

A DIFFUSION ANALYSIS APPROACH TO TE MODE PROPAGATION IN RANDOMLY PERTURBED OPTICAL WAVEGUIDES*

EMMANUEL PERREY-DEBAIN[†] AND I. DAVID ABRAHAMS[‡]

Abstract. The aim of this work is to model the evolution of the modal distribution of the electromagnetic field as it propagates along a randomly deformed multimode optical waveguide. When the number of guided modes becomes large we can regard the discrete set of modes as a quasi continuum. In some cases, nearest neighbor coupling predominates over other power transfer mechanisms and the coupling process can be ideally described in terms of a diffusion equation. The theory is applied to the propagation of guided transverse electric (TE) field waves in a slab waveguide with parabolic refractive index profile. Numerical simulations are in good agreement with theoretical results, and the error is shown to behave as the inverse of the number of guided modes. The technique allows the prediction of the long-distance modal distribution for a very large number of guided modes within fixed computational resources.

Key words. random waveguide, modal diffusion, optical fiber

AMS subject classifications. 78A45, 78A48, 78A50

DOI. 10.1137/060673874

1. Introduction. While more and more sophisticated methods for the manufacture and control of graded index multimode fibers are being implemented, the random variations of the optical and geometrical properties of fibers from the ideal model are impossible to avoid. These small imperfections influence the signal propagation as a result of mode coupling, and their cumulative effects may become important after a long propagation length. Thus, they need to be taken into account when calculating the power attenuation, the signal distortion, and the bandwidth of the fiber [1, 2, 3]. The statistical treatment of wave propagation in random waveguides has been the topic of numerous papers, and a complete survey would merit a separate article; some interesting references can be found in [4].

The most common approach consists in deriving and solving the coupled power equations describing the evolution of the average power carried by the propagating modes. The earliest investigations of mode coupling in optical waveguides were concerned with the excess losses which result from the coupling of guided modes to radiation modes [5, 6]. Rowe and Young [7] showed that when random perturbations are present in a two-mode waveguide, one can derive coupled power equations for the power in each mode. Marcuse [8] generalized this result to any number of guided modes, and an excellent summary of this work can be found in his textbook [9]. When the number of guided modes becomes too large, a direct algebraic treatment of the coupled system is ruled out because of the computational overhead. In some cases,

*Received by the editors November 1, 2006; accepted for publication (in revised form) August 7, 2007; published electronically November 30, 2007. This research was carried out as part of the activities of the Smith Institute Knowledge Transfer Network (<http://www.smithinst.ac.uk>).

<http://www.siam.org/journals/siap/68-2/67387.html>

[†]Laboratoire Roberval, Université de Technologie de Compiègne, BP 60319-60203, Compiègne Cedex, France (emmanuel.perrey-debain@utc.fr). This author's research was supported by a post-doctoral grant from the UK Engineering and Physical Sciences Research Council (EPSRC) and Photon Design (Oxford, UK).

[‡]School of Mathematics, University of Manchester, Oxford Road, Manchester, M13 9PL, England (i.d.abrahams@maths.manchester.ac.uk).

however, nearest neighbor coupling predominates over other power transfer mechanisms and, within appropriate limits, the coupling process can be ideally described in terms of a diffusion equation in which the mode number is treated as a continuous variable. This idea originated in the mid-seventies for dealing with the specific problem of random bends [10, 11, 12].

The diffusion analysis approach to mode propagation in optical fibers relies on many simplifications and assumptions which render the theory's validity difficult to estimate. Curiously enough, no progress has been made since the mid-seventies and, until recently, Gloge's diffusion theory [10] has been the starting point for evaluating mode conversion in step-index multimode fibers [13, 14]. In this paper (and in a forthcoming article discussing the three-dimensional waveguide [15]), we aim at offering a new contribution to the diffusion approach [16] by treating the problem in a much more rigorous manner. It is found that, for the specific case of a slab waveguide with a parabolic index profile, the coupled power equations system can be approximated as a diffusion equation with an approximation error of order $\mathcal{O}(N^{-1})$, where N is the number of modes. In practice, the theory leads to a numerically tractable problem for predicting the long-distance modal distribution of the transverse electric (TE) field for any waveguide supporting a sufficiently large number of modes. Furthermore, it allows one to identify nondiffusive regimes in which the modal power distribution is not the solution of a diffusion equation and which exhibits irregular behavior.

The structure of the paper is as follows. The statement of the problem is presented in section 2. In section 3, the standard coupled power equations for the slab waveguide are stated, and a continuous model is derived in section 4. We finally compare the theoretical results with numerical solutions of the diffusion equation for various cases in section 5.

2. Problem statement. We aim to study the propagation of a monochromatic TE field $E_Y = E(X, Z)e^{-i\omega t}$ in a weakly guiding two-dimensional dielectric waveguide whose parabolic graded-index profile n is affected by a small random perturbation, say, δn . The field is governed by the time-harmonic wave equation

$$(2.1) \quad \frac{\partial^2 E}{\partial X^2} + \frac{\partial^2 E}{\partial Z^2} + \kappa^2 n^2(X)E = \kappa^2 \delta n^2(X, Z)E,$$

where κ is the vacuum wavenumber, Z is the guide axis, and X is the transverse coordinate. The refractive index of the unperturbed waveguide has the parabolic profile

$$(2.2) \quad n^2(X) = n_0^2(1 - 2\Delta(X/a)^2)$$

in the waveguide region, $|X| \leq a$, and $n^2(X) = n_c^2$ in the infinite cladding, $|X| > a$. The profile height parameter $\Delta = (n_0^2 - n_c^2)/2n_0^2$ is assumed to be small, and backscattering is ignored so that, under appropriate scaling, the problem can be conveniently formulated [4] as the following Schrödinger-type equation for the amplitude $\Psi = Ee^{-i\kappa n_0 Z}$:

$$(2.3) \quad 2i \frac{\partial \Psi}{\partial z} = -\frac{\partial^2 \Psi}{\partial x^2} + v(x)\Psi + \delta v(x, z)\Psi,$$

where $z = \sqrt{2\Delta}Z/a$, $x = \sqrt{V}X/a$, and $\delta v = V(n_0^2 - n_c^2)^{-1}\delta n^2$. Here V denotes the usual waveguide parameter $V = \kappa n_0 a \sqrt{2\Delta}$ and v stands for the quadratic potential with finite depth: $v(x) = Vf(\bar{x})$, where $\bar{x} = x/\sqrt{V} = X/a$ and f is the normalized

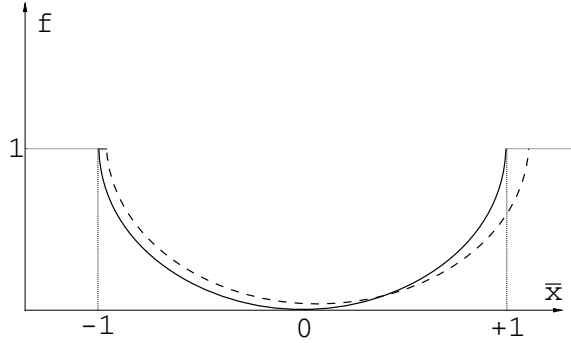


FIG. 2.1. Profile of the normalized index of refraction; solid line indicates the unperturbed waveguide; dashed line indicates the waveguide under small deformation.

quadratic profile (see Figure 2.1)

$$(2.4) \quad f(\bar{x}) = \left\{ \begin{array}{ll} \bar{x}^2, & |\bar{x}| \leq 1 \\ 1, & |\bar{x}| > 1 \end{array} \right\}.$$

Let us now introduce ϵ , the relative amplitude of the perturbation of the slab profile. We can simulate small deviations from the ideal profile (2.4) as follows:

$$\tilde{f}(\bar{x}, z) = f(\bar{x}) + \delta f(\bar{x}, z) = (1 + \epsilon g_2(z))(\bar{x} + \epsilon g_1(z)/2)^2 + \epsilon g_3(z), \quad a^-(z) \leq \bar{x} \leq a^+(z).$$

The functions $g_q(z)$'s are random processes with amplitudes that do not exceed unity. In practice, g_1 simulates random oscillations of the center of the waveguide around the optical axis (microbending), g_2 is a random change of waveguide width, and g_3 is a random variation of the average refractive index. Note that the core-cladding interface is perturbed slightly from $\bar{x} = \pm 1$ to $\bar{x} = a^\pm(z)$. However, we will be interested in the behavior of the wave field away from the interface and so, as is usual, we will ignore the effects of the core-cladding interface on the mode intercoupling in this article. To first order, we find that the nondimensionalized perturbation to the refractive index is

$$(2.5) \quad \tilde{f}(\bar{x}, z) - f(\bar{x}) = \delta f(\bar{x}, z) = \epsilon(g_1(z)\bar{x} + g_2(z)\bar{x}^2 + g_3(z)) + O(\epsilon^2).$$

We can be more general, and henceforth take $\delta f = \epsilon g$, where the normalized random perturbation g is assumed to have the separable form

$$(2.6) \quad g(\bar{x}, z) = \sum_{q=1}^Q g_q(z)\phi_q(\bar{x}),$$

in which the g_q 's are real-valued zero-mean, *independent*, stationary, and ergodic processes with respect to the waveguide axis coordinate z , and $\phi_q(\bar{x})$ are *deterministic* functions that can be referred to as the “perturbation modes.” These functions are assumed to be sufficiently regular that they can be conveniently formulated as the truncated Taylor series

$$(2.7) \quad \phi_q(\bar{x}) = \sum_{n=0}^{\eta_q} b_{q,n}\bar{x}^n, \quad \text{where} \quad b_{q,n} = \frac{1}{n!} \left(\frac{d^n \phi_q}{d\bar{x}^n} \right)_{\bar{x}=0}.$$

3. Coupled mode theory. We wish to solve the parabolic equation (2.3), given the initial conditions $\Psi(x, 0)$ at the input of the waveguide. For sufficiently small perturbations, it is possible to express the field distribution in the waveguide by using standard perturbation theory [17, 18, 19], i.e., Ψ is expanded in the eigenfunction basis of the unperturbed waveguide as

$$(3.1) \quad \Psi(x, z) = \sum_{\nu} a_{\nu}(z) \psi_{\nu}(x) e^{-i\beta_{\nu} z} + \sum \int a(\beta, z) \psi(\beta, x) e^{-i\beta z} d\beta.$$

The first summation extends over all the discrete spectrum of *guided modes* satisfying the eigenmode problem

$$(3.2) \quad -\frac{d^2 \psi_{\nu}}{dx^2} + v(x) \psi_{\nu} = 2\beta_{\nu} \psi_{\nu},$$

with the boundary condition $\psi_{\nu}(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Eigenvalues of (3.2) are real positive quantities lying in the range $0 < \beta_{\nu} < \beta_c$ ($\beta_c = V/2$ is the cut-off wavenumber). They characterize the number of oscillations of the eigenfunctions along the transverse section of the waveguide. For a very large number of modes ($V \gg 1$), the highly oscillating wavefunction ψ_{ν} are conveniently described by the WKB approximation, and the largest permitted eigenvalue below cut-off can be shown to be approximately given by the upper bound, $\max_{\nu} \beta_{\nu} \approx \beta_c$. The integral in (3.1) extends over modes of the continuum (radiation modes), and the summation sign in front of the integral indicates summation over even and odd modes. These modes are oscillatory solutions of (3.2) and do not have the evanescent behavior (in x) of the guided mode fields. To be consistent with the forward scattering approximation introduced earlier, we restrict the integration domain to small propagation constants: $\beta_c \leq \beta \ll \beta_c/\Delta$. The expansion coefficients $a_{\nu}(z)$ and $a(\beta, z)$ are unknown functions of z . Using orthogonality properties of both guided and radiation modes, (2.3) is transformed into the system of ordinary differential equations

$$(3.3) \quad \begin{aligned} \frac{da_{\nu}}{dz} &= -i\epsilon\beta_c \sum_{\mu} C_{\nu,\mu}(z) e^{i(\beta_{\nu}-\beta_{\mu})z} a_{\mu}(z) \\ &\quad - i\epsilon\beta_c \sum \int a(\beta, z) C_{\nu}(\beta, z) e^{i(\beta_{\nu}-\beta)z} d\beta, \end{aligned}$$

where coupling coefficients are given by the overlap integrals

$$(3.4) \quad C_{\nu,\mu}(z) = \int_{-\infty}^{\infty} \psi_{\nu}(x) g(\bar{x}, z) \psi_{\mu}(x) dx$$

and

$$(3.5) \quad C_{\nu}(\beta, z) = \int_{-\infty}^{\infty} \psi_{\nu}(x) g(\bar{x}, z) \psi(\beta, x) dx.$$

Though feasible, a numerical solution of the coupled mode system (3.3) can be obtained at a heavy price, which could be well above standard computational resources. In fact, the solution of (3.3) contains more information (i.e., regarding the phase) than is required. It is now well established [9] that, under some additional assumptions, system (3.3) can be averaged over an ensemble of N_w similar waveguide realizations. More precisely, if the guided modes are weakly coupled over a distance which is large

compared to the correlation length of the random process g , then the average power $A_\nu(z) = \langle |a_\nu(z)|^2 \rangle = \lim_{N_w \rightarrow \infty} \frac{1}{N_w} \sum_{N_w} |a_\nu(z)|^2$ carried by mode ν can be shown to satisfy the system of master equations

$$(3.6) \quad \frac{1}{\epsilon^2} \frac{dA_\nu}{dz} = \sum_\mu W_{\nu \rightarrow \mu} (A_\mu - A_\nu) - \alpha_\nu A_\nu,$$

where the transition probability matrix coefficients $W_{\nu \rightarrow \mu}$ are given from the spectral density of $C_{\nu,\mu}(z)$ evaluated at the wavenumber spacing $|\beta_\nu - \beta_\mu|$, i.e.,

$$(3.7) \quad W_{\nu \rightarrow \mu} = 2\beta_c^2 \int_0^\infty \langle C_{\nu,\mu}(0)C_{\nu,\mu}(z) \rangle \cos[(\beta_\nu - \beta_\mu)z] dz.$$

These are, by definition, positive quantities, and thus $W_{\nu \rightarrow \mu} \geq 0$. Note that a similar derivation can be found in the context of quantum mechanics [20] and in acoustics [21]. The power loss coefficients α_ν are positive quantities taking into account the coupling between mode ν to the continuum of radiation modes. A rigorous analysis of the radiation loss is a very difficult task as it requires both an accurate description of the perturbation in the vicinity of the core-cladding interface as well as precise knowledge of the mathematical form for the guided and radiation modes close to cut-off. Nevertheless, for waveguides supporting a sufficiently large number of modes, only highest order modes near cut-off carry nonnegligible energy near the interface and therefore suffer from very high losses. To simplify the analysis we will assume that $\alpha_\nu = \infty$ when $\nu \geq \beta_c$, which means that high order modes carry no power: $A_{\nu \geq \beta_c} = 0$. These assumptions were introduced by Marcuse [11] for the parabolic index fiber and were recently found to be in agreement with measurements carried out by Golowich et al. [3]. Due to the symmetry of the matrix coefficients $W_{\nu \rightarrow \mu}$, the solution of (3.6) is given explicitly by

$$(3.8) \quad \mathbf{A}(z) = \mathbf{U} \exp(\epsilon^2 \mathbf{\Lambda} z) \mathbf{U}^T \mathbf{A}(0),$$

where vector $\mathbf{A}(z) = (A_1(z), A_2(z), \dots)^T$, and $\mathbf{A}(0)$ contains the initial conditions at $z = 0$, i.e.,

$$(3.9) \quad A_\nu(0) = |a_\nu(0)|^2 = \left| \int_{-\infty}^\infty \Psi(x, 0) \psi_\nu(x) dx \right|^2.$$

The diagonal matrix $(\mathbf{\Lambda})_{i,i} = \lambda_i$, $i = 1, 2, \dots$, contains the real eigenvalues in descending order, and the column vectors $\mathbf{U}^{(i)} = (U_1^{(i)}, U_2^{(i)}, \dots)^T$ are the eigenmodes of the real symmetric system with eigenvalues λ_i :

$$(3.10) \quad \mathbf{W} \mathbf{U}^{(i)} = \lambda_i \mathbf{U}^{(i)},$$

with the cut-off condition that $U_\nu^{(i)} = 0$ when $\nu \geq \beta_c$. Note that the transition probability operator \mathbf{W} is defined, from (3.6), as

$$(3.11) \quad (\mathbf{W} \mathbf{U}^{(i)})_\nu = \sum_\mu W_{\nu \rightarrow \mu} (U_\mu^{(i)} - U_\nu^{(i)}) - \alpha_\nu U_\nu^{(i)}.$$

Due to the special structure of (3.11) and the positivity of the off-diagonal terms, Dozier and Tappert [21] showed that Gerschgorin discs with radius $R_\nu = \sum_{\mu \neq \nu} W_{\nu \rightarrow \mu}$

lie in the left part of the complex plane, and hence all eigenvalues, λ_i , are negative. The special case, $\lambda_1 = 0$, can be referred to as the adiabatic case and corresponds to the long-distance solutions $\lim_{z \rightarrow \infty} A_\nu(z) = (\sum_\mu 1)^{-1} \sum_\mu A_\mu(0)$, i.e., an equipartition of energy is achieved whatever the initial conditions. This is a consequence of neglecting the radiation loss. This ideal scenario was considered in [21] but is obviously unrealistic in our context as losses from the highest order modes are unavoidable.

4. The continuous model.

4.1. Simplification when $V \gg 1$. Numerical diagonalization of the transition probability operator \mathbf{W} (3.11) becomes impractical for very large V . Nevertheless, progress can be made if we are interested only in the lower $|\lambda_i|$ corresponding to long-distance solutions. To achieve this, we need to find a continuum analogue of (3.10). Let us first observe that in the limit of large V , the set of orthonormal functions ψ_ν satisfying (3.2) are the classical harmonic oscillator bases [18, 20]:

$$(4.1) \quad \psi_\nu(x) = \frac{1}{\sqrt{\pi^{1/2} 2^\nu \nu!}} H_\nu(x) e^{-x^2/2}, \quad \nu = 0, 1, 2, \dots, \quad \text{with} \quad \beta_\nu = \nu + 1/2,$$

where H_ν denotes the usual Hermite polynomials. These are good approximations to the exact solutions, except for modes near cut-off, $\beta_\nu \approx \beta_c$. We assume that (4.1) is valid for all modes below cut-off; these modes are unaffected by the interface core-cladding as the power carried in this region is negligible and the evaluation of the coupling coefficients can be greatly simplified by extending the perturbation (2.6) over the whole real line as

$$(4.2) \quad C_{\nu,\mu}(z) = \sum_{q=1}^Q \sum_{n=0}^{\eta_q} g_q(z) b_{q,n} \int_{-\infty}^{\infty} \psi_\nu(x) \bar{x}^n \psi_\mu(x) dx.$$

To make some progress, we can observe that Hermite polynomials fall into the class of orthogonal polynomials satisfying a three-term recurrence relation which, in terms of the normalized function ψ_ν , reads

$$(4.3) \quad x\psi_\nu(x) = \frac{1}{\sqrt{2}} (\sqrt{\nu} \psi_{\nu-1}(x) + \sqrt{\nu+1} \psi_{\nu+1}(x)).$$

Using purely algebraic arguments, the n th power of the two-term recurrence operator (4.3) is established in [22]. This leads to following result.

LEMMA 4.1. *Given positive integers $(\zeta, n) \in \mathbb{N}^2$, the following integration formula holds:*

$$(4.4) \quad \int_{-\infty}^{\infty} \psi_\nu(x) x^n \psi_{\nu+\zeta}(x) dx = 2^{-\frac{n}{2}} F_\zeta(\nu) G_{\zeta,n}(\nu),$$

where

$$(4.5) \quad F_\zeta(\nu) = \prod_{l=1}^{\zeta} \sqrt{\nu+l} \quad \text{and} \quad G_{\zeta,n}(\nu) = \sum_{\sigma \in \frac{n-\zeta}{2} \cap \mathbb{N}} \sum_{\mathbf{i}_\sigma \in \mathcal{I}_\sigma^n} \prod_{l=1}^{\sigma} (\nu+1+i_l-l)$$

and \mathcal{I}_σ^n is the set of indices $\mathbf{i}_\sigma = (i_1, \dots, i_\sigma) \in \mathbb{N}^\sigma$ associated with the nested sum

$$(4.6) \quad \sum_{\mathbf{i}_\sigma \in \mathcal{I}_\sigma^n} = \sum_{i_\sigma=0}^{n-\sigma} \sum_{i_{\sigma-1}=0}^{i_\sigma} \cdots \sum_{i_2=0}^{i_3} \sum_{i_1=0}^{i_2}.$$

By common convention, the products above take the value unity when the lower limit exceeds the upper, and the notation $\sum_{\sigma \in \xi \cap \mathbb{N}}$ indicates that σ takes the value of ξ when ξ is an integer, or else the sum is zero. Note that the parameter ν has been written as an argument of F and G because we will soon generalize it to take noninteger values. Several other quantities will soon be defined which will also use this convention.

The result (4.4) shows that, for the ideal modes just described, the coupling coefficients between modes ν and $\nu + \zeta$ (ζ positive) take the form

$$(4.7) \quad C_{\nu, \nu+\zeta}(z) = \sum_{q=1}^Q g_q(z) \Phi_{q, \zeta}(\nu),$$

where

$$(4.8) \quad \Phi_{q, \zeta}(\nu) = F_\zeta(\nu) \sum_{n \geq 0} b_{q, n} (2V)^{-\frac{n}{2}} G_{\zeta, n}(\nu),$$

and $b_{q, n}$ is written in (2.7). The factorization of the quantity $F_\zeta(\nu)$ in (4.8) is a key result since it allows us to define the polynomial series w_ζ defined over the real line $\tilde{\nu} \in \mathbb{R}$ as

$$(4.9) \quad w_\zeta(\tilde{\nu}) = \sum_{q=1}^Q \Phi_{q, |\zeta|}^2(\tilde{\nu} - |\zeta|/2) \Gamma_q(\zeta),$$

where ζ now belongs to the whole integer set, $\zeta \in \mathbb{Z}$, and Γ_q stands for the spectral density function

$$(4.10) \quad \Gamma_q(\zeta) = 2 \int_0^\infty \langle g_q(0) g_q(z) \rangle \cos(\zeta z) dz.$$

Finally the transition probability matrix coefficients $W_{\nu \rightarrow \nu+\zeta}$ are given from the regular function w_ζ evaluated at the midpoint $\nu + \zeta/2$, i.e.,

$$(4.11) \quad W_{\nu \rightarrow \nu+\zeta} = \beta_c^2 w_\zeta(\nu + \zeta/2).$$

This is a key result of this article; it relates the transition probability matrix to a regular function over continuous arguments. This fact will be used shortly in obtaining a Taylor series expansion.

Let us now introduce η as the maximum exponent in the truncated Taylor expansion (2.7), i.e., $\eta = \max_{1 \leq q \leq Q} \{\eta_q\}$. By virtue of (4.5), the transition matrix has a band-diagonal structure: $W_{\nu \rightarrow \nu+\zeta} = 0$ when $|\zeta| > \eta$, and furthermore the roots of w_ζ are such that

$$(4.12) \quad W_{\nu \rightarrow -1} = \dots = W_{\nu \rightarrow \nu-\eta} = 0.$$

Thus, there is no coupling with negative indices and (4.11) is exact for all guided modes. Let us now introduce a real analytic function \tilde{U}_i which interpolates the discrete values of the elements of the column vector $\mathbf{U}^{(i)}$,

$$(4.13) \quad \tilde{U}_i(\nu) = U_\nu^{(i)} \quad \text{in the interval} \quad 0 \leq \nu < \beta_c,$$

with the cut-off condition $\tilde{U}_i(\nu) = 0$ in the interval $\beta_c \leq \nu \leq \beta_c + \eta$. This last condition does not need to be satisfied for $\nu > \beta_c + \eta$ due to the band-diagonal structure of

the transition probability matrix. Similarly, we introduce a regular function $\tilde{\gamma}$ such that $\tilde{\gamma}(\nu) = \alpha_\nu$ (the attenuation factor) for all modes below cut-off $0 \leq \nu < \beta_c$. Note that such interpolating functions exist, and it suffices to consider the Lagrange interpolation polynomial, for instance. We can now exploit the equality (4.11) to give the continuum analogue of (3.11). Direct application of Taylor’s theorem for real analytic functions yields the following result.

LEMMA 4.2. *Let $\tilde{U}_i(\tilde{\nu})$ be the real analytic function introduced earlier. Then we have*

$$\begin{aligned}
 (\mathbf{WU}^{(i)})_\nu &= \sum_{\zeta=-\eta}^{\eta} W_{\nu \rightarrow \nu+\zeta} \left(\tilde{U}_i(\nu + \zeta) - \tilde{U}_i(\nu) \right) - \tilde{\gamma}(\nu)\tilde{U}_i(\nu) \\
 (4.14) \qquad &= \beta_c^2 \frac{d}{d\tilde{\nu}} \left(D(\tilde{\nu}) \frac{d\tilde{U}_i}{d\tilde{\nu}} \right) \Bigg|_{\tilde{\nu}=\nu} - \tilde{\gamma}(\nu)\tilde{U}_i(\nu) + \mathcal{R}(\tilde{U}_i)(\nu),
 \end{aligned}$$

where $D(\tilde{\nu}) = \sum_{\zeta=1}^{\eta} \zeta^2 w_\zeta(\tilde{\nu})$ and the residual term $\mathcal{R}(\tilde{U}_i)$ is given by the Taylor series

$$\mathcal{R}(\tilde{U}_i) = 4\beta_c^2 \sum_{(n,m) \in \mathbb{N}_*^2} \sum_{\zeta=1}^{\eta} \frac{(\zeta/2)^{2(n+m+1)}}{(2n+1)!(2m+1)!} \frac{d^{2m+1}}{d\tilde{\nu}^{2m+1}} \left(w_\zeta \frac{d^{2n+1}\tilde{U}_i}{d\tilde{\nu}^{2n+1}} \right),$$

where $\mathbb{N}_*^2 = \mathbb{N}^2 \setminus \{(0, 0)\}$.

Proof. First, we have the following equality, taking an expansion about the midpoint $\nu + \zeta/2$:

$$(4.15) \quad \tilde{U}_i(\nu + \zeta) - \tilde{U}_i(\nu) = \zeta \frac{d\tilde{U}_i}{d\tilde{\nu}}(\nu + \zeta/2) + 2 \sum_{n=1}^{\infty} \frac{(\zeta/2)^{2n+1}}{(2n+1)!} \frac{d^{2n+1}\tilde{U}_i}{d\tilde{\nu}^{2n+1}}(\nu + \zeta/2).$$

Now, functions w_ζ are polynomial series and so, given (4.11), we find

$$\begin{aligned}
 \left(w_\zeta \frac{d\tilde{U}_i}{d\tilde{\nu}} \right) (\nu + \zeta/2) - \left(w_\zeta \frac{d\tilde{U}_i}{d\tilde{\nu}} \right) (\nu - \zeta/2) &= \zeta \frac{d}{d\tilde{\nu}} \left(w_\zeta \frac{d\tilde{U}_i}{d\tilde{\nu}} \right) (\nu) \\
 &+ 2 \sum_{n=1}^{\infty} \frac{(\zeta/2)^{2n+1}}{(2n+1)!} \frac{d^{2n+1}}{d\tilde{\nu}^{2n+1}} \left(w_\zeta \frac{d\tilde{U}_i}{d\tilde{\nu}} \right) (\nu).
 \end{aligned}$$

Repeating this operation for the residual term on the right-hand side of (4.15) yields the expected result. \square

The reader’s attention is drawn to the fact that the leading term on the right-hand side of (4.14) is just the diffusion operator acting on \tilde{U}_i . We comment on this further below.

4.2. Asymptotic series expansion. We are interested in the eigenmode solutions of (3.10) when the number of guided modes is sufficiently large. To do this, we introduce the small parameter $\varepsilon = 1/V$. The asymptotic approach starts by writing every quantity as a power series of ε , and equating coefficients of like powers to yield a hierarchy of equations. Let us first observe that by introducing the normalized variables $u = \beta_c^{-1}\tilde{\nu}$, where $\beta_c^{-1} = 2/V = 2\varepsilon \ll 1$, function w_ζ admits the regular series expansion

$$(4.16) \quad w_\zeta(\tilde{\nu}) = W_\zeta^0(u) + \varepsilon W_\zeta^1(u) + \varepsilon^2 W_\zeta^2(u) + \dots + \varepsilon^\eta W_\zeta^\eta(u),$$

where

$$(4.17) \quad W_\zeta^\alpha(u) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\alpha!} \frac{\partial^\alpha w_\zeta(u(2\varepsilon)^{-1})}{\partial \varepsilon^\alpha}, \quad \alpha = 0, 1, 2, \dots, \eta,$$

are all polynomial series with respect to the continuous variable u . The first two terms are given explicitly by

$$(4.18) \quad W_\zeta^0(u) = u^\zeta \sum_{q=1}^Q \Gamma_q(\zeta) [P_{q,\zeta}^I(u)]^2$$

and

$$(4.19) \quad W_\zeta^1(u) = u^{\zeta-1} \sum_{q=1}^Q \Gamma_q(\zeta) P_{q,\zeta}^I(u) [\zeta P_{q,\zeta}^I(u) + 4P_{q,\zeta}^{II}(u)],$$

where

$$(4.20) \quad P_{q,\zeta}^I(u) = \sum_{\sigma=0}^{\lfloor (\eta_q - \zeta)/2 \rfloor} 2^{-n} b_{q,n} C_\sigma^n u^\sigma |_{n=\zeta+2\sigma}$$

and

$$(4.21) \quad P_{q,\zeta}^{II}(u) = \sum_{\sigma=1}^{\lfloor (\eta_q - \zeta)/2 \rfloor} 2^{-n} b_{q,n} A_\sigma^n u^\sigma |_{n=\zeta+2\sigma}$$

with

$$C_\sigma^n = \text{card}(\mathcal{I}_\sigma^n) = \frac{n!}{\sigma!(n-\sigma)!} \quad \text{and} \quad A_\sigma^n = \sum_{i_\sigma \in \mathcal{I}_\sigma^n} \sum_{l=1}^\sigma (1 + i_l - l - n/2 + \sigma).$$

Note that the quantity $\lfloor x \rfloor$ in these equations indicates the floor of x , i.e., the largest integer less than or equal to the real number x , and η_q (from (2.7)) is the maximum value of n with nonzero $b_{q,n}$. The expansion (4.16) suggests writing a solution $\tilde{U}_i(\tilde{\nu})$ in the form of an asymptotic series expansion

$$(4.22) \quad \tilde{U}_i(\tilde{\nu}) = U_{i,0}(u) + \varepsilon U_{i,1}(u) + \varepsilon^2 U_{i,2}(u) + \dots$$

and

$$(4.23) \quad \lambda_i = \lambda_{i,0} + \varepsilon \lambda_{i,1} + \varepsilon^2 \lambda_{i,2} + \dots$$

Similarly, we may assume that

$$(4.24) \quad \tilde{\gamma}(\tilde{\nu}) = \gamma_0(u) + \varepsilon \gamma_1(u) + \varepsilon^2 \gamma_2(u) + \dots$$

Substituting these expansions into (3.10) yields a series of diffusion equations (for brevity we restrict ourselves to writing just the leading order and the first order corrections):

$$(4.25) \quad \frac{d}{du} \left(D_0(u) \frac{dU_{i,0}}{du} \right) - \gamma_0 U_{i,0} = \lambda_{i,0} U_{i,0},$$

$$(4.26) \quad \frac{d}{du} \left(D_0(u) \frac{dU_{i,1}}{du} \right) - \gamma_0 U_{i,1} = \lambda_{i,0} U_{i,1} + \lambda_{i,1} U_{i,0} + \gamma_1 U_{i,0} - \frac{d}{du} \left(D_1(u) \frac{dU_{i,0}}{du} \right),$$

where functions

$$(4.27) \quad D_0(u) = \sum_{\zeta=1}^{\eta} \zeta^2 W_{\zeta}^0(u) \quad \text{and} \quad D_1(u) = \sum_{\zeta=1}^{\eta} \zeta^2 W_{\zeta}^1(u)$$

can be interpreted as diffusion coefficients controlling the average transfer of modal power at the mode “number” $\tilde{\nu} = \beta_c u$. At subsequent orders, formulae are more complicated due to the presence of the residual term $\mathcal{R}(\tilde{U}_i)$. Each diffusion equation in the family must be solved over the unit interval $[0, 1]$ and all have boundary data $U_{i,\alpha}(1) = 0$, $\alpha \in \mathbb{N}$. The boundary condition at the origin emerges naturally after realizing that there is no transfer of modal energy from negative indices; this implies that

$$(4.28) \quad \left(D_0 \frac{dU_{i,\alpha}}{du} \right)_{u=0} = 0.$$

This boundary condition is in fact the continuous analogue of the no-coupling condition (4.12) and is automatically satisfied for any *regular* solution since, by construction (see (4.18)), we have $D_0(0) = 0$.

4.3. Nature of the leading order solution. In this section, we are interested in *regular* solutions of the leading order eigenmode satisfying

$$(4.29) \quad \frac{d}{du} \left(D_0 \frac{d\varphi}{du} \right) - \gamma_0 \varphi = \lambda \varphi, \quad \varphi(1) = 0.$$

From the remark below (3.11) we also require that $\lambda < 0$. Since $D_0(u)$ is a strictly positive polynomial in $(0, 1]$ with $D_0(0) = 0$, (4.29) is a singular Sturm–Liouville eigenvalue problem and $u = 0$ is a singular endpoint. The regularity of the eigensolution therefore depends upon the behavior of $D_0(u)$ as u tends to zero. Fortunately, (4.29) admits exact analytical solutions for monomial perturbations $g(\bar{x}, z) = g_n(z)\bar{x}^n$ (recall that \bar{x} is the scaled transverse coordinate given in section 2 by $\bar{x} = X/a$) because in these cases

$$(4.30) \quad D_0(u) = d_n u^n \quad \text{with} \quad d_n = 2^{-2n} \sum_{\sigma=0}^{\lfloor \frac{n-1}{2} \rfloor} (n - 2\sigma)^2 \Gamma_n(n - 2\sigma) (C_{\sigma}^n)^2 > 0;$$

by neglecting the loss term γ_0 , it can be shown that the general family of solutions (up to the normalization constant) is

$$(4.31) \quad \varphi(u) = u^{(1-n)/2} J_{\pm\vartheta}(\omega_{n,i}^{\pm} u^{(2-n)/2}), \quad \vartheta = \frac{n-1}{n-2} \quad (n \geq 3),$$

where $\omega_{n,i}^{\pm}$ satisfies $J_{\pm\vartheta}(\omega_{n,i}^{\pm}) = 0$ and $J_{\pm\vartheta}$ is the Bessel function of the first kind of order $\pm\vartheta$. The particular case $n = 3$ yields $\vartheta = 2$ so an additional independent solution is given by $\varphi(u) = u^{-1} Y_2(y_{2,i} u^{-1/2})$, where Y_2 denotes the usual Bessel function of the second kind of order 2 and $y_{2,i}$ are the zeros of Y_2 . When $n = 2$, (4.29) is the classical Euler–Cauchy equation with general solution $\varphi(u) = u^r$, where r satisfies the associated characteristic equation $r^2 + r - \lambda/d_2 = 0$. A quick inspection reveals that, as λ is negative, $\text{Re}(r) < 0$. To summarize, for the specific case where the diffusion coefficient has the simple form $D_0(u) = d_n u^n$ and losses are neglected

($\gamma_0 = 0$), solutions of (4.29) are all singular at the origin except when $n = 1$, for which there exists a unique regular solution of the form

$$(4.32) \quad \varphi(u) = J_0(j_{0,i}\sqrt{u}),$$

where $j_{0,i}$ are the zeros of J_0 in ascending order. The other independent solution is $\varphi(u) = Y_0(y_{0,i}\sqrt{u})$, where Y_0 denotes the usual Bessel function of the second kind of order 0 and $y_{0,i}$ are the zeros of Y_0 . This result suggests that regular solutions are expected, provided the diffusion coefficient has *linear* behavior as $u \rightarrow 0$. This is confirmed by the following proposition.

LEMMA 4.3. *There exists a unique power series solution to the Sturm–Liouville problem (4.29), provided $D_0(u) \sim u$ as $u \rightarrow 0$, which is equivalent to*

$$(4.33) \quad \Gamma_q(1) b_{q,1}^2 \neq 0, \quad q = 1, \dots, Q.$$

Proof. The diffusion coefficient $D_0(u)$ has the general polynomial form

$$(4.34) \quad D_0(u) = \sum_{n=1}^{\eta} d_n u^n,$$

where the first coefficient is explicitly given by $d_1 = \frac{1}{4} \sum_{q=1}^Q \Gamma_q(1) b_{q,1}^2$. Without loss of generality, we may assume that $\gamma_0(u)$ has a power series expansion. Substituting the Frobenius–Fuchs series

$$(4.35) \quad \varphi(u) = u^c \sum_{j=0}^{\infty} a_j u^j, \quad a_0 \neq 0,$$

in (4.29) leads to the indicial equation: $d_1 a_0 c^2 = 0$ and the existence of a power series solution is guaranteed if $d_1 \neq 0$, which is equivalent to (4.33) since the Γ_q 's are positive functions. Furthermore, the series is unique due to Fuchs' theorem. Note the associated eigenvalues can be checked to be real negative since the regularity of φ implies that

$$(4.36) \quad \lambda \int_0^1 \varphi^2 du = - \int_0^1 D_0 \left(\frac{d\varphi}{du} \right)^2 du - \int_0^1 \gamma_0 \varphi^2 du,$$

where all the integrals are positive; the result is apparent by inspection. This is consistent with that found for the discrete eigenvalue problem (3.10).

If $d_1 = 0$ and $d_2 \neq 0$, then the indicial equation becomes

$$(4.37) \quad c^2 + c - \frac{\lambda + \gamma_0(0)}{d_2} = 0.$$

In the limit of large V , the energy carried by the fundamental mode $\nu = 0$ is vanishingly small at the core-cladding interface, so $\lim_{V \rightarrow \infty} \tilde{\gamma}(0) = 0$ and therefore $\gamma_0(0) = 0$. Now, given the fact that $\lambda < 0$ and $d_2 > 0$, roots of the quadratic form have strictly negative real part, and this leads to singular solutions. When $d_1 = d_2 = 0$, $u = 0$ is an irregular singular point and there is no series solution. \square

In order to give a physical explanation of the condition (4.33), for simplicity let us assume that the waveguide is affected by a single perturbation mode, i.e.,

$g(\bar{x}, z) = g_1(z)\phi_1(\bar{x})$. As the number of guided modes tends to infinity, the modal distribution tends to the solution of a diffusion equation provided that (see (4.10))

$$(4.38) \quad \Gamma_1(1) = 2 \int_0^\infty \langle g_1(0)g_1(z) \rangle \cos(z) dz \neq 0 \quad \text{and} \quad b_{1,1} = \left(\frac{d\phi_1}{d\bar{x}} \right)_{\bar{x}=0} \neq 0.$$

The first inequality is nothing other than the well-known “resonance” condition [9] to ensure the coupling between two adjacent modes with equal spacing in β space; in other words, the perturbation g_1 must have spatial frequency support at $\beta_{\nu+1} - \beta_\nu = 1$. The second condition means that the perturbation cannot be locally flat in the vicinity of the waveguide axis. If it is, then as $V \rightarrow \infty$ the lowest order modes, localized very near the axis, will not “see” any perturbation at all and there will be no coupling and therefore no modal diffusion. This behavior is illustrated numerically in the last section of this paper.

4.4. Regular solution and first order correction. Given a random perturbation satisfying (4.33), we call $\{U_{i,0}\}_{i=1}^\infty$ the set of regular solutions of the self-adjoint eigenvalue problem (4.25). Assuming that the associated eigenvalues are all distinct, the following orthogonality property holds:

$$(4.39) \quad \int_0^1 U_{i,0} U_{k,0} du = \|U_{i,0}\|^2 \delta_{i,k},$$

where $\|\cdot\|$ stands for the usual energy norm of $L^2([0,1])$. The orthogonality of the eigenfunctions is in line with the orthogonality of the eigenvectors

$$(4.40) \quad \mathbf{U}^{(i)} \cdot \mathbf{U}^{(k)} = \delta_{i,k}.$$

Let us define vectors $\mathbf{V}_\alpha^{(i)}$, $\alpha = 0, 1, 2, \dots$, as the discrete versions of their continuous counterpart: $(\mathbf{V}_\alpha^{(i)})_\nu = U_{i,\alpha}(\nu/\beta_c)$ for all guided modes. Rewriting the perturbation expansion (4.22) in its vectorial form gives

$$(4.41) \quad \mathbf{U}^{(i)} = \mathbf{V}_0^{(i)} + \varepsilon \mathbf{V}_1^{(i)} + \varepsilon^2 \mathbf{V}_2^{(i)} + \dots$$

According to (4.40), the norm of the leading order solution $\|U_{i,0}\|$ must be chosen such that $\|\mathbf{V}_0^{(i)}\|_2^2 = 1 + \mathcal{O}(\varepsilon)$. This can easily be shown to be satisfied by simply taking

$$(4.42) \quad \|U_{i,0}\| = \beta_c^{-\frac{1}{2}} = \sqrt{2\varepsilon}^{\frac{1}{2}}.$$

The first order correction is explicitly obtained by expanding $U_{i,1}$ in the leading order orthogonal basis $U_{i,0}$, i.e.,

$$(4.43) \quad U_{i,1}(u) = \sum_{k=1}^\infty v_{i,k} U_{k,0}(u).$$

Substituting (4.43) into (4.26) and using orthogonality properties yields

$$(4.44) \quad v_{i,k} = \frac{(2\varepsilon)^{-1}}{\lambda_{i,0} - \lambda_{k,0}} \int_0^1 \left[\frac{d}{du} \left(D_1 \frac{dU_{i,0}}{du} \right) U_{k,0} - \gamma_1 U_{i,0} U_{k,0} \right] du, \quad k \neq i,$$

and

$$(4.45) \quad \lambda_{i,1} = (2\varepsilon)^{-1} \int_0^1 \left[\frac{d}{du} \left(D_1 \frac{dU_{i,0}}{du} \right) U_{i,0} - \gamma_1 U_{i,0} U_{i,0} \right] du.$$

The diagonal correction terms $v_{i,i}$ stem from the discrete normalization (4.40). To first order, this is equivalent to the condition

$$(4.46) \quad \|\mathbf{V}_0^{(i)} + \varepsilon \mathbf{V}_1^{(i)}\|_2^2 = 1 + \mathcal{O}(\varepsilon^2).$$

The correspondence between the discrete and continuous norms is given by the composite trapezoidal rule:

$$\int_0^{\beta_c} (U_{i,0}(\tilde{\nu}/\beta_c) + \varepsilon U_{i,1}(\tilde{\nu}/\beta_c))^2 d\tilde{\nu} = \|\mathbf{V}_0^{(i)} + \varepsilon \mathbf{V}_1^{(i)}\|_2^2 - \frac{1}{2}(U_{i,0}(0) + \varepsilon U_{i,1}(0))^2 + T^{(i)},$$

where the quadrature error $T^{(i)}$ is bounded by

$$|T^{(i)}| \leq \frac{\varepsilon}{6} \max_{u \in [0,1]} \left| \frac{d^2(U_{i,0} + \varepsilon U_{i,1})^2}{du^2}(u) \right|.$$

Thus, due to the normalization (4.42), $T^{(i)} \sim \mathcal{O}(\varepsilon^2)$. Moreover, by construction

$$\int_0^{\beta_c} (U_{i,0}(\tilde{\nu}/\beta_c) + \varepsilon U_{i,1}(\tilde{\nu}/\beta_c))^2 d\tilde{\nu} = 1 + \varepsilon 2v_{i,i} + \mathcal{O}(\varepsilon^2),$$

and so the normalization condition (4.46) is satisfied if

$$(4.47) \quad v_{i,i} = -\frac{\varepsilon^{-1}}{4} U_{i,0}^2(0).$$

Note that the orthogonality of the first order eigenvectors is checked in Appendix A, confirming the above analysis.

5. Numerical experiments.

5.1. Linear perturbation. In this section we shall focus on the linear perturbation $g(\bar{x}, z) = g_1(z)\bar{x}$. This arises from random changes in the direction of the waveguide axis. From (4.20)–(4.21), $P_{1,1}^I(u) = 2^{-1}$ and $P_{1,1}^{II}(u) = 0$, which, taking $Q = \eta = 1$ in (4.18), (4.19), and (4.27), leads to the simple form for the diffusion coefficients: $D_0(u) = u\Gamma_1(1)/4$ and $D_1(u) = \Gamma_1(1)/4$. To simplify the analysis, the spectral density of g_1 is chosen so that (see (4.10)) $\Gamma_1(1) = 4$. This choice gives

$$(5.1) \quad D_0(u) = u \quad \text{and} \quad D_1(u) = 1.$$

By neglecting the radiation losses, we get the leading order solution $U_{i,0} \equiv \varphi_i$ (see Appendix B). The leading order eigenvalue is $\lambda_{i,0} = -j_{0,i}^2/4$, and we show in the appendix that the integral (4.45) can be evaluated analytically to yield $\lambda_{i,1} = -\lambda_{i,0}$. However, an analytical form for (4.44) could not be found and so numerical integration has to be performed. To summarize, the eigenvector solution of the original eigenvalue problem (3.10) is, to first order,

$$(5.2) \quad \mathbf{U}^{(i)} = \mathbf{V}_0^{(i)} + \varepsilon \sum_{k=1}^{\infty} v_{i,k} \mathbf{V}_0^{(k)} + \mathcal{O}(\varepsilon^2),$$

TABLE 5.1
Evolution of the first eigenvalue.

V	Discrete system	Continuous model
10	-1.317801	-1.301216
100	-1.431525	-1.431338
500	-1.442912	-1.442904
2500	-1.445218	-1.445218

TABLE 5.2
Evolution of the second eigenvalue.

V	Discrete system	Continuous model
10	-7.067015	-6.856034
100	-7.544146	-7.541637
500	-7.602682	-7.602579
2500	-7.614772	-7.614768

where $(\mathbf{V}_0^{(i)})_\nu = (\varphi_i(\nu/\beta_c))$. The off-diagonal terms are given explicitly by

$$(5.3) \quad v_{i,k} = \frac{1 - \frac{j_{0,k}}{j_{0,i}} \int_0^1 u^{-1} J_1(\sqrt{u}j_{0,i}) J_1(\sqrt{u}j_{0,k}) du}{\left(\frac{j_{0,k}^2}{j_{0,i}^2} - 1\right) |J_1(j_{0,i}) J_1(j_{0,k})|}$$

and $v_{i,i} = -1/(2J_1^2(j_{0,i}))$. The eigenvalues are, from (4.23), found to be

$$(5.4) \quad \lambda_i = -\frac{j_{0,i}^2}{4}(1 - \varepsilon) + \mathcal{O}(\varepsilon^2).$$

In this example, the associated original discrete system (3.10) may be written explicitly as

$$(5.5) \quad (\nu + 1)(U_{\nu+1}^{(i)} - U_\nu^{(i)}) + \nu(U_{\nu-1}^{(i)} - U_\nu^{(i)}) = \lambda_i U_\nu^{(i)}.$$

So, mode coupling occurs only between adjacent modes, and the description of the power coupling process in terms of a diffusion equation is clearly validated since (5.5) is nothing but the finite difference discretization of (B.1).

Tables 5.1 and 5.2 display the values of the first two eigenvalues calculated from the original discrete system (5.5) and using the first order approximation (5.4). The number of digits of accuracy given by the continuous model is in agreement with the expected V^{-2} law; recall that the above expressions are correct to $\mathcal{O}(\varepsilon^2)$, $\varepsilon \rightarrow 0$, which is equivalent to $\mathcal{O}(V^{-2})$, $V \rightarrow \infty$. This is clearly confirmed in Figure 5.1, where the evolution of the quadratic errors (in percentages) for the leading order solutions

$$E_0^{(i)} = 100 \times \|\mathbf{U}^{(i)} - \mathbf{V}_0^{(i)}\|_2$$

and for the first order solutions

$$E_1^{(i)} = 100 \times \|\mathbf{U}^{(i)} - \mathbf{V}_0^{(i)} - \varepsilon \mathbf{V}_1^{(i)}\|_2$$

are plotted against the waveguide parameter V . Note that the first order correction vector $\mathbf{V}_1^{(i)}$ is computed with only the first 30 terms in the infinite sum in (5.2).

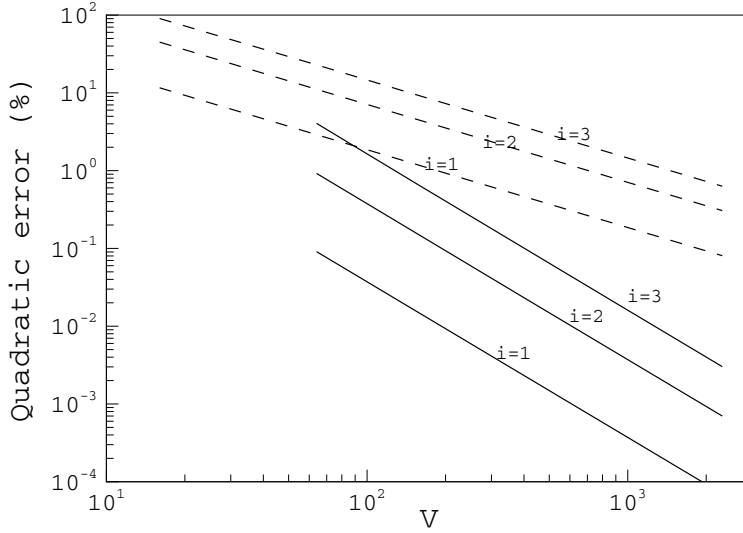


FIG. 5.1. Error behavior with respect to V for the first three eigenmodes; dashed lines denote leading order solution, and straight lines denote first order solution.

5.2. General case: The Rayleigh–Ritz method. We consider a random perturbation satisfying (4.33). Regular solutions of the Sturm–Liouville eigenvalue problem (4.25) can be numerically recovered using the Rayleigh–Ritz method [23]. For the sake of notational simplicity, we define the symmetric positive bilinear form

$$(5.6) \quad \mathcal{A}(\varphi, \varphi') = \int_0^1 \left(D_0(u) \frac{d\varphi}{du} \frac{d\varphi'}{du} + \gamma_0(u) \varphi \varphi' \right) du.$$

Solutions of (4.25) are stationary points of the energy functional $\mathcal{E}[U_{i,0}] = \mathcal{A}(U_{i,0}, U_{i,0})$ subject to the normalization constraint $\|U_{i,0}\| = \beta_c^{-\frac{1}{2}}$. The set of functions $\{\varphi_k\}_{k=1}^\infty$ forms a complete orthogonal system on $L^2([0, 1])$ satisfying the Dirichlet boundary condition at the endpoint $u = 1$ (see Appendix B) and can therefore serve as a natural basis for an approximate solution $U_{i,0}^K$; i.e., we consider the truncated generalized Fourier expansion

$$(5.7) \quad U_{i,0}^K(u) = \sum_{k=1}^K c_{i,k}^K \varphi_k(u).$$

Following standard variational techniques, the stationary points are reached at the approximate eigenvalue $|\lambda_{i,0}^K| = \beta_c \mathcal{E}[U_{i,0}^K]$, where the expansion coefficients $c_{i,k}^K$ satisfy the matrix eigenvalue problem,

$$(5.8) \quad \sum_{k=1}^K c_{i,k}^K \mathcal{A}(\varphi_k, \varphi_l) = -\lambda_{i,0}^K c_{i,l}^K, \quad l = 1, 2, \dots, K.$$

The symmetry of the bilinear form implies that the basis set $(U_{i,0}^K)_{i=1, \dots, K}$ is orthogonal in $L_2([0, 1])$. Furthermore, the true solution is recovered by taking the limit

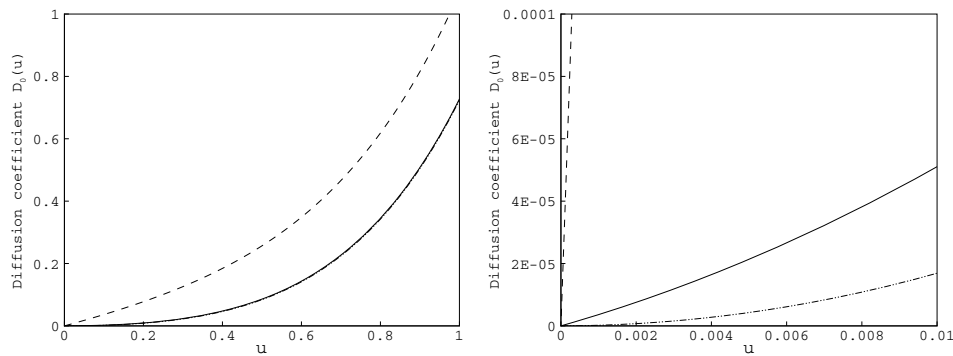


FIG. 5.2. Leading order diffusion coefficient $D_0(u)$. Dashed line: $\tau = 1$; continuous line: $\tau = 0.01$; dash-dot-dot line: $\tau = 0.0001$.

$U_{i,0} = \lim_{K \rightarrow \infty} U_{i,0}^K$. Similarly, $\lambda_{i,0} = \lim_{K \rightarrow \infty} \lambda_{i,0}^K$ and from the Rayleigh–Ritz principle, approximate eigenvalues $|\lambda_{i,0}^K|$ are upper bounds for the true eigenvalues $|\lambda_{i,0}|$ of the infinite-dimensional problem. The convergence of the method depends upon the properties of the perturbation such as its shape and its power spectrum. In most cases of practical interest, lowest order eigenvalues are expected to be obtained at a modest computational price (say $K \leq 100$). For the sake of illustration, we consider perturbations given by the general form

$$(5.9) \quad g(\bar{x}, z) = \tau g_1(z)\bar{x} + \sum_{q=2}^5 g_q(z)\bar{x}^q.$$

In the current analysis, the random functions g_q are assumed to be statistically identical and satisfy the Gaussian distribution: $\langle g_q(0)g_q(z) \rangle = e^{-z^2}$, $q = 1, \dots, 5$, which gives $\Gamma_q(\zeta) = \sqrt{\pi}e^{-\zeta^2/4}$. The corresponding diffusion coefficient is the polynomial of degree 5,

$$(5.10) \quad D_0(u) = \tau \frac{\sqrt{\pi}e^{-1/4}}{4} u + \sum_{n=2}^5 d_n u^n.$$

The other coefficients are given explicitly in (4.30). In Figure 5.2 are plotted two graphs of D_0 against u for three values of τ . The tiny difference between the two curves $\tau = 0.01$ and $\tau = 0.0001$ can be identified on the magnified figure on the right. When $\tau = 0$, criterion (4.33) is not satisfied and there is no continuous counterpart to the discrete eigenmode. Thus, the coefficient τ can be interpreted as a diffusion parameter and the modal distribution in the waveguide is expected to “lose its regularity” when $\tau \rightarrow 0$. This behavior is revealed in Figures 5.3–5.4, where the first and fifth eigenmodes obtained from the discrete system (3.10) and from the continuous model (4.25) are shown. In all cases illustrated we considered $\beta_c = V/2 = 500$ guided modes, and the eigenfunctions $U_{i,0}^K$ are computed with $K = 50$ basis functions. From (4.44), it can be shown that the amplitude of the first order correction terms $v_{i,k}$ will grow as $\tau \rightarrow 0$. Thus the leading order solution $U_{i,0}$ will be a good approximation only if the number of guided modes is sufficiently large so that $\varepsilon \|U_{i,1}\| \ll \|U_{i,0}\|$. This explains the discrepancy observed when $\tau = 0.0001$.

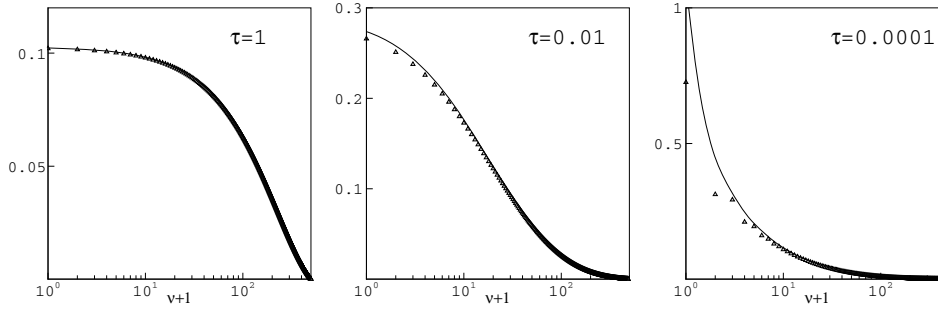


FIG. 5.3. Influence of the diffusion parameter τ on the “regularity” of the first eigenmode; unbroken line denotes the continuous model $U_{1,0}$, and triangles denote the discrete eigenmode $\mathbf{U}^{(1)}$. Number of guided modes: 500.

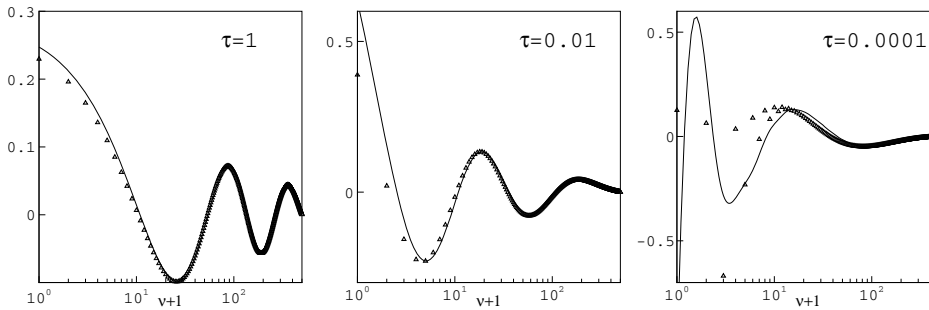


FIG. 5.4. Influence of the diffusion parameter τ on the “regularity” of the fifth eigenmode; unbroken line denotes the continuous model $U_{5,0}$, and triangles denote the discrete eigenmode $\mathbf{U}^{(5)}$. Number of guided modes: 500.

TABLE 5.3

Long distance power distribution. Fraction of energy carried by the lowest modes ($\beta_c = 500$).

τ	$\nu = 0, 1, \dots, 10$	$\nu = 0, 1, \dots, 100$
1	33%	82%
0.01	70%	98%
0.0001	95%	99.8%

The diffusion parameter has noticeable consequences on the modal distribution of $\mathbf{U}^{(1)}$ and therefore the long-distance power distribution

$$(5.11) \quad \mathbf{A}(z) \approx \mathbf{U}^{(1)} \exp(\epsilon^2 \lambda_1 z) \left[\mathbf{U}^{(1)} \right]^T \mathbf{A}(0) \quad \text{as } z \rightarrow \infty.$$

This is clearly illustrated in Figure 5.3, where τ has a significant effect on the modal distribution. Since the fundamental mode $U_{1,0}$ is a positive function in $[0, 1]$ and $\lambda_{1,0}U_{1,0}(0) = \tau(dU_{1,0}/du)_{u=0}$ (see Appendix A), the optical power is likely to be concentrated among the lowest order modes as $\tau \rightarrow 0$. This is confirmed in Table 5.3, where the fraction of energy carried by the lowest modes is shown for various values of τ .

6. Conclusion. In this paper we have analyzed the evolution of the modal power distribution of the transverse electric field as it propagates along a multimode slab

waveguide with quadratic refractive index profile and with small random deformations. We showed that for waveguides supporting a sufficiently large number of guided modes, the mode coupling mechanism can be ideally described as a diffusion equation. Even for a moderate number of modes, the regular expansion method allows us to obtain very accurate solutions when first order correction terms are taken into account. In practice, the technique described herein provides excellent qualitative predictions for the long-distance modal distribution within fixed computational resources regardless of the number of modes. Furthermore, we were able to identify nearly nondiffusive regimes in which the modal power distribution is not the solution of a diffusion equation and exhibits irregular behavior. In these latter scenarios, we observed strong focusing effects of the wave field in the vicinity of the waveguide axis. Work is almost complete on applying the present technique to three-dimensional fibers of circular cross-section with parabolic index profile [15]. We have good reason to believe that the results demonstrated in this paper could be generalized to other graded-index fibers, and this will be the subject of future work.

Appendix A. Orthogonality of the first order eigenvectors. The purpose of this appendix is to prove that the eigenmodes of the discrete system, taken to first order in ε , are orthogonal to $\mathcal{O}(\varepsilon^2)$. We commence by integrating by parts (4.44). This yields

$$(A.1) \quad v_{i,j} + v_{j,i} = -\frac{\beta_c D_1(0)}{\lambda_{i,0} - \lambda_{j,0}} \left(\frac{dU_{i,0}}{du} U_{j,0} - \frac{dU_{j,0}}{du} U_{i,0} \right)_{u=0}, \quad i \neq j.$$

Moreover, the eigenvalue equation (4.25), together with $\gamma_0(0) = 0$ and $D_0(0) = 0$, implies

$$(A.2) \quad \lambda_{i,0} U_{i,0}(0) = \left(\frac{dD_0}{du} \frac{dU_{i,0}}{du} \right) \Big|_{u=0},$$

and by virtue of (4.20) and (4.21), it can be shown that $W_\zeta^1(0) = (dW_\zeta^0/du)_{u=0}$. Thus, from (4.27),

$$(A.3) \quad D_1(0) = \frac{dD_0}{du}(0),$$

and so we can construct the identity

$$(A.4) \quad (\lambda_{i,0} - \lambda_{j,0})(U_{i,0}U_{j,0})_{u=0} = D_1(0) \left(\frac{dU_{i,0}}{du} U_{j,0} - \frac{dU_{j,0}}{du} U_{i,0} \right)_{u=0}.$$

Therefore, from (A.1) and (A.4) we arrive at the result

$$(A.5) \quad v_{i,j} + v_{j,i} = -\beta_c (U_{i,0}U_{j,0})_{u=0}.$$

Finally, using $\beta_c^{-1} = 2\varepsilon$ and applying the composite trapezoidal rule yields a relationship between the discrete and continuous eigenmode products, namely,

$$(A.6) \quad \begin{aligned} & (\mathbf{V}_0^{(i)} + \varepsilon \mathbf{V}_1^{(i)}) \cdot (\mathbf{V}_0^{(j)} + \varepsilon \mathbf{V}_1^{(j)}) = \frac{1}{2} (U_{i,0}U_{j,0})_{u=0} + \mathcal{O}(\varepsilon^2) \\ & + \int_0^{\beta_c} (U_{i,0}(\tilde{\nu}/\beta_c) + \varepsilon U_{i,1}(\tilde{\nu}/\beta_c))(U_{j,0}(\tilde{\nu}/\beta_c) + \varepsilon U_{j,1}(\tilde{\nu}/\beta_c)) d\tilde{\nu}. \end{aligned}$$

This reduces to

$$(A.7) \quad (\mathbf{V}_0^{(i)} + \varepsilon \mathbf{V}_1^{(i)}) \cdot (\mathbf{V}_0^{(j)} + \varepsilon \mathbf{V}_1^{(j)}) = \varepsilon(v_{i,j} + v_{j,i}) + \frac{1}{2}(U_{i,0}U_{j,0})_{u=0} + \mathcal{O}(\varepsilon^2) = \mathcal{O}(\varepsilon^2)$$

by virtue of (A.5) and completes the exercise.

Appendix B. Microbending solution. This appendix proves completeness of the regular orthogonal eigenfunctions satisfying

$$(B.1) \quad \frac{d}{du} \left(u \frac{d\varphi}{du} \right) = \lambda\varphi, \quad \varphi(1) = 0,$$

together with the normalization condition (4.42), $\|\varphi_i\| = \beta_c^{-\frac{1}{2}}$. We find that (see section 4.3) these functions are

$$(B.2) \quad \varphi_i(u) = \beta_c^{-\frac{1}{2}} \frac{J_0(j_{0,i}\sqrt{u})}{|J_1(j_{0,i})|},$$

where J_n is the Bessel function of the first kind of order n , $j_{0,i}$ is the location of the i th zero of J_0 , and each $\varphi_i(u)$ has the associated eigenvalue $\lambda_{i,0} = -j_{0,i}^2/4$.

B.1. Completeness. By construction, the set of eigenfunctions $\{\varphi_i\}_{i=1}^\infty$ defines an orthogonal system on $L^2([0, 1])$. The system is complete if the Dalzell-type criterion [24] is satisfied, i.e.,

$$(B.3) \quad S = 2\beta_c \sum_{i=1}^\infty \int_0^1 \left| \int_0^t \varphi_i(u) du \right|^2 dt = 1.$$

A straightforward calculation yields

$$(B.4) \quad S = \sum_{i=1}^\infty \frac{8}{3j_{0,i}^2} \left(1 + \frac{4}{j_{0,i}^2} \right).$$

Now, let $\alpha \geq 2$ be an integer; then the Cauchy residue theorem gives the identity

$$(B.5) \quad I_{N,\alpha} = \oint_{C_N} \frac{d \ln J_0(z)}{dz} \frac{dz}{z^\alpha} = 4\pi i \sum_{i=1}^N \frac{1}{j_{0,i}^\alpha} + 2\pi i \operatorname{Res} \left\{ \frac{d \ln J_0(z)}{z^\alpha dz}; z = 0 \right\},$$

where the closed contour C_N is the circle centered at the origin with radius R_N chosen such that $j_{0,N} < R_N < j_{0,N+1}$. Since $\lim_{N \rightarrow \infty} I_{N,\alpha} = 0$, we get, setting respectively $\alpha = 2$ and $\alpha = 4$,

$$(B.6) \quad \sum_{i=1}^\infty \frac{1}{j_{0,i}^2} = \frac{1}{4} \quad \text{and} \quad \sum_{i=1}^\infty \frac{1}{j_{0,i}^4} = \frac{1}{32}.$$

Substitution of these sums into (B.4) completes the result.

B.2. First order eigenvalues. The first order eigenvalues are specified by (4.45); we can evaluate them as follows. First, multiply (B.1) by $d\varphi/du$ and integrate over 0 to 1 to give

$$(B.7) \quad \int_0^1 \frac{d\varphi}{du} \frac{d}{du} \left(u \frac{d\varphi}{du} \right) du = \frac{\lambda}{2} \int_0^1 \frac{d\varphi^2}{du} du.$$

Integrating by parts twice then yields

$$(B.8) \quad \frac{1}{2} \left(\frac{d\varphi}{du} \right)_{u=1}^2 + \frac{1}{2} \int_0^1 \left(\frac{d\varphi}{du} \right)^2 du = -\frac{\lambda}{2} \varphi^2(0).$$

Integrating by parts again and using the equality $(d\varphi/du)_{u=0} = \lambda\varphi(0)$ gives finally

$$(B.9) \quad \int_0^1 \varphi \frac{d^2\varphi}{du^2} du = \left(\frac{d\varphi}{du} \right)_{u=1}^2 = -\frac{\lambda}{\beta_c}.$$

Substituting this into (4.45), using (A.3) and setting $\gamma_1 = 0$ (lossless case), yields

$$(B.10) \quad \lambda_{i,1} = -\lambda_{i,0}.$$

Acknowledgments. The authors are most grateful to Professor John Love (ANU, Australia), Dr. David Allwright and Dr. John Ockendon (OCIAM, University of Oxford, UK), and Dr. Dominic Gallagher and Dr. Tom Felici (Photon Design) for their substantial input to the project.

REFERENCES

- [1] R. OLSHANSKY, *Propagation in glass optical waveguides*, Rev. Modern Phys., 51 (1979), pp. 341–367.
- [2] A. F. GARITO, J. WANG, AND R. GAO, *Effects of random perturbations in plastic optical fibers*, Science, 281 (1998), pp. 962–967.
- [3] S. E. GOLOWICH, W. WHITE, W. A. REED, AND E. KNUDSEN, *Quantitative estimates of mode coupling and differential modal attenuation in perfluorinated graded-index plastic optical fiber*, J. Lightwave Tech., 21 (2003), pp. 111–121.
- [4] J. GARNIER, *Light propagation in square law media with random imperfections*, Wave Motion, 31 (2000), pp. 1–19.
- [5] D. MARCUSE, *Mode conversion caused by surface imperfections of a dielectric slab waveguide*, in Integrated Optics, IEEE Press, New York, 1972.
- [6] D. MARCUSE, *Radiation losses of dielectric waveguides in terms of the power spectrum of the wall distortion function*, in Integrated Optics, IEEE Press, New York, 1972.
- [7] H. E. ROWE AND D. T. YOUNG, *Transmission distortion in random multimode waveguides*, IEEE Trans. Microwave Theory, MTT-20 (1972), pp. 349–365.
- [8] D. MARCUSE, *Derivation of coupled power equations*, Bell System Tech. J., 51 (1972), pp. 229–237.
- [9] D. MARCUSE, *Theory of Dielectric Optical Waveguides*, Academic Press, New York, 1991.
- [10] D. GLOGE, *Optical power flow in multimode fibers*, Bell System Tech. J., 51 (1972), pp. 1767–1783.
- [11] D. MARCUSE, *Losses and impulse response of a parabolic index fiber with random bends*, Bell System Tech. J., 52 (1973), pp. 1423–1427.
- [12] R. OLSHANSKY, *Mode coupling effects in graded-index optical fibers*, Appl. Optics, 14 (1975), pp. 935–945.
- [13] J. ZUBIA, G. DURANA, G. ADABALDETRERU, J. ARRUE, M. A. LOSADA, AND M. LOPEZ-HIGUERA, *A new method to calculate mode conversion coefficients in SI multimode optical fibers*, J. Lightwave Tech., 16 (1998), pp. 1195–1202.
- [14] A. DJORDJEVITCH AND S. SAVOVIĆ, *Numerical solution of the power flow equation in step-index plastic optical fibers*, J. Opt. Soc. Amer. B, 21 (2004), pp. 1437–1442.
- [15] E. PERREY-DEBAIN AND I. D. ABRAHAMS, *A continuous model for mode mixing in graded-index multimode fibres with random imperfections*, Proc. Roy. Soc. A, submitted.
- [16] E. PERREY-DEBAIN AND I. D. ABRAHAMS, *A diffusion analysis approach for multimode random optical waveguides*, in Proceedings of the 7th International Conference on Mathematical and Numerical Aspects of Waves (WAVES 2005), Brown University, CT, 2005, pp. 267–269.

- [17] A. W. SNYDER AND J. D. LOVE, *Optical Waveguide Theory*, Chapman and Hall, London, 1983.
- [18] L. D. LANDAU AND E. M. LIFSHITZ, *Quantum Mechanics (Non-Relativistic Theory)*, 3rd ed., Pergamon Press, Oxford, 1977.
- [19] M. SKOROBOGATY, S. G. JOHNSON, S. A. JACOBS, AND Y. FINK, *Dielectric profile variations in high-index-contrast waveguides, coupled mode theory, and perturbation expansions*, Phys. Rev. E (3), 67 (2003), article 046613.
- [20] J. GARNIER, *Energy distribution of the quantum harmonic oscillator under random time-dependent perturbation*, Phys. Rev. E (3), 60 (1999), pp. 3676–3687.
- [21] L. B. DOZIER AND F. D. TAPPERT, *Statistics of normal mode amplitudes in a random ocean, I. Theory*, J. Acoust. Soc. Amer., 63 (1978), pp. 353–365.
- [22] E. PERREY-DEBAIN AND I. D. ABRAHAMS, *A band factorization technique for transition matrix element asymptotics*, Comput. Phys. Comm., 175 (2006), pp. 315–322.
- [23] J. D. PRYCE, *Numerical Solution of Sturm-Liouville Problems*, Clarendon Press, Oxford, 1993.
- [24] J. R. HIGGINS, *Completeness and Basis Properties of Sets of Special Functions*, Cambridge University Press, Cambridge, UK, 2004.