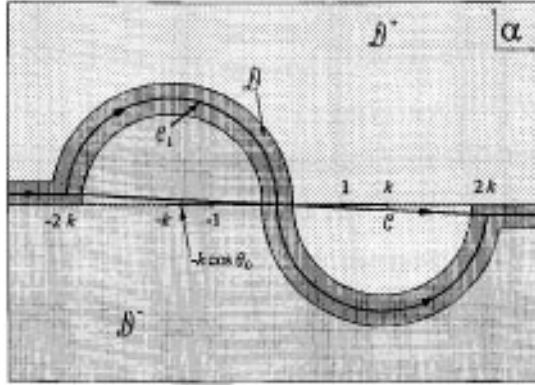


FIG. 2. The original integration contour \mathcal{C} passing just above the left-hand branch cuts and below the right-hand cuts. The bold lines indicate the finite branch cuts of $f(\alpha)$, and the dashed lines are the cuts of $\gamma(\alpha)$. Also shown are the deformed integration path \mathcal{C}_1 employed in expressions (117), (118) and the regions of regularity \mathcal{D}^\pm with their common strip of analyticity \mathcal{D} .

as $0 < \theta_0 \leq \pi/2$. All the singularities in $\mathbf{K}(\alpha)$ and $\mathbf{F}^+(\alpha)$ are shown on Figure 2, and therefore $\Psi_1(\alpha, Y), \Psi_2(\alpha, Y)$ will have these, and only these, cuts and pole. The inverse integral

$$(40) \quad \psi_j(x, y) = \frac{1}{2\pi} \int_{\mathcal{C}} e^{-i\alpha x \omega / c_1} \Psi_j(\alpha, y \omega / c_1) d\alpha, \quad j = 1, 2,$$

has the contour \mathcal{C} , which can therefore be deformed away from the branch points to a typical contour \mathcal{C}_1 say, lying within a strip of arbitrary width \mathcal{D} (Figure 2). Then $\Psi^+(\alpha)$ and $\mathbf{F}^+(\alpha)$ are functions analytic in the region above and including \mathcal{D} , say, \mathcal{D}^+ , and \mathcal{D}^- is the region below and including \mathcal{D} . The vector function $\Psi^-(\alpha)$ is analytic in \mathcal{D}^- and

$$(41) \quad \mathcal{D} \equiv \mathcal{D}^+ \cap \mathcal{D}^-.$$

The reasons for choosing the path \mathcal{C}_1 will be made clear in the next sections, but it suffices here to state that the path should be bounded (minimum distance unity) away from the real-line segments between $+1 \leftrightarrow +k$ and $-1 \leftrightarrow -k$.

4. Kernel factorization using Padé approximants.

4.1. Commutative partial decomposition. The boundary value problem defined in section 2 was reduced in the previous section to a two-dimensional matrix Wiener–Hopf equation (34) valid in the S -shaped strip of analyticity \mathcal{D} (see Figure 2). To solve this equation for the unknown vector functions $\Psi^+(\alpha), \Psi^-(\alpha)$, which are analytic in $\mathcal{D}^+, \mathcal{D}^-$, respectively, it is necessary to factorize $\mathbf{K}(\alpha)$ into the product form

$$(42) \quad \mathbf{K}(\alpha) = \begin{pmatrix} 1 & \mu\gamma(\alpha) \\ -\mu/\delta(\alpha) & 1 \end{pmatrix} = \mathbf{K}^-(\alpha)\mathbf{K}^+(\alpha).$$

Here again the \pm superscripts denote analyticity in \mathcal{D}^\pm . Furthermore, $\det(\mathbf{K}^\pm(\alpha))$ must be zero free in \mathcal{D}^\pm so that $[\mathbf{K}^\pm(\alpha)]^{-1}$ are also analytic in their respective half-planes. The matrix $\mathbf{K}(\alpha)$ (42) is the simplest possible nontrivial example of a Wiener–Hopf kernel with two branch cut pairs. To the author’s knowledge, no procedure has

yet been published in the literature for factorization of this class of matrix kernels. In this section a simple approximate solution scheme is presented which yields an *explicit noncommutative* decomposition (42). The accuracy of the procedure can be improved almost indefinitely, and it will be shown that, without much effort, extremely good agreement with an exact factorization is possible ($< 10^{-7}\%$). Thus, the solution can, for all intents and purposes, be treated as an exact matrix factorization.

To proceed, it is useful to first examine the factorization of $\mathbf{K}(\alpha)$ when media 1 and 2 have the same propagation speeds, i.e., $\gamma(\alpha) = \delta(\alpha) = (1 - \alpha^2)^{1/2}$. Denoting this as $\bar{\mathbf{K}}(\alpha)$, then

$$(43) \quad \bar{\mathbf{K}}(\alpha) = \begin{pmatrix} 1 & \mu\gamma(\alpha) \\ -\mu/\gamma(\alpha) & 1 \end{pmatrix} = \bar{\mathbf{K}}^-(\alpha)\bar{\mathbf{K}}^+(\alpha),$$

which, when $\mu = 1$, is the problem studied previously by various authors, including Rawlins [31], Hurd [21], and Daniele [14]. This kernel has a commutative factorization, which is perhaps best written in the Khrapkov form [26],

$$(44) \quad \bar{\mathbf{K}}^\pm(\alpha) = a^\pm(\alpha) \left(\cos[\Delta(\alpha)b^\pm(\alpha)]\mathbf{I} + \frac{1}{\Delta(\alpha)} \sin[\Delta(\alpha)b^\pm(\alpha)]\bar{\mathbf{J}}(\alpha) \right),$$

where $a^\pm(\alpha), b^\pm(\alpha)$ are scalar functions with the indicated analyticity properties, \mathbf{I} is the 2×2 identity matrix, $\bar{\mathbf{J}}(\alpha)$ is the entire matrix

$$(45) \quad \bar{\mathbf{J}}(\alpha) = \begin{pmatrix} 0 & \gamma^2 \\ -1 & 0 \end{pmatrix},$$

and

$$(46) \quad \bar{\mathbf{J}}^2 = -\Delta^2(\alpha)\mathbf{I}.$$

The aforementioned constraint on $\det(\bar{\mathbf{K}}^\pm(\alpha)) = a^\pm(\alpha)$ implies further that $a^\pm(\alpha)$ are zero as well as singularity free in \mathcal{D}^\pm . From (45), $\Delta(\alpha) = \gamma(\alpha)$, and multiplying $\bar{\mathbf{K}}^-(\alpha)$ with $\bar{\mathbf{K}}^+(\alpha)$ and equating with (43) gives

$$(47) \quad a^+(\alpha)a^-(\alpha) \cos[\gamma(\alpha)(b^+(\alpha) + b^-(\alpha))] = 1,$$

$$(48) \quad a^+(\alpha)a^-(\alpha) \sin[\gamma(\alpha)(b^+(\alpha) + b^-(\alpha))] = \mu.$$

By rearrangement, two scalar factorizations are required, namely, the product split

$$(49) \quad [a^+(\alpha)a^-(\alpha)]^2 = 1 + \mu^2$$

and the sum split

$$(50) \quad \tan[\gamma(\alpha)(b^+(\alpha) + b^-(\alpha))] = \mu,$$

or

$$(51) \quad b^+(\alpha) + b^-(\alpha) = \frac{\tan^{-1}(\mu)}{\gamma(\alpha)}.$$

The first of these is trivial, where for symmetry in a^+, a^- we choose

$$(52) \quad a^+(\alpha) = a^-(\alpha) = (1 + \mu^2)^{1/4};$$

the second is a standard sum decomposition (see page 21 of Noble [29]),

$$(53) \quad b^\pm(\alpha) = \frac{2 \tan^{-1}(\mu)}{\pi \gamma(\alpha)} \tan^{-1} \left(\frac{1 - \alpha}{1 + \alpha} \right)^{\pm 1/2}.$$

Not only has a factorization been achieved, but every element of $\mathbf{K}^\pm(\alpha)$ has algebraic behavior as $|\alpha| \rightarrow \infty$. This is an essential requirement for a successful solution of the Wiener–Hopf equation.

Returning now to the factorization of the kernel in (42), it is desirable that the construction for $\mathbf{K}^-(\alpha)$, $\mathbf{K}^+(\alpha)$ reduces to that in (44) as $k \rightarrow 1$. Hence, $\mathbf{K}(\alpha)$ may be written as

$$(54) \quad \mathbf{K}(\alpha) = \mathbf{I} + \mu \frac{f(\alpha)}{\gamma(\alpha)} \mathbf{J}(\alpha),$$

where

$$(55) \quad \mathbf{J}(\alpha) = \begin{pmatrix} 0 & \gamma^2(\alpha)/f(\alpha) \\ -f(\alpha) & 0 \end{pmatrix},$$

$$(56) \quad f(\alpha) = (\gamma(\alpha)/\delta(\alpha))^{1/2} = \left(\frac{\alpha^2 - 1}{\alpha^2 - k^2} \right)^{1/4},$$

and, *crucially*, as before,

$$(57) \quad \mathbf{J}^2(\alpha) = -\gamma^2(\alpha)\mathbf{I}.$$

That Riemann surface of $f(\alpha)$ is chosen which has $f(0) = 1/\sqrt{k}$, and so $f(\alpha) \equiv 1$ when $k = 1$ as required. Thus, the Khrapkov factorization forms

$$(58) \quad \mathbf{Q}^\pm(\alpha) = r^\pm(\alpha) \left(\cos(\gamma(\alpha)s^\pm(\alpha))\mathbf{I} + \frac{1}{\gamma(\alpha)} \sin(\gamma(\alpha)s^\pm(\alpha))\mathbf{J}(\alpha) \right)$$

can be obtained in an identical fashion to that shown previously, and $r^\pm(\alpha)$, $s^\pm(\alpha)$ will be functions regular in \mathcal{D}^\pm . However, $\mathbf{Q}^\pm(\alpha)$ are *not analytic in \mathcal{D}^\pm* because $\mathbf{J}(\alpha)$ is not entire. From the choice of cut locations shown in Figure 2, $f(\alpha)$ (and hence $\mathbf{J}(\alpha)$) has finite branch cuts between $1 \leq |\alpha| \leq k$. For the moment this fact will be ignored but will be addressed later in this section. From (58), the commutative factorization gives

$$(59) \quad \mathbf{K}(\alpha) = \mathbf{Q}^-(\alpha)\mathbf{Q}^+(\alpha) = r^+(\alpha)r^-(\alpha) \left(\cos[\gamma(\alpha)(s^+(\alpha) + s^-(\alpha))]\mathbf{I} + \frac{1}{\gamma(\alpha)} \sin[\gamma(\alpha)(s^+(\alpha) + s^-(\alpha))]\mathbf{J}(\alpha) \right),$$

which equates with (54) to yield

$$(60) \quad (r^+(\alpha))^2(r^-(\alpha))^2 = 1 + \mu^2 f^2(\alpha) = \frac{\mu^2 \gamma(\alpha) + \delta(\alpha)}{\delta(\alpha)},$$

$$(61) \quad s^+(\alpha) + s^-(\alpha) = \frac{1}{\gamma(\alpha)} \tan^{-1}(\mu f(\alpha)).$$

It should be stated that, as $f(\alpha)$ has the finite cuts mentioned above and $f(0) = 1/\sqrt{k}$, then

$$(62) \quad \Re(f(\alpha)) \geq 0$$

for all α and hence $\tan^{-1}(\mu f(\alpha))$ is a *single-valued* function. Therefore, Cauchy’s integral theorem can be applied to obtain the sum factors

$$(63) \quad s^\pm(\alpha) = \pm \frac{1}{2\pi i} \int_{\mathcal{C}_1} \frac{\tan^{-1}(\mu f(\zeta))}{\gamma(\zeta)(\zeta - \alpha)} d\zeta$$

(see Theorem B of Noble [29]) in which α lies above the contour for the + function and below for the – function. Similarly, it is a simple matter to prove

$$(64) \quad r^\pm(\alpha) = (1 + \mu^2)^{1/4} \exp \left\{ \frac{\pm 1}{4\pi i} \int_{\mathcal{C}_1} \frac{\ln[(1 + \mu^2 f^2(\zeta))/(1 + \mu^2)]}{\zeta - \alpha} d\zeta \right\},$$

which converges as written, and again α lies above \mathcal{C}_1 for $r^+(\alpha)$ and below \mathcal{C}_1 for $r^-(\alpha)$. Note that this form of $r^\pm(\alpha)$ is zero as well as singularity free in \mathcal{D}^\pm as required. Both (63) and (64) are simple and relatively quick to evaluate numerically, as will be discussed in the following sections.

4.2. Approximate noncommutative factorization. The commutative factors (58) have now been constructed, but they require modification in order to remove the branch cut singularities in $\mathbf{J}(\alpha)$. To (approximately) achieve this end in a remarkably simple fashion, it is convenient to replace $f(\alpha)$ in (55) by its [N/N] Padé approximant; that is,

$$(65) \quad f(\alpha) = (\gamma(\alpha)/\delta(\alpha))^{1/2} \approx f_N(\alpha) = \frac{P_N(\alpha)}{Q_N(\alpha)},$$

where $P_N(\alpha), Q_N(\alpha)$ are the polynomials

$$(66) \quad P_N(\alpha) = a_0 + a_1\alpha^2 + a_2\alpha^4 + \dots + a_N\alpha^{2N},$$

$$(67) \quad Q_N(\alpha) = 1 + b_1\alpha^2 + b_2\alpha^4 + \dots + b_N\alpha^{2N},$$

and N is any positive integer. Note that there is no need to replace the explicit $f(\alpha)$ term in (54) by $f_N(\alpha)$. The notation used above is that employed by Baker [9] except for the labeling of the coefficients (which is amended due to the absence of odd powers of α). Because workers in diffraction theory will, in general, be unfamiliar with Padé approximants, their main properties and derivation are summarized briefly in Appendix A (see also the relevant work by Bruno and Reitich [10]). It suffices here to mention that Padé approximants are determined uniquely (for each N) from the Taylor series expansion of $f(\alpha)$ about the origin (or any other regular point), and they are also far superior to the Taylor series expansion for obtaining an accurate approximation to the original function in the whole of the complex plane (away from singularities). The polynomials in the numerator and denominator are chosen here to have the same order in α to ensure that $f_N(\alpha)$ tends to a constant at infinity. Note that the zeros of $P_N(\alpha), Q_N(\alpha)$ are all simple, all lie on the real line between $1 < |\alpha| < k$ (i.e., where the branch cuts are in $f(\alpha)$), and as N increases so the poles and zeros of the approximant crowd on these intervals. Choosing $P_N(\alpha)$ and

$Q_N(\alpha)$ by any other method, for example by a collocation procedure, that is, enforcing $P_N(\alpha)/Q_N(\alpha) = f(\alpha)$ at a finite number of points on \mathcal{C}_1 , will be a nonunique process and in general leads to poles and zeros off the real line. As will be shown in the following sections, excellent agreement is obtained between $\mathbf{K}(\alpha)$ and its approximant counterpart ($< 10^{-7}\%$) for even moderate values of N . This is because the Padé factorization appears to be an optimal approximate solution of the exact integral equations defining the decomposition and rapidly tends to the exact solution as $N \rightarrow \infty$ (see the complementary article on scalar approximant factorizations by the author [3]).

The commutative factors (59), with $f(\alpha)$ approximated by (65), are now analytic in their respective half-planes $\mathcal{D}^+, \mathcal{D}^-$ except for simple poles at the zeros of $P_N(\alpha)$ and $Q_N(\alpha)$ in $\mathbf{J}(\alpha)$. The approximation of $\mathbf{J}(\alpha)$ is written as

$$(68) \quad \mathbf{J}_N(\alpha) = \begin{pmatrix} 0 & \gamma^2(\alpha)/f_N(\alpha) \\ -f_N(\alpha) & 0 \end{pmatrix},$$

and $\mathbf{K}_N(\alpha), \mathbf{Q}_N^\pm(\alpha)$, etc., are as given by (54) and (58) but with $\mathbf{J}(\alpha)$ replaced by $\mathbf{J}_N(\alpha)$. It is now a straightforward matter to remove the offending poles in $\mathbf{Q}_N^\pm(\alpha)$. To do this a new matrix $\mathbf{M}(\alpha)$, say, can be constructed which yields

$$(69) \quad \mathbf{K}_N^+(\alpha) = \mathbf{M}^{-1}(\alpha)\mathbf{Q}_N^+(\alpha),$$

$$(70) \quad \mathbf{K}_N^-(\alpha) = \mathbf{Q}_N^-(\alpha)\mathbf{M}(\alpha),$$

where $\mathbf{K}_N^\pm(\alpha)$ are analytic in \mathcal{D}^\pm as required and because of this form

$$(71) \quad \mathbf{K}_N^-(\alpha)\mathbf{K}_N^+(\alpha) = \mathbf{Q}_N^-(\alpha)\mathbf{M}(\alpha)\mathbf{M}^{-1}(\alpha)\mathbf{Q}_N^+(\alpha) = \mathbf{K}_N(\alpha).$$

Thus, finding $\mathbf{M}(\alpha)$ will lead directly to the exact explicit noncommutative factorization elements of $\mathbf{K}_N(\alpha)$, namely $\mathbf{K}_N^\pm(\alpha)$. Obviously $\mathbf{M}(\alpha)$ must be a matrix with meromorphic elements, and the poles must correspond to the zeros of $P_N(\alpha), Q_N(\alpha)$. In Appendix B such a matrix is determined directly by enforcing regularity conditions. However, here it is simpler to construct the elements of $\mathbf{M}(\alpha)$ by posing the ansatz

$$(72) \quad \mathbf{M}(\alpha) = \begin{pmatrix} 1 + \sum_{n=1}^N \frac{B_n}{\alpha^2 - p_n^2}, & \alpha \sum_{n=1}^N \frac{\bar{B}_n}{\alpha^2 - p_n^2} \\ -\alpha \sum_{n=1}^N \frac{A_n}{\alpha^2 - q_n^2}, & 1 - \sum_{n=1}^N \frac{\bar{A}_n}{\alpha^2 - q_n^2} \end{pmatrix},$$

in which p_n, q_n are the locations of the N positive real zeros of $P_N(\alpha), Q_N(\alpha)$, respectively, ordered as

$$(73) \quad 1 < p_1 < p_2 < p_3 < \dots < p_N < k, \quad 1 < q_1 < q_2 < q_3 < \dots < q_N < k.$$

The negative real zeros of $P_N(\alpha), Q_N(\alpha)$ are located at $-p_n, -q_n$, and $A_n, B_n, \bar{A}_n, \bar{B}_n$ are as yet undetermined constants. Note that Wiener–Hopf product factorizations are nonunique, and so there is *no loss of generality* in choosing $\mathbf{M}(\alpha)$ as above. Indeed, Appendix B is presented in order to justify the precise form of $\mathbf{M}(\alpha)$. Before expanding the right-hand sides of (69), (70) it will be necessary to write $f_N(\alpha)$ and its inverse in partial fraction form. That is,

$$(74) \quad f_N(\alpha) = C + \sum_{n=1}^N \frac{\alpha_n}{\alpha^2 - q_n^2},$$

$$(75) \quad \frac{1}{f_N}(\alpha) = \frac{1}{C} + \sum_{n=1}^N \frac{\beta_n}{\alpha^2 - p_n^2},$$

where C is the value of $f_N(\alpha)$ at infinity,

$$(76) \quad C = a_N/b_N \approx 1,$$

and the partial sum coefficients are easily shown to be

$$(77) \quad \alpha_n = 2q_n P_N(q_n)/Q'_N(q_n), \quad \beta_n = 2p_n Q_N(p_n)/P'_N(p_n),$$

the ' denoting differentiation with respect to α .

Now, substituting the approximant form of (58) and (72) into (70) and expanding give the (1, 1) element of $\mathbf{K}_N^-(\alpha)$:

$$(78) \quad r^-(\alpha) \cos[\gamma(\alpha)s^-(\alpha)] \left(1 + \sum_{n=1}^N \frac{B_n}{\alpha^2 - p_n^2} \right) + r^-(\alpha) \frac{\gamma(\alpha)}{f_N(\alpha)} \sin[\gamma(\alpha)s^-(\alpha)] \left(-\alpha \sum_{n=1}^N \frac{A_n}{\alpha^2 - q_n^2} \right).$$

This expression does not have poles at $\alpha = \pm q_n$ because they are cancelled by the zeros of $1/f_N(\alpha)$. It will have poles at $-p_m, m = 1, \dots, N$, i.e., in \mathcal{D}^- , unless its residues are zero. This is achieved by setting

$$(79) \quad r^-(-p_m) \left(\cos[\gamma(-p_m)s^-(-p_m)] \frac{B_m}{(-2p_m)} + \frac{\beta_m}{(-2p_m)} \gamma(-p_m) \sin[\gamma(-p_m)s^-(-p_m)] p_m \sum_{n=1}^N \frac{A_n}{p_m^2 - q_n^2} \right) = 0$$

for each m , or, as $\gamma(-p_m) = \gamma(p_m), s^-(-p_m) = s^+(p_m)$,

$$(80) \quad B_m = p_m \beta_m \gamma(p_m) \tan[\gamma(p_m)s^+(p_m)] \sum_{n=1}^N \frac{A_n}{q_n^2 - p_m^2}, \quad m = 1, \dots, N.$$

Repeating for the (2, 1) element of $\mathbf{K}^-(\alpha)$ gives, for regularity in \mathcal{D}^- ,

$$(81) \quad A_m = \frac{\alpha_m}{q_m \gamma(q_m)} \tan[\gamma(q_m)s^+(q_m)] \left(1 + \sum_{n=1}^N \frac{B_n}{q_m^2 - p_n^2} \right), \quad m = 1, \dots, N.$$

One of the unknown column vectors— B_m , say—can be eliminated between these two expressions, to yield the solution of the system as

$$(82) \quad \mathbf{A} = (\mathbf{I} - \mathbf{C})^{-1} \mathbf{c}$$

in matrix form, where \mathbf{I} is the $N \times N$ identity matrix, \mathbf{c} has components

$$(83) \quad c_m = \frac{\alpha_m}{q_m \gamma(q_m)} \tan[\gamma(q_m)s^+(q_m)], \quad m = 1, \dots, N,$$

and the square matrix \mathbf{C} is given by

$$(84) \quad C_{mn} = \sum_{\ell=1}^N \frac{c_m d_\ell}{(q_m^2 - p_\ell^2)(q_n^2 - p_\ell^2)}, \quad m = 1, \dots, N, \quad n = 1, \dots, N,$$

with

$$(85) \quad d_m = p_m \gamma(p_m) \beta_m \tan[\gamma(p_m) s^+(p_m)], \quad m = 1, \dots, N.$$

Thus, (82) is a simple and efficient form for calculating the vector of coefficients \mathbf{A} (see section 6). The other vector is determined via the relation

$$(86) \quad \mathbf{B} = \mathbf{D}\mathbf{A},$$

where the $N \times N$ matrix \mathbf{D} has elements

$$(87) \quad D_{mn} = \frac{d_m}{q_n^2 - p_m^2}, \quad m = 1, \dots, N, \quad n = 1, \dots, N.$$

The procedure outlined above is repeated to eliminate poles in \mathcal{D}^- from the second column of $\mathbf{K}_N^-(\alpha)$. Omitting details it is found that

$$(88) \quad \bar{\mathbf{B}} = (\mathbf{I} - \bar{\mathbf{C}})^{-1} \bar{\mathbf{d}},$$

in matrix form, where $\bar{\mathbf{C}}$ is the square matrix

$$(89) \quad \bar{C}_{mn} = \sum_{\ell=1}^N \frac{\bar{d}_m \bar{c}_\ell}{(p_m^2 - q_\ell^2)(p_n^2 - q_\ell^2)}, \quad m = 1, \dots, N, \quad n = 1, \dots, N,$$

and $\bar{\mathbf{c}}, \bar{\mathbf{d}}$ are

$$(90) \quad \bar{c}_m = c_m q_m^2, \quad \bar{d}_m = d_m / p_m^2, \quad m = 1, \dots, N$$

in which c_m, d_m are given in (83), (85). The remaining vector of coefficients, $\bar{\mathbf{A}}$, is determined via

$$(91) \quad \bar{\mathbf{A}} = \bar{\mathbf{D}}\bar{\mathbf{B}},$$

where

$$(92) \quad \bar{D}_{mn} = \frac{\bar{c}_m}{q_m^2 - p_n^2}, \quad m = 1, \dots, N, \quad n = 1, \dots, N.$$

This completes the construction of $\mathbf{M}(\alpha)$ from the enforcement of analyticity of $\mathbf{K}_N^-(\alpha)$ in \mathcal{D}^- , but it does not necessarily enforce regularity of $\mathbf{K}_N^+(\alpha)$ in \mathcal{D}^+ . However, the particular choice of ansatz for $\mathbf{M}(\alpha)$, (72), can be shown by direct substitution into (69) to yield identical values of the coefficients $\mathbf{A}, \mathbf{B}, \bar{\mathbf{A}}, \bar{\mathbf{B}}$ (82), (86), (88), (91) when poles are eliminated in \mathcal{D}^+ . Furthermore, it is easily shown from (81) and (91) that

$$(93) \quad \det(\mathbf{M}(\alpha)) = \left(1 + \sum_{n=1}^N \frac{B_n}{\alpha^2 - p_n^2}\right) \left(1 - \sum_{n=1}^N \frac{\bar{A}_n}{\alpha^2 - q_n^2}\right) + \alpha^2 \left(\sum_{n=1}^N \frac{\bar{B}_n}{\alpha^2 - p_n^2}\right) \left(\sum_{n=1}^N \frac{A_n}{\alpha^2 - q_n^2}\right)$$

has no singularities at $\pm p_n, \pm q_n, n = 1, \dots, N$. Thus, it is an entire function which by inspection tends to the constant value unity as $|\alpha| \rightarrow \infty$, and so by Liouville’s theorem

$$(94) \quad \det(\mathbf{M}) \equiv 1.$$

Therefore (69), (70) are the noncommutative matrix factors of the approximant kernel $\mathbf{K}_N(\alpha)$, (71), and they, together with their inverses, are regular in $\mathcal{D}^+, \mathcal{D}^-$, respectively. We have thus constructed approximate factorizations of the original matrix kernel (35), and it is clear, although not yet proven (see [3] for proof in the scalar case), that

$$(95) \quad \mathbf{K}_N^- \rightarrow \mathbf{K}^-, \quad \mathbf{K}_N^+ \rightarrow \mathbf{K}^+$$

as $N \rightarrow \infty$, where $\mathbf{K}^- \mathbf{K}^+$ is the exact noncommutative factorization of $\mathbf{K}(\alpha)$. Our approximate solution should therefore be able to be made indefinitely accurate by increasing N , and in the numerical evaluation section it will be shown that the approximate factorization matrices appear to converge very rapidly to the exact result as N increases.

5. Solution of the boundary value problem. In the previous section an approximate noncommutative factorization of the Wiener–Hopf kernel into two matrices, $\mathbf{K}^-(\alpha)$ and $\mathbf{K}^+(\alpha)$, regular in \mathcal{D}^- and \mathcal{D}^+ , respectively, was obtained. These can be employed in the Wiener–Hopf functional equation (34), which may be arranged into the form

$$(96) \quad \mathbf{K}^+(\alpha)\Psi^+(\alpha) - \frac{2i\mu}{(\alpha + k \cos \theta_0)_+} [\mathbf{K}^-(-k \cos \theta_0)]^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \mathbf{E}(\alpha) \\ \equiv [\mathbf{K}^-(\alpha)]^{-1}\Psi^-(\alpha) + \frac{2i\mu}{(\alpha + k \cos \theta_0)_+} \{[\mathbf{K}^-(\alpha)]^{-1} - [\mathbf{K}^-(-k \cos \theta_0)]^{-1}\} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

for $\alpha \in \mathcal{D}$. The last term on each side of (96) is included to move the simple pole at $\alpha = -k \cos \theta_0$, which was defined in section 3 to lie in \mathcal{D}^- , from the right-hand to the left-hand side. Thus, the left-hand side of (96) is analytic in \mathcal{D}^+ , and the right-hand side is analytic in \mathcal{D}^- , and so by analytic continuation both sides define a vector function entire in the whole complex α plane $\mathcal{D}^+ \cup \mathcal{D}^-(\alpha)$, $\mathbf{E}(\alpha)$, say. The precise form of this vector is determined by examining the behavior of both sides of (96) as $|\alpha| \rightarrow \infty$ in their respective half-planes of analyticity. First, by inspection of (72) it is clear that

$$(97) \quad \mathbf{M}(\alpha) = \mathbf{I} + \mathcal{O}(\alpha^{-1}), \quad |\alpha| \rightarrow \infty$$

in $\mathcal{D}^+ \cup \mathcal{D}^-$, and by standard techniques we can show that

$$(98) \quad r^\pm(\alpha) = (1 + \mu^2)^{1/4} + \mathcal{O}(\alpha^{-1}), \quad |\alpha| \rightarrow \infty.$$

Similarly, asymptotic analysis of (63) reveals

$$(99) \quad s^\pm(\alpha) = \pm \frac{i\tau}{\alpha} \ln \alpha + \mathcal{O}(\alpha^{-1}), \quad |\alpha| \rightarrow \infty$$

in the respective half-planes of analyticity, where

$$(100) \quad \tau = \frac{1}{\pi} \tan^{-1} \mu.$$

In view of the fact that the density ratio has range $0 < \mu < \infty$, we find

$$(101) \quad 0 < \tau < 1/2.$$

Thus,

$$(102) \quad \cos(\gamma(\alpha)s^\pm(\alpha)) \sim K\alpha^\tau; \quad \frac{1}{\gamma(\alpha)} \sin(\gamma(\alpha)s^\pm(\alpha)) \sim \pm iK\alpha^{\tau-1}$$

as $|\alpha| \rightarrow \infty$ in \mathcal{D}^\pm ; and K is some constant. With these expansions, together with (74), (76), substituted into (58) and (97), we find

$$(103) \quad \mathbf{K}^\pm(\alpha) \sim (1 + \mu^2)^{1/4} K \begin{pmatrix} \alpha^\tau & \pm i\alpha^{\tau+1} \\ \mp i\alpha^{\tau-1} & \alpha^\tau \end{pmatrix}$$

and also

$$(104) \quad [\mathbf{K}^\pm(\alpha)]^{-1} \sim (1 + \mu^2)^{-1/4} K \begin{pmatrix} \alpha^\tau & \mp i\alpha^{\tau+1} \\ \pm i\alpha^{\tau-1} & \alpha^\tau \end{pmatrix}$$

for $|\alpha| \rightarrow \infty$ in \mathcal{D}^\pm .

The next element is to estimate the sizes of $\Psi^+(\alpha)$, $\Psi^-(\alpha)$ for large $|\alpha|$. If we express the dimensionless coordinates in polar form,

$$(105) \quad X = R \cos \theta, \quad Y = R \sin \theta,$$

then it is straightforward to perform a local analysis around $R = 0$ which reveals that

$$(106) \quad \psi_1(R, \theta) \sim K_1 \sin(\tau\pi) R^\tau \sin \tau(\pi - \theta),$$

$$(107) \quad \psi_2(R, \theta) \sim K_1 \cos(\tau\pi) R^\tau \cos \tau(\pi + \theta) - 2$$

as $R \rightarrow 0$ for finite energy density at the edge. Here K_1 is some unknown constant and τ is the edge exponent written in (100). Note that in the case of equal densities in media 1 and 2, $\mu = 1$ and so $\tau = 1/4$ as already found by other authors (see e.g., Rawlins [31]). There is a direct relationship between the small X expansion of a function and the large α behavior of its half-range Fourier transform. Referring the reader to the Abelian theorem quoted in Noble [29, equation (1.74)], and omitting all detail, we find

$$(108) \quad \Psi^+(\alpha) \sim K_1 \sin(\tau\pi) \cos(\tau\pi) \Gamma(\tau + 1) e^{i\tau\pi/2} \begin{pmatrix} -\alpha^{-\tau} \\ i\alpha^{-\tau-1} \end{pmatrix}$$

for $|\alpha| \rightarrow \infty$ in \mathcal{D}^+ and

$$(109) \quad \Psi^-(\alpha) \sim \begin{pmatrix} 0 \\ -2i\mu\alpha^{-1} \end{pmatrix} + K_1 \sin(\tau\pi) \Gamma(\tau + 1) e^{-i\tau\pi/2} \begin{pmatrix} -\alpha^{-\tau} \\ i\alpha^{-\tau-1} \end{pmatrix}$$

for $|\alpha| \rightarrow \infty$ in \mathcal{D}^- , where $\Gamma(z)$ is the gamma function.

Both sides of (96) can now be estimated for large $|\alpha|$. From (103) and (108) the left-hand side is

$$(110) \quad \mathcal{O} \left(\begin{pmatrix} 1 \\ \alpha^{-1} \end{pmatrix} \right), \quad |\alpha| \rightarrow \infty, \alpha \in \mathcal{D}^+,$$

and the right-hand side is

$$(111) \quad \mathcal{O} \left(\begin{array}{c} 1 \\ \alpha^{-1} \end{array} \right), \quad |\alpha| \rightarrow \infty, \alpha \in \mathcal{D}^-.$$

Therefore, by Liouville's theorem both sides must be equal to

$$(112) \quad \mathbf{E}(\alpha) \equiv \begin{pmatrix} C \\ 0 \end{pmatrix},$$

where C is a constant. This gives the column vector $\Psi^+(\alpha)$ as

$$(113) \quad \Psi^+(\alpha) = [\mathbf{K}^+(\alpha)]^{-1} \left\{ \begin{pmatrix} C \\ 0 \end{pmatrix} + \frac{2i\mu}{(\alpha+k \cos \theta_0)_+} [\mathbf{K}^-(-k \cos \theta_0)]^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

which will have growth

$$(114) \quad \mathcal{O} \left(\begin{array}{c} \alpha^\tau \\ \alpha^{\tau-1} \end{array} \right)$$

as $|\alpha| \rightarrow \infty$ unless C is chosen appropriately. To enforce the behavior written in (108) it is straightforward to show that C must be

$$(115) \quad C = -2\mu(0, 1)[\mathbf{K}^-(-k \cos \theta_0)]^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and so

$$(116) \quad \Psi^+(\alpha) = 2\mu[\mathbf{K}^+(\alpha)]^{-1} \begin{pmatrix} \frac{i}{(\alpha+k \cos \theta_0)_+} & -1 \\ 0 & \frac{i}{(\alpha+k \cos \theta_0)_+} \end{pmatrix} [\mathbf{K}^-(-k \cos \theta_0)]^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This choice of C also ensures the correct growth (109) for $\Psi^-(\alpha)$. The scattered potentials are now known in terms of this column vector via equations (21), (22), (30), and (31). From (40) we write the solution to the boundary value problem in dimensional coordinates as

$$(117) \quad \psi_1(x, y) = \frac{\mu}{2\pi} \int_{\mathcal{C}_1} e^{-i\alpha x\omega/c_1 - \gamma(\alpha)y\omega/c_1} (0, 1)\Psi^+(\alpha) d\alpha, \quad y > 0,$$

$$(118) \quad \psi_2(x, y) = \frac{1}{2\pi} \int_{\mathcal{C}_1} \frac{1}{\delta(\alpha)} e^{-i\alpha x\omega/c_1 + \delta(\alpha)y\omega/c_1} (1, 0)\Psi^+(\alpha) d\alpha, \quad y < 0,$$

where \mathcal{C}_1 lies in the strip \mathcal{D} as indicated on Figure 2. This contour can be moved back to \mathcal{C} , or indeed to any path as long as it does not cross poles or branch cuts during the deformation. *This shows that the choice of path \mathcal{C}_1 was chosen for convenience, offering a way of estimating the error of using a Padé approximant solution for $\mathbf{K}^\pm(\alpha)$ (see the next section), and is not germane to the solution method.*

By direct substitution of (117) and (118) into conditions (10)–(15) it is clear that the above solution does indeed satisfy all elements of the boundary value problem. However, $\mathbf{K}^\pm(\alpha)$ are known only approximately through the kernel factorizations $\mathbf{K}_N^\pm(\alpha)$ derived in the last section. We show by numerical example in the following section that $\mathbf{K}_N^\pm(\alpha)$ converge very rapidly to $\mathbf{K}^\pm(\alpha)$ in their respective half-planes of regularity as N increases.

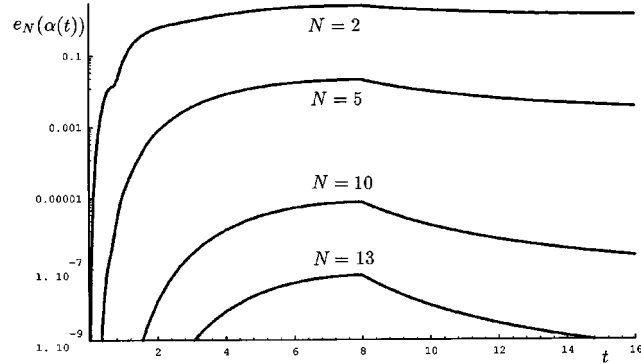


FIG. 3. The percentage error $e_N(\alpha(t))$, (120), plotted on a Log-scale versus the parameter t , (121), for $N = 2, 5, 10, 13$, $k = 4$.

As a final point to note, if we require $\mathbf{K}^+(\alpha)$ in \mathcal{D}^- , and in particular near to the branch cut between $-k < \alpha < -1$, then the approximant solution will not be accurate unless N is large. This is easily overcome, however, by employing the following form for $\mathbf{K}^+(\alpha)$:

$$(119) \quad \mathbf{K}^+(\alpha) = [\mathbf{K}^-(\alpha)]^{-1} \mathbf{K}(\alpha) \approx [\mathbf{K}_N^-(\alpha)]^{-1} \mathbf{K}(\alpha), \quad \alpha \in \mathcal{D}^-,$$

where $\mathbf{K}_N^-(\alpha)$ is now an excellent approximation to $\mathbf{K}^-(\alpha)$ for moderate N in the lower half-plane. The same technique can be employed if we require $\mathbf{K}^-(\alpha)$ in \mathcal{D}^+ , and the substitution (119) will be employed in the following section.

6. Accuracy of approximant solution and far-field evaluation. In this section we will examine the accuracy and convergence of the approximant factorizations and illustrate the results by evaluating the far-field diffraction pattern in medium 1 for different Padé numbers. In section 4 an *exact* factorization of the matrix kernel $\mathbf{K}_N(\alpha)$ was obtained, namely $\mathbf{K}_N^\pm(\alpha)$. But $\mathbf{K}_N(\alpha)$ is an approximation to the original kernel $\mathbf{K}(\alpha)$, differing only in the replacement of $f(\alpha)$ in $\mathbf{J}(\alpha)$ by its Padé approximant $f_N(\alpha)$, (65). If $f_N(\alpha)$ is an accurate approximation to $f(\alpha)$ on \mathcal{C}_1 , which is any infinite line in \mathcal{D} , then obviously $\mathbf{K}_N(\alpha)$ will accurately approximate $\mathbf{K}(\alpha)$ on this line. We define the percentage error as

$$(120) \quad e_N(\alpha) = 100 \times |[f_N(\alpha) - f(\alpha)]/f(\alpha)|, \quad \alpha \in \mathcal{D}.$$

Further, from Abrahams [3] (see also Koiter [25] and Noble [29, Chapter 4.5]), it is expected that $\mathbf{K}_N^+(\alpha)$ approximates $\mathbf{K}^+(\alpha)$ in the region of regularity, $\alpha \in \mathcal{D}^+$, with error less than or equal to $\max\{e_N(\alpha) : \alpha \in \mathcal{D}\}$. The same applies to $\mathbf{K}_N^-(\alpha)$ in \mathcal{D}^- . We therefore need only to calculate the error $e_N(\alpha)$ along \mathcal{C}_1 for different Padé numbers N in order to bound the errors of the approximant factors $\mathbf{K}_N^\pm(\alpha)$ in \mathcal{D}^\pm .

Numerical experiments for a range of values of the density ratio, i.e., μ , and sound speed ratio, k , were performed. All gave similar results, and, as expected, the nearer k approached unity the better the accuracy of the Padé approximants. However, this convergence property was not found to be very strongly dependent on k . Hence, in what follows the entirely typical values $\mu = 1.5$, and $k = 4$ are selected. Figure 3 illustrates the (percentage) error in the Padé approximants on the right half of the \mathcal{C}_1 path (only half shown because of symmetry) using the parametric representation

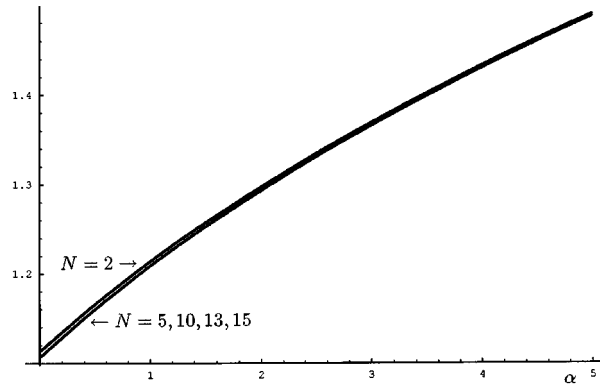


FIG. 4. The (1, 1) element of $\mathbf{K}_N^+(\alpha)$ plotted on the real line, $0 < \alpha < 5$, for $N = 2, 5, 10, 13, 15$, $\mu = 1.5$, $k = 4$.

$e_N(\alpha(t))$, where t is

$$(121) \quad \alpha = \begin{cases} k(1 + e^{i\pi(t/(2k)-1)}), & 0 < t < 2k, \\ t, & t > 2k. \end{cases}$$

The ordinate is presented on a logarithmic scale because of the dramatic increase in accuracy with increasing N , and four approximant errors are plotted, namely, for $N = 2, 5, 10, 13$. For $N = 2$ the results reveal that the maximum possible error in the factorization matrices $\mathbf{K}_N^\pm(\alpha)$ is 2.5%. However, increasing to a still modest approximant number of $N = 10$ (i.e., 10 zeros and poles to replace each finite branch cut) the upper bound on the error is a mere .00001%. At $N = 13$ the error is at most $6 \times 10^{-8}\%$ and for $N \approx 20$ the error is smaller than the rounding error in double precision on most microcomputers!

Figure 3 indicates the strong convergence of the approximant matrices to the exact solution. Yet from [3] we can expect the product factorization matrices to converge even quicker than that shown, and Figure 4 bears this out. The first element of $\mathbf{K}_N^+(\alpha)$ is plotted along the real line segment $0 < \alpha < 5$ for $N = 2, 5, 10, 13, 15$ and for $k = 4$, $\mu = 1.5$. All these functions are very close in value, and indeed only the $N = 2$ result can be discriminated in this figure. It is clear, therefore, that significant convergence has taken place by $N = 5$. The plot of the imaginary part of the (1,2) element of $\mathbf{K}_N^+(\alpha)$ along the line $.01 + it$, $0 < t < 5$, Figure 5, further demonstrates this point, and on examination of the numerical results it is found that $\mathbf{K}_5^+(\alpha)$ and $\mathbf{K}_{13}^+(\alpha)$ never differ by more than .00002% for α lying in the first quadrant of the complex plane. As regards speed of computation, all calculations were carried out using Mathematica on a 90-Mhz Pentium personal computer. This generated $f_N(\alpha)$ and all the coefficients of $\mathbf{M}(\alpha)$ in a few seconds for small systems $N = 2, 5$ and in 160 seconds for $N = 13$. The far-field diffraction pattern shown in Figure 6 took approximately 4 minutes to generate, and this time was not found to alter significantly with Padé number N .

As a final illustration, and to reveal the likely convergence properties (as N increases) for the solutions ψ_1, ψ_2 , we will examine the far-field behavior. Since this paper does not aim to present an exhaustive study of the physical model, we will confine our attention to $y > 0$, and for simplicity of analysis we again choose $\mu = 1.5$, $k = 4$, and incident wave angle (7) $\theta_0 = \pi/3$. This means the pole at $\alpha = -k \cos \theta_0 = -2$ lies to the left of the branch point at -1 , and the inverse contour path passes above

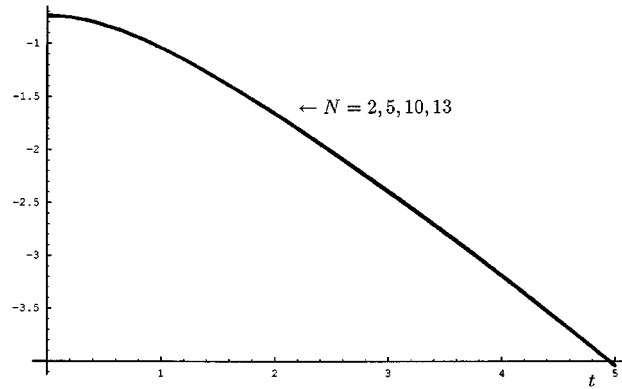


FIG. 5. The imaginary part of the (1,2) element of $\mathbf{K}_N^+(\alpha)$ plotted along the vertical line, $\alpha = .01 + it$, $0 < t < 5$, for $N = 2, 5, 10, 13$, $\mu = 1.5$, $k = 4$.

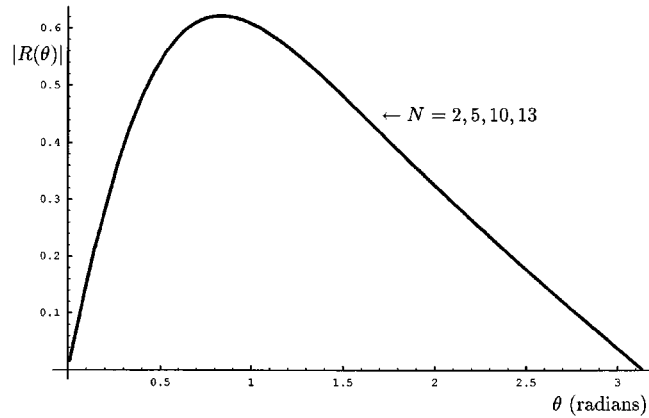


FIG. 6. The modulus of the diffraction coefficient, (125), versus θ (radians) in the upper medium plotted for different approximant values $N = 2, 5, 10, 13$ and $\mu = 1.5$, $k = 4$.

it. We now perform the standard treatment on the integral in (117) to employ the method of stationary phase: first we change to polar coordinates

$$(122) \quad x = r \cos \theta, \quad y = r \sin \theta, \quad 0 < \theta < \pi, r > 0$$

and make the substitution $\alpha = -\cos(\theta + it)$ after deforming the contour path into a hyperbola passing through the point $\alpha = -\cos \theta$. No poles or other singularities are crossed in this deformation. The integral may now be rewritten as

$$(123) \quad \psi_1 = \frac{i\mu}{2\pi} \int_{-\infty}^{\infty} e^{i\frac{r\omega}{c_1} \cosh t} (0, 1) \Psi^+(-\cos(\theta + it)) \sin(\theta + it) dt,$$

where t lies on the real line. The dominant contribution to this integral comes from near $t = 0$, and by usual arguments we find

$$(124) \quad \psi_1(r, \theta) \sim R(\theta) \frac{e^{ir\omega/c_1}}{\sqrt{r\omega/c_1}}, \quad r \rightarrow \infty,$$

where the diffraction coefficient is

$$(125) \quad R(\theta) = \frac{\mu e^{3i\pi/4} \sin \theta}{\sqrt{2\pi}} (0, 1) \Psi^+(-\cos \theta), \quad 0 < \theta < \pi,$$

and $\Psi^+(\alpha)$ is written in (116). Figure 6 is a plot of $|R(\theta)|$ over all $0 < \theta < \pi$ and for $N = 2, 5, 10, 13$. As can be seen, it is impossible to distinguish between the different curves. That is, the approximate solutions, including that for $N = 2$, give results which are remarkably close to each other! This indicates that the upper bounds on the errors of the approximate factorizations are indeed too pessimistic for most results of physical importance (Abrahams [3]).

As a cautionary note, it must be stated that numerical difficulties are encountered if the Padé number is too large. This is because, for large N , the poles and zeros crowd onto the branch cut intervals, thus making an accurate determination of their locations both essential and rather delicate. Typically the Mathematica routine NSolve started to introduce inaccuracies at around $N \approx 20$, but with a little care this technical difficulty can be overcome until a much larger Padé number. However, we conclude that, from the numerical evidence, the Padé factorization matrices $\mathbf{K}_N^\pm(\alpha)$ converge very rapidly with N to the exact solutions $\mathbf{K}^\pm(\alpha)$ and so this difficulty is entirely irrelevant. The last figure suggests that using a Padé number of, say, $N = 5$ or 10 will yield both an accurate and a computationally efficient solution of the boundary value problem, and even one as small as $N = 2$ will probably suffice for most purposes.

7. Concluding remarks. This paper has presented a new method for obtaining approximate factorizations of Wiener–Hopf kernels using Padé approximants. As well as illustrating a technique for matrix kernels, it is also of potentially significant use in scalar problems (Abrahams [3]). This is because the approximant factorization form eliminates the necessity of using the Cauchy integral representation and so removes the problem of singularities of the integrand lying near to or on the integration path. The scalar Padé approximant representation is the natural extension of Koiter’s method [25], and it can make significant savings in computer time for calculations involving complex scalar kernels (e.g., that found in [11]). The present approach can also be contrasted with an approximate scheme for scalar factorization suggested by Carrier [12] and used recently on a specific problem by Dahl and Frisk [13]. The simplicity of the factorization procedure shown in section 4 is due to the fact that quotients of polynomials (Padé approximants) are trivial to decompose into product terms with zeros and poles in respective half-planes, and the residues at these points are simple to evaluate. Further, by increasing the approximant number N the accuracy can be raised to very high levels, and also there is a comprehensive literature on the full behavior of approximants. The rapid convergence (with N) of the factorization forms to the exact solution, shown here for a specific example, is entirely typical. This is because Padé approximants *very accurately* imitate the behavior of complex functions with very general singularity structure away from these singularities.

In the previous section an upper bound on the error of the approximation, $\mathbf{K}_N^\pm(\alpha)$, $\alpha \in \mathcal{D}^\pm$, was given as $\max\{e_N(\alpha) : \alpha \in \mathcal{D}\}$. The proof that the error is greatest on the boundary of the half-plane domain is given in [3], and in this article an S-shaped boundary (Figure 2) was taken as it was bounded away from the finite branch cuts of $f(\alpha)$ (56). A better (i.e., lower) upper bound on the maximum error may be found by deforming the strip \mathcal{D} in any way as long as it does not cross any singularities of $\mathbf{K}(\alpha)$. Thus, the operation of deformation of \mathcal{D} performed in this article is merely employed to obtain an upper error bound and *does not alter the error in approximating $\mathbf{K}^\pm(\alpha)$*

itself. The final solution is completely independent of this choice of path. As was shown in the numerical examples, the error of approximation of $\mathbf{K}^\pm(\alpha)$ decreases rapidly as α moves to an interior point of \mathcal{D}^\pm . Fortunately, we almost always require these kernel factors at interior points and so good accuracy is always expected even for moderate Padé number N .

A crucial step in the formulation of the approximate factorization is to ensure that the matrix $\mathbf{J}(\alpha)$ (55) has the correct behavior. That is, the elements have polynomial form as $|\alpha| \rightarrow \infty$, i.e., any branch point at infinity has vanishing coefficient, and $\mathbf{J}^2(\alpha)$ is proportional to the identity matrix. All problems so far examined by the author have been able to be cast into this form without difficulty, which is why the method propounded should have wide applicability. It is interesting to note that Wickham's integral equation approach [33] also involves removing terms which are the correct factors at infinity in the transform space. These rather complex terms as expressed in his formulation are nothing other than the simple commutative factors $\mathbf{Q}^\pm(\alpha)$ in the present setting! Wickham then shows that the coupled second-kind integral equations remaining are solvable but require a good deal of effort to evaluate, and this must then be followed by the nontrivial task of constructing a second eigenvector. The present approach circumvents the need to tackle the integral equations for the exact elements of $\mathbf{M}(\alpha)$, and this is its key simplifying step. However, in order to understand why the Padé representation seems to be, in some sense, an *optimal* approximation, ongoing research is concerned with the relationship between collocation solutions of the exact integral equations and the Padé approximants.

To illustrate the important range of physical problems now amenable to solution, the author has examined elastic wave scattering in an elastic half-space lying in $y < 0$, half the surface of which ($x < 0, y = 0$) is clamped. The remaining half of the surface $x > 0$ is traction free, and a Rayleigh wave is incident from infinity along this surface. The interesting variation in the reflection coefficient of the surface Rayleigh wave as a function of Poisson's ratio is calculated in Abrahams [4]. Other work in progress [3] includes the approximate factorization of scalar kernels arising in fluid-structural diffraction problems [11], and other models in static and dynamic elasticity.

Appendix A. Derivation of Padé approximants. Given a function $f(z)$, regular at the origin, with a Taylor series expansion

$$(A.126) \quad f(z) = \sum_{n=0}^{\infty} e_n z^n,$$

say, then the $[L/M]$ Padé approximant to $f(z)$ is

$$(A.127) \quad [L/M] = \frac{a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_L z^L}{1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots + b_M z^M}.$$

The $[L/M]$ approximant has the same Taylor series expansion as $f(z)$ up to and including order $L + M$, i.e.,

$$(A.128) \quad \sum_{n=0}^{\infty} e_n z^n - [L/M] = \mathcal{O}(z^{L+M+1}).$$

Generally speaking, $[L/M]$ Padé approximants exist and are uniquely determined by L, M , and the first $L + M + 1$ coefficients of $f(z)$, namely, the e_n . Multiplying (A.128)

by the denominator polynomial of $[L/M]$ and equating coefficients of $z^{L+1}, z^{L+2}, \dots, z^{L+M}$ give an algebraic system of equations:

$$(A.129) \quad \mathbf{E}\mathbf{b} = -\mathbf{e}_L,$$

where

$$(A.130) \quad \mathbf{b} = (b_1, b_2, \dots, b_M)^T,$$

$$(A.131) \quad \mathbf{e}_j = (e_{j+1}, e_{j+2}, \dots, e_{j+M})^T,$$

and \mathbf{E} is the $M \times M$ matrix formed from the column vectors

$$(A.132) \quad \mathbf{E} = (\mathbf{e}_{L-1}, \mathbf{e}_{L-2}, \dots, \mathbf{e}_{L-M}).$$

Note that we define $e_n \equiv 0$ for $n < 0$ in the above expressions. Inverting (A.129) gives all the denominator coefficients, $b_n, 1 \leq n \leq M$, and, once determined, the a_n s are found by equating coefficients of $z^n, 0 \leq n \leq L$. It must be stated that particular $[L/M]$ approximants of a given function may not exist if the system (A.129) is not invertible. Details of these cases, and methods for predicting the special $[L/M]$ values, can be found in Baker [9]. For further information on approximants, see [18, 19].

The essential property of Padé approximants is that they give an approximation to $f(z)$ far beyond the radius of convergence of its Taylor series expansion. This is due to the fact that $f(w)$, where

$$(A.133) \quad w(z) = \frac{z}{1 + cz}, \quad c = \text{constant},$$

will yield the identical $[L/M]$ approximant to that of $f(z)$ for any c . This invariance property ensures good agreement between the function and its approximant everywhere in the z -plane except near to the singularities of $f(z)$. Obviously approximants can only have pole-type singularities, and these may have to represent branch cuts, essential singularities, etc. In practice, this does not present any difficulty, and the Padé singularities cluster along cuts or near to actual point singularities of $f(z)$. As a final point, we could have derived the approximants by equating them with a Taylor expansion at any regular point of $f(z)$, not just at the origin. This would lead to a slightly different Padé approximant than that deduced from $z = 0$, but by a simple transformation in independent variable, the above procedure can still be followed.

In this article the function $f(\alpha)$, (65), which we approximate, has constant behavior as $|\alpha| \rightarrow \infty$. Thus, to mimic its behavior everywhere, including the point at infinity, it is necessary to employ $[N/N]$ Padé approximants. Further, as $f(\alpha)$ is only a function of α^2 , in section 4 we use a slightly different notation to that specified in this appendix. The differences should be transparent to the reader.

Appendix B. Construction of the meromorphic matrix $\mathbf{M}(\alpha)$. In section (4.2) the noncommutative matrix factors \mathbf{K}_N^\pm were constructed using the meromorphic matrix $\mathbf{M}(\alpha)$, (72). In the text $\mathbf{M}(\alpha)$ was posed as an ansatz with unknown coefficients, but for general matrix Wiener–Hopf kernels the form of the regularizing matrix $\mathbf{M}(\alpha)$ will not be easy to spot. This appendix demonstrates a procedure which will *construct* $\mathbf{M}(\alpha)$ and its coefficients $A_n, B_n, \bar{A}_n, \bar{B}_n$ for the given Wiener–Hopf kernel (35), and it can be applied in an identical fashion for any kernel.

Our aim is to construct the noncommutative factors \mathbf{K}_N^\pm once the partial commutative split matrices \mathbf{Q}_N^\pm are determined. From (71) we know that

$$(B.134) \quad \mathbf{Q}_N^+[\mathbf{K}_N^+]^{-1} = [\mathbf{Q}_N^-]^{-1}\mathbf{K}_N^-,$$

and if we write the first column of $[\mathbf{K}_N^+]^{-1}$ as $\frac{1}{r^+(\alpha)}(a^+(\alpha), b^+(\alpha))^T$ and the first column of \mathbf{K}_N^- as $r^-(\alpha)(a^-(\alpha), b^-(\alpha))^T$, then

$$(B.135) \quad \begin{aligned} & \left(\cos(\gamma(\alpha)s^+(\alpha))\mathbf{I} + \frac{1}{\gamma(\alpha)} \sin(\gamma(\alpha)s^+(\alpha))\mathbf{J}_N(\alpha) \right) \begin{pmatrix} a^+(\alpha) \\ b^+(\alpha) \end{pmatrix} \\ & = \left(\cos(\gamma(\alpha)s^-(\alpha))\mathbf{I} - \frac{1}{\gamma(\alpha)} \sin(\gamma(\alpha)s^-(\alpha))\mathbf{J}_N(\alpha) \right) \begin{pmatrix} a^-(\alpha) \\ b^-(\alpha) \end{pmatrix}. \end{aligned}$$

The left-hand side of this equation is analytic in \mathcal{D}^+ except for simple poles at $\alpha = p_n$, from the zeros of $f_N(\alpha)$ in $\mathbf{J}_N(\alpha)$, and at $\alpha = q_n$ from the poles of $f_N(\alpha)$ (see (73)). Similarly the right-hand side is analytic in \mathcal{D}^- except for poles at $\alpha = -p_n, -q_n$. It is simple to show, from (75), that the top element of the left-hand side of (B.135) has residue

$$(B.136) \quad \frac{\beta_m}{2p_m} \gamma(p_m) \sin[\gamma(p_m)s^+(p_m)] b^+(p_m) = \frac{B_m}{2p_m},$$

say, at $\alpha = p_m$, and the top element of the right-hand side has residue

$$(B.137) \quad \frac{\beta_m}{2p_m} \gamma(p_m) \sin[\gamma(p_m)s^+(p_m)] b^-(-p_m) = \frac{C_m}{2p_m},$$

say, at $\alpha = -p_m$, where in (B.137) the symmetry property $\gamma(p_m) = \gamma(-p_m)$, $s^+(p_m) = s^-(-p_m)$ has been employed. Thus, the top line of (B.135) can be written as

$$(B.138) \quad \begin{aligned} & a^+(\alpha) \cos(\gamma(\alpha)s^+(\alpha)) + b^+(\alpha) \frac{\gamma(\alpha)}{f_N(\alpha)} \sin(\gamma(\alpha)s^+(\alpha)) \\ & - \sum_{n=1}^N \left(\frac{B_n}{\alpha - p_n} + \frac{C_n}{\alpha + p_n} \right) \frac{1}{2p_n} \\ & = a^-(\alpha) \cos(\gamma(\alpha)s^-(\alpha)) - b^-(\alpha) \frac{\gamma(\alpha)}{f_N(\alpha)} \sin(\gamma(\alpha)s^-(\alpha)) \\ & - \sum_{n=1}^N \left(\frac{B_n}{\alpha - p_n} + \frac{C_n}{\alpha + p_n} \right) \frac{1}{2p_n}, \end{aligned}$$

where the left (right) side is now regular in \mathcal{D}^+ (\mathcal{D}^-). Hence by the usual analytic continuation arguments both sides of this equation must be equal to an entire function, which, *without loss of generality*, we can choose to be a constant, C_1 , say (for a similar exercise see Abrahams [2]). Similarly, the bottom equation of (B.135) must have poles at $\alpha = \pm q_n$ subtracted from both sides:

$$(B.139) \quad \begin{aligned} & b^+(\alpha) \cos(\gamma(\alpha)s^+(\alpha)) - a^+(\alpha) \frac{f_N(\alpha)}{\gamma(\alpha)} \sin(\gamma(\alpha)s^+(\alpha)) \\ & + \frac{1}{2} \sum_{n=1}^N \left(\frac{A_n}{\alpha - q_n} + \frac{D_n}{\alpha + q_n} \right) \\ & = b^-(\alpha) \cos(\gamma(\alpha)s^-(\alpha)) + a^-(\alpha) \frac{f_N(\alpha)}{\gamma(\alpha)} \sin(\gamma(\alpha)s^-(\alpha)) \\ & + \frac{1}{2} \sum_{n=1}^N \left(\frac{A_n}{\alpha - q_n} + \frac{D_n}{\alpha + q_n} \right), \end{aligned}$$

where use has been made of (74), and

$$(B.140) \quad A_m = \frac{\alpha_m}{q_m \gamma(q_m)} \sin[\gamma(q_m) s^+(q_m)] a^+(q_m),$$

$$(B.141) \quad D_m = \frac{\alpha_m}{q_m \gamma(q_m)} \sin[\gamma(q_m) s^+(q_m)] a^-(q_m).$$

Again both sides of (B.139) are analytic in overlapping half-planes and so are equal to the constant C_2 , say. Hence (B.138) and (B.139) give

$$(B.142) \quad \begin{pmatrix} a^+(\alpha) \\ b^+(\alpha) \end{pmatrix} = [\mathbf{Q}^+(\alpha)]^{-1} \begin{pmatrix} C_1 + \sum_{n=1}^N \left(\frac{B_n}{\alpha - p_n} + \frac{C_n}{\alpha + p_n} \right) \frac{1}{2p_n} \\ C_2 - \frac{1}{2} \sum_{n=1}^N \left(\frac{A_n}{\alpha - q_n} + \frac{D_n}{\alpha + q_n} \right) \end{pmatrix},$$

$$(B.143) \quad \begin{pmatrix} a^-(\alpha) \\ b^-(\alpha) \end{pmatrix} = \mathbf{Q}^-(\alpha) \begin{pmatrix} C_1 + \sum_{n=1}^N \left(\frac{B_n}{\alpha - p_n} + \frac{C_n}{\alpha + p_n} \right) \frac{1}{2p_n} \\ C_2 - \frac{1}{2} \sum_{n=1}^N \left(\frac{A_n}{\alpha - q_n} + \frac{D_n}{\alpha + q_n} \right) \end{pmatrix},$$

and all that remains is to determine the coefficients $A_n - D_n$. Taking the top line of (B.142) first, we can obtain a relation between the unknown coefficients if $\alpha \rightarrow q_m$. Omitting all details, we find

$$(B.144) \quad a^+(q_m) = \frac{1}{\cos[\gamma(q_m) s^+(q_m)]} \left(C_1 + \sum_{n=1}^N \left(\frac{B_n}{q_m - p_n} + \frac{C_n}{q_m + p_n} \right) \frac{1}{2p_n} \right)$$

or, from (B.140),

$$(B.145) \quad A_m = \frac{\alpha_m}{q_m \gamma(q_m)} \tan[\gamma(q_m) s^+(q_m)] \left(C_1 + \sum_{n=1}^N \left(\frac{B_n}{q_m - p_n} + \frac{C_n}{q_m + p_n} \right) \frac{1}{2p_n} \right).$$

Similarly, the top line of (B.143) gives, as $\alpha \rightarrow -q_m$, the identity

$$(B.146) \quad D_m = \frac{\alpha_m}{q_m \gamma(q_m)} \tan[\gamma(q_m) s^+(q_m)] \left(C_1 - \sum_{n=1}^N \left(\frac{B_n}{q_m + p_n} + \frac{C_n}{q_m - p_n} \right) \frac{1}{2p_n} \right),$$

and the bottom rows of (B.142), (B.143) also reveal

$$(B.147) \quad B_m = \beta_m \gamma(p_m) \tan[\gamma(p_m) s^+(p_m)] \left(C_2 - \frac{1}{2} \sum_{n=1}^N \left(\frac{A_n}{p_m - q_n} + \frac{D_n}{p_m + q_n} \right) \right),$$

$$(B.148) \quad C_m = \beta_m \gamma(p_m) \tan[\gamma(p_m) s^+(p_m)] \left(C_2 + \frac{1}{2} \sum_{n=1}^N \left(\frac{A_n}{p_m + q_n} + \frac{D_n}{p_m - q_n} \right) \right).$$

The two constants C_1, C_2 are still arbitrary and so, again without loss of generality, we can take two cases: first $C_1 = 1, C_2 = 0$, and second $C_1 = 0, C_2 = 1$. When $C_2 = 0$, the symmetry in equations (B.145)–(B.148) gives

$$(B.149) \quad A_m = D_m, \quad B_m = -C_m,$$

and so we obtain the pair of coupled algebraic equations

$$(B.150) \quad A_m = \frac{\alpha_m}{q_m \gamma(q_m)} \tan[\gamma(q_m) s^+(q_m)] \left(1 + \sum_{n=1}^N \frac{B_n}{q_m^2 - p_n^2} \right),$$

$$(B.151) \quad B_m = -p_m \beta_m \gamma(p_m) \tan[\gamma(p_m) s^+(p_m)] \sum_{n=1}^N \frac{A_n}{p_m^2 - q_n^2}$$

for $m = 1, \dots, N$. These agree exactly with (81), (80). The alternative choice of constants, $C_1 = 0$, $C_2 = 1$, is easily shown to give the coupled equations for \bar{A}_m (defined here as $q_m A_m$), \bar{B}_m (defined here as B_m/p_m) found in section 4. Therefore, forming $\mathbf{K}_N^-(\alpha)$ from (B.143) by taking the two column vectors with the above choice of constants, we obtain

$$(B.152) \quad \mathbf{M}(\alpha) = \begin{pmatrix} 1 + \sum_{n=1}^N \frac{B_n}{\alpha^2 - p_n^2}, & \alpha \sum_{n=1}^N \frac{\bar{B}_n}{\alpha^2 - p_n^2} \\ -\alpha \sum_{n=1}^N \frac{A_n}{\alpha^2 - q_n^2}, & 1 - \sum_{n=1}^N \frac{\bar{A}_n}{\alpha^2 - q_n^2} \end{pmatrix},$$

which is the ansatz defined in (72).

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