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Diffraction of flexural waves by cracks in orthotropic thin elastic plates. I Formal solution

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The problem of flexural wave diffraction by a semi-infinite crack in an infinite orthotropic thin plate is considered. Such models have application to the ultrasonic non-destructive inspection of thin components, such as aeroplane wings. For simplicity, the plate is modelled using Kirchhoff theory, and the crack is chosen to be aligned along one of the principal directions of material orthotropy. For incident plane waves, an exact analytical expression for the scattered field is derived by means of the Wiener–Hopf technique. In this model problem, the Wiener–Hopf kernel is scalar and its factorization is expressed in terms of simple, definite, non-singular contour integrals. A detailed numerical evaluation of the solution will be provided in the second part of this work.

Keywords: orthotropic plate; thin elastic plate; diffraction; scattering; crack; Wiener–Hopf technique

1. Introduction

There is considerable interest in the ultrasonic non-destructive evaluation of thin elastic components, such as metal panels used in the fabrication of aircraft fuselages and wings or in submarine hulls. Such measurement requires an understanding of the way in which cracks, voids, aperities or other inclusions scatter the incoming ultrasonic elastic wave radiation. Even very simple models, for example, an infinite thin plate containing a semi-infinite straight crack or a single spherical inclusion, are difficult to solve exactly (see e.g. Norris & Wang 1994). In recent years, an increasing number of engineering applications has started to employ composite materials in the fabrication process, because they tend to offer superior characteristics, such as increased strength or rigidity, over traditional materials. This, unfortunately, means that such bodies are typically anisotropic in nature, thereby making an analysis of their propagation properties significantly more difficult. A detailed account of the mechanics and theory of such materials is given, for example, by Jones (1975). Often, an anisotropic thin plate will exhibit two orthogonal lines of symmetry (e.g. for fibre reinforced

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materials), in which case it can be classified as orthotropic. Most fibre-reinforced laminae fall into this category. The lines of symmetry are known as *principal directions*, and if the spatial axes are aligned with these, then the system is said to be *specially orthotropic*.

In this, the first of a two-part paper, we obtain a formal solution to the problem of diffraction by a semi-infinite crack in a specially orthotropic thin plate. Our axes are determined by the orientation of the crack, which must lie along one of the principal directions. Essentially, this is a generalization of the work of Norris & Wang (1994), who examined flexural wave diffraction from semi-infinite cracks and rigid strips in isotropic Kirchhoff plates. Here, we will be using more widely applicable versions of the governing equation and constitutive relations (Timoshenko 1940) than those employed by Norris & Wang, and so the model requires substantially more effort in effecting its solution. The method of solution will be that devised by Wiener & Hopf (1931), a technique that has proved highly successful in solving a great variety of wave diffraction problems, in diverse fields such as acoustics, elasticity, electromagnetics and geophysics (Noble 1988).

The solution procedure consists of introducing an incident flexural wave on the plate, and rewriting, after suitable non-dimensionalization, the boundary-value problem in terms of a scattered bending wave term (§2). This section also indicates the behaviour of the field local to the crack tip, and decomposes the scattered field into symmetric and antisymmetric parts. Fourier transforms are then introduced in §3 and applied to the governing equation and boundary conditions. In the following section (§4), these are then reduced to a pair of scalar Wiener–Hopf equations corresponding to symmetric and antisymmetric bending wave solutions. A Wiener–Hopf equation is a functional equation defined in a strip of the complex plane involving two complex functions, one analytic in the region above and including the strip, and the other analytic in the region below and including the strip. These functions are otherwise unknown, except for their behaviour at infinity corresponding to contraints on the bending displacement and moment at the crack tip (§2*c*). Rearrangement of the Wiener–Hopf equation so that each side is analytic in an overlapping half-plane allows analytic continuation to be used to solve for both the unknowns. The key step in this rearrangement process is to factorize a specific function, the Wiener–Hopf kernel, into a product of two terms, one analytic in the upper region and the other analytic in the overlapping lower region. This step, which is of considerable complexity in the present problem, is discussed in detail in §4*c*. Once the Wiener–Hopf solutions are obtained, Fourier inverses are invoked to write down the formal solution of the symmetric and antisymmetric parts of the flexural wave field (§4*d*,*e*). We verify that the combined solution satisfies the governing equations and crack edge conditions in §5*a*.

Note that in this article we employ a version of the Wiener–Hopf technique similar to that used by Abrahams (1997), and adhere where possible to the plate theory notation defined by Timoshenko (1940). This means that the now commonplace convention of using lower case letters to represent physical functions, and corresponding capitals for their Fourier transforms, unfortunately cannot be used. Instead, all of our commonly used notation conventions are described in detail at the beginning of §3. A major advantage of our chosen technique, as opposed to the ‘classical’ method employed by

Noble (1988), is that the necessity of adding a damping term to the incident wave field is removed, leading to a simpler representation in Fourier space, with many singularities occupying positions on the real line. Since the fourth-order nature of the problem leads to the appearance of unusually complicated multivalued functions, this simplification is of some importance. At the outset, we assume convergence of Fourier integrals involving certain unknown functions, and confirm these assumptions when the solution is obtained. In simple problems, a WKJB method (Nayfeh 2000) is sometimes employed in order to confirm the existence of Fourier representations for physical quantities a priori (Billingham & King 2000). This is not practical here, since obtaining the appropriate asymptotic expansions would be arduous at best. The Wiener–Hopf kernel is scalar, and its factorization is expressed in terms of definite, non-singular contour integrals. The final solution embodies all of the features possessed by the result of Norris & Wang (1994), but also has certain unique characteristics that arise from the greater complexity of the orthotropic governing equation and constitutive relations. This article will conclude in §5*b* with a few remarks on this point. A full and detailed analysis and discussion of this solution is, in itself, a complicated exercise, and so will be the focus for the second part of this work.

2. The boundary-value problem

(*a*) *Governing equation and incoming wave*

Consider an orthotropic Kirchhoff thin plate of infinite extent lying in the (x, y)-plane, where (x, y, z) are orthogonal Cartesian coordinates. The governing equation for transverse flexural displacements W is given (e.g. Timoshenko 1940) by

$$D_x \frac{\partial^4 W}{\partial x^4} + 2(D_1 + 2D_{xy}) \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 W}{\partial y^4} + \rho h \frac{\partial^2 W}{\partial t^2} = 0, \tag{2.1}$$

where the coordinates are aligned with the principal axes of orthotropy of the material. Here, ρ and h are the plate density and thickness, respectively, and D_x, D_y, D_1 and D_{xy} are bending stiffnesses that describe the particular composite material (Timoshenko 1940). These obey the following inequalities (Norris 1994)

$$D_x > 0, \quad D_y > 0, \quad D_{xy} > 0, \quad D_x D_y > D_1^2, \tag{2.2}$$

and since the combination occurs frequently, we write

$$H = D_1 + 2D_{xy}. \tag{2.3}$$

Although the inequalities (2.2) permit non-positive values of H, these correspond to materials with negative Poisson ratios and therefore tend to be unimportant for ordinary engineering applications. Consequently, for ease of exposition we restrict our attention to cases in which

$$H > 0, \tag{2.4}$$

since these can be solved concurrently.

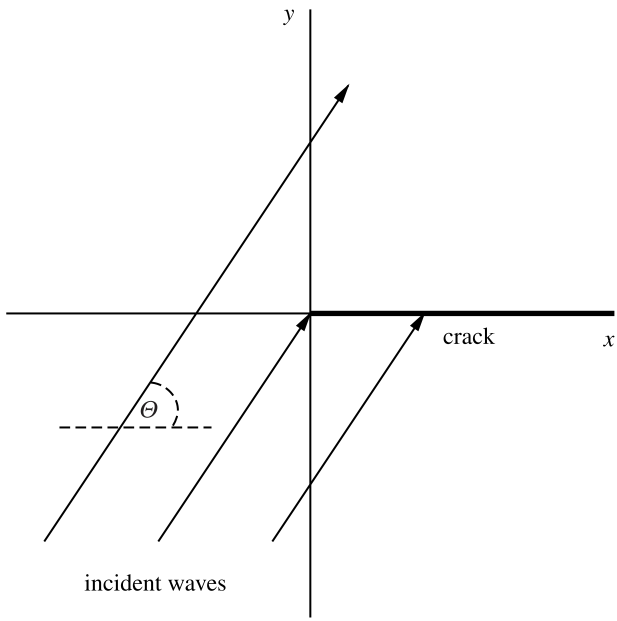

Figure 1. The thin elastic plate with flexural waves incident at angle Θ and with a semi-infinite crack along $x \geq 0$, $y=0$.

Now, suppose that flexural waves of unit magnitude, angular frequency ω and form

$$W^{\text{inc}} = \text{Re}\left[e^{ik(x\cos\Theta + y\sin\Theta)}e^{-i\omega t}\right] \tag{2.5}$$

are incident on an infinitely thin crack which lies along the half-line $x \geq 0$, $y=0$ (see figure 1). Here, $\text{Re}[Z]$ indicates the real part of the complex quantity Z. The incidence angle Θ is chosen to lie in the interval $[0,\pi)$, without loss of generality, since solutions for $\Theta \in (-\pi, 0)$ can be determined by symmetry. We seek an expression for the scattered field W, which is isolated by writing

$$W = W^{\text{t}} - W^{\text{inc}}, \tag{2.6}$$

in which the superscript 't' refers to 'total'. In view of this definition, it is clear that W must consist entirely of waves that are either outgoing or decaying away from the crack tip as $\sqrt{x^2+y^2} \to \infty$. It is convenient, for ease of presentation, to now introduce non-dimensional spatial and temporal coordinates, based on the length-scale of flexural waves propagating in the x direction, and the angular frequency ω, respectively. Thus, we define

$$x_* = \left(\frac{\rho h \omega^2}{D_x}\right)^{1/4} x, \quad y_* = \left(\frac{\rho h \omega^2}{D_x}\right)^{1/4} y, \quad t_* = \omega t,$$

and in addition, we suppress the time harmonic factor by writing

$$W^{\text{inc}} = \text{Re}[W_*^{\text{inc}}\, e^{-it_*}],$$

with similar notation applying to the scattered and total fields. Using these in equation (2.1), we obtain the scaled governing equation

$$\mathcal{G} W_* = 0, \tag{2.7}$$

in which

$$\mathcal{G} = \left[D_x\left(\frac{\partial^4}{\partial x_*^4} - 1\right) + 2H\frac{\partial^4}{\partial x_*^2 \partial y_*^2} + D_y\frac{\partial^4}{\partial y_*^4}\right]. \tag{2.8}$$

In terms of the new variables, the incident field becomes

$$W_*^{\text{inc}}(x_*, y_*) = \mathrm{e}^{\mathrm{i}k_*(x_* \cos \Theta + y_* \sin \Theta)}, \tag{2.9}$$

where the non-dimensional wavenumber k_* is obtained by substitution into equation (2.7); thus,

$$k_* = \left(\frac{D_x}{D_x \cos^4 \Theta + 2H \cos^2 \Theta \sin^2 \Theta + D_y \sin^4 \Theta}\right)^{1/4}. \tag{2.10}$$

Note that k_* depends on the angle of incidence Θ, as we should expect for waves in anisotropic media (Lighthill 2002). In subsequent sections, we shall always use the scaled variables, omitting the asterisk, and once a solution has been determined we can reintroduce the exponential time factor and then take the real part to obtain the scattered flexural displacement.

Before proceeding to discuss the boundary conditions on the crack faces, there are several points worth mentioning regarding the scaling of the variables x_*, y_* given above. The reader may observe that a natural alternative way of non-dimensionalizing the governing equation would be to define

$$y_* = \left(\frac{\rho h \omega^2}{D_y}\right)^{1/4} y,$$

with x scaled as above, which would yield the modified form to equation (2.7)

$$\left[\frac{\partial^4}{\partial x_*^4} - 1 + \frac{2H}{\sqrt{D_x D_y}}\frac{\partial^4}{\partial x_*^2 \partial y_*^2} + \frac{\partial^4}{\partial y_*^4}\right] W_* = 0. \tag{2.11}$$

This has the immediate effect of reducing the number of parameters to one in the governing equation, and for the incident wave equation (2.9) the wavenumber becomes

$$k_* = \left(\cos^4 \Theta + \left(2H/\sqrt{D_x D_y}\right)\cos^2 \Theta \sin^2 \Theta + \sin^4 \Theta\right)^{-1/4}. \tag{2.12}$$

Hence, in this scaled coordinate system, a material for which $H = \sqrt{D_x D_y}$ would yield $k_* = 1$, i.e. it appears isotropic in nature. However, this coincidence of parameter values is unlikely to occur in practice, and when $H \neq \sqrt{D_x D_y}$ no advantage is gained by employing this (anisotropic) form of k_* in equation (2.12) over that chosen in equation (2.10). Furthermore, the ratio D_x/D_y, although scaled from the governing equation (2.11), would appear as a new parameter in the crack edge conditions. Thus, the benefits of remaining in a coordinate system in which Θ is the physical wave angle, rather than a distorted azimuthal coordinate, outweigh any advantages of the alternative scaling. An interesting point regarding the biharmonic part of the operator (2.11)

$$\frac{\partial^4}{\partial x_*^4} + \frac{2H}{\sqrt{D_x D_y}}\frac{\partial^4}{\partial x_*^2 \partial y_*^2} + \frac{\partial^4}{\partial y_*^4} \tag{2.13}$$

is that for $H \le \sqrt{D_x D_y}$ it possesses the real factors

$$\frac{\partial^2}{\partial x_*^2} \pm \left(2 - \frac{2H}{\sqrt{D_x D_y}}\right)^{1/2} \frac{\partial^2}{\partial x_* \partial y_*} + \frac{\partial^2}{\partial y_*^2}, \tag{2.14}$$

whereas for $H > \sqrt{D_x D_y}$, these factors are

$$\left(\frac{H}{\sqrt{D_x D_y}} \pm \sqrt{\frac{H^2}{D_x D_y} - 1}\right) \frac{\partial^2}{\partial x_*^2} + \frac{\partial^2}{\partial y_*^2}. \tag{2.15}$$

This might indicate that the nature of the solution of the boundary-value problem changes fundamentally as H passes through the value $\sqrt{D_x D_y}$, especially as the principal axes for the operators in the former case (2.14) are now not aligned with the x and y axes. As will be shown in §3, this change does indeed result in a change in the singularity structure in an associated (Fourier transformed) complex plane; however, this does not result in any significant alteration of the scattered field in the medium. Furthermore, the authors have found no useful physical relationship between the features of the solution of the boundary-value problem and the nature of the second-order differential operators (2.14) and (2.15).

(*b*) *Boundary conditions on the crack faces*

A free edge on a Kirchhoff plate requires two boundary, or edge, conditions. The first is the vanishing of the bending moment M_y and the second requires that V_y, a combination of the twisting moment and vertical shear force (Graff 1991), is also zero along the edge. These are given in terms of displacements in the orthotropic case by Timoshenko (1940). We introduce a non-dimensionalizing factor of $1/D_x$, and dispense with the subscript y, since we need only determine edge conditions along $y=0$. Thus, we define differential operators

$$\mathcal{M} = -\frac{1}{D_x}\left[D_1 \frac{\partial^2}{\partial x^2} + D_y \frac{\partial^2}{\partial y^2}\right] \tag{2.16}$$

and

$$\mathcal{V} = -\frac{1}{D_x}\left[(D_1 + 4D_{xy}) \frac{\partial^2}{\partial x^2} + D_y \frac{\partial^2}{\partial y^2}\right], \tag{2.17}$$

and introduce

$$Y(x, y) = \frac{\partial W}{\partial y}(x, y), \tag{2.18}$$

so that the scattered bending moment is given by

$$M = \mathcal{M} W; \tag{2.19}$$

we also have

$$V = \mathcal{V} Y, \tag{2.20}$$

with similar notation applying to the incident and total fields. The Kirchhoff edge conditions for the scattered response are now easily obtained. The first is given by

$$M(x,0) = -M^{\mathrm{inc}}(x,0), \quad x \geq 0, \tag{2.21}$$

and the second by

$$V(x,0) = -V^{\mathrm{inc}}(x,0), \quad x \geq 0. \tag{2.22}$$

These conditions apply on both faces of the crack, i.e. for $y=0^+$ and $y=0^-$.

(*c*) *The tip condition*

In order to obtain unique solutions, we must specify the asymptotic behaviour of the scattered field around the crack tip at the origin. A suitable physical condition is given by Norris & Wang (1994), namely, that the strain energy density S must be integrable in the region of the tip. That is, the integral

$$I = \int_0^{2\pi} \int_0^{\epsilon} Sr \, \mathrm{d}r \, \mathrm{d}\theta,$$

must be finite, where $\epsilon > 0$ is a small positive constant and standard polar coordinates have been used. Thus, for small r, we must have $S \sim F(\theta) r^{\beta}$, where $\beta > -2$ and F is some regular function of θ. Now, the strain energy density in an orthotropic plate (Timoshenko 1940; Norris 1994) is given by

$$S = D_1 \frac{\partial^2 W}{\partial x^2} \frac{\partial^2 W}{\partial y^2} + 2D_{xy} \left(\frac{\partial^2 W}{\partial x \partial y} \right)^2 + \frac{1}{2} \left[D_x \left(\frac{\partial^2 W}{\partial x^2} \right)^2 + D_y \left(\frac{\partial^2 W}{\partial y^2} \right)^2 \right],$$

which is a quadratic form in the second derivatives of W. Therefore, in view of the standard differential transformations

$$\frac{\partial}{\partial x} \equiv \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta}, \qquad \frac{\partial}{\partial y} \equiv \sin\theta \frac{\partial}{\partial r} - \frac{\cos\theta}{r} \frac{\partial}{\partial \theta},$$

it is seen that first derivatives of the displacement W remain finite in the neighbourhood of the origin. Consequently, as $r \to 0$, we have

$$W(r,\theta) \sim G(\theta) r^{\mu} + C \tag{2.23}$$

and

$$Y(r,\theta) \sim [\mu G(\theta) \sin\theta + G'(\theta) \cos\theta] r^{\mu-1}, \tag{2.24}$$

where G is a regular function of the variable θ, C is a constant and

$$\mu \geq 1. \tag{2.25}$$

In particular, equations (2.23) and (2.24) give the leading order behaviour of $W(x,0)$ and $Y(x,0)$ as $x \to 0^+$, which is required later. Note that choosing $\theta=0$ and $\theta=2\pi$ will, in general, lead to differing expressions, although the asymptotic dependence upon x is unaffected by this.

(d) Symmetry

Following Norris & Wang (1994), we exploit the linearity of the equations to split the problem into two parts. Thus, if $f(x, y)$ is a function of the variables x and y, then by defining

$$f^{\mathrm{S}}(x, y) = \tfrac{1}{2}[f(x, y) + f(x, -y)]$$

and

$$f^{\mathrm{A}}(x, y) = \tfrac{1}{2}[f(x, y) - f(x, -y)],$$

we decompose $f(x, y)$ into a sum of two functions, one symmetric ($f^{\mathrm{S}}(x, y)$) about the line $y=0$, and the other antisymmetric ($f^{\mathrm{A}}(x, y)$). Applying this procedure to the incident flexural displacement field (2.9), substituting into equations (2.19) and (2.20) and referring to equations (2.21) and (2.22), we obtain edge conditions on the half line $x\geq 0$, $y=0$, for the scattered response in the symmetric and antisymmetric problems. The total number of conditions for each problem is then increased to three by means of elementary results concerning derivatives of odd and even continuous functions (in y). Thus, along the line $-\infty < x < \infty$, we find that

$$V^{\mathrm{S}}(x, 0) = 0 \tag{2.26S}$$

and

$$M^{\mathrm{A}}(x, 0) = 0. \tag{2.26A}$$

Also, for $-\infty < x < 0$, we obtain

$$Y^{\mathrm{S}}(x, 0) = 0 \tag{2.27S}$$

and

$$W^{\mathrm{A}}(x, 0) = 0. \tag{2.27A}$$

Finally, along the half line $0 \leq x < \infty$, the conditions are

$$M^{\mathrm{S}}(x, 0) = -\frac{k^2}{D_x}(D_1 \cos^2 \Theta + D_y \sin^2 \Theta)\mathrm{e}^{\mathrm{i}kx \cos \Theta} \tag{2.28S}$$

and

$$V^{\mathrm{A}}(x, 0) = -\frac{\mathrm{i}k^3}{D_x}\sin \Theta[(D_1 + 4D_{xy})\cos^2 \Theta + D_y \sin^2 \Theta]\mathrm{e}^{\mathrm{i}kx \cos \Theta}. \tag{2.28A}$$

When no ambiguity can arise, we will omit the superscripts S and A. Also, we need only consider the half plane $y\geq 0$, since the solution for $y<0$ is given immediately by symmetry.

3. Fourier representation

(a) Notation

We now introduce several conventions that serve to clarify our notation in subsequent sections. Firstly, when representing a function $f(x, y)$ as a Fourier integral, we write

$$f(x, y) = \frac{1}{2\pi}\int_{\mathcal{C}} \hat{f}(\alpha, y)\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha,$$

where the contour $\mathcal{C}$ is shown schematically in figure 2. This curve traverses the real line for large $|\alpha|$, and is indented above any singularities on the real line in $\alpha \le \alpha_0$, where

$$\alpha_0 = -k \cos \Theta, \tag{3.1}$$

and below singularities on the real line for $\alpha > \alpha_0$. Note that the restriction (2.4) placed on H ensures that $|\alpha_0| < 1$. The values of several functions at this point turn out to be of some importance, so we shall often write $\hat{f}^{\,0}$ to mean $\hat{f}(\alpha_0)$. The choice of integration path, and the validity of operations carried out on such expressions will be confirmed a posteriori. The regions $\mathcal{D}_+$ and $\mathcal{D}_-$ occupy the upper and lower half planes, as shown in figure 2, and share the common overlap region $\mathcal{D}$ in which runs the contour $\mathcal{C}$. Any function of the complex variable α that is analytic in $\mathcal{D}_+$ shall receive a subscript '+', and a function that is analytic in $\mathcal{D}_-$ will be denoted with a subscript '−'. These generally arise from either a sum or product factorization (Noble 1988); the context will make clear which of these is the case. For a transformed function $\hat{f}(\alpha, y)$, we require only the sum factorization along the line $y=0$, so we use the compact notation

$$\hat{f}_+ + \hat{f}_- = \hat{f}(\alpha, 0).$$

Indeed, it is often convenient to omit the arguments of certain functions, so for a transformed function $\hat{f}(\alpha, y)$ we adopt the convention

$$\hat{f} = \hat{f}(\alpha, y),$$

while any other function shown without argument shall be understood to be dependent upon the complex variable α only. Finally, since the work which follows involves frequent usage of fractional powers, we introduce the following conventions regarding these objects.

(i) The surd symbol $\sqrt{\cdot}$ shall only be used when taking the positive root of a positive real number.
(ii) If f is a quantity whose argument has a fixed value ϑ in the range $(-\pi,\pi]$, then for any $q \in \mathbb{R}$, $[f(\alpha)]^q$ has argument $q\vartheta$.

(b) *Flexural displacement and governing equation*

We now express the flexural displacement as an integral transform; thus

$$W(x, y) = \frac{1}{2\pi} \int_{\mathcal{C}} \hat{W}(\alpha, y) \mathrm{e}^{-\mathrm{i}\alpha x} \, \mathrm{d}\alpha. \tag{3.2}$$

Applying the differential operator $\mathcal{G}$ of equation (2.8) under the integral sign, we deduce from equation (2.7) that

$$\frac{1}{2\pi} \int_{\mathcal{C}} \left[D_x(\alpha^4 - 1)\hat{W} - 2H\alpha^2 \frac{\partial^2 \hat{W}}{\partial y^2} + D_y \frac{\partial^4 \hat{W}}{\partial y^4} \right] \mathrm{e}^{-\mathrm{i}\alpha x} \, \mathrm{d}\alpha = 0.$$

Since this must hold for all x, it is clear that

$$D_x(\alpha^4 - 1)\hat{W} - 2H\alpha^2 \frac{\partial^2 \hat{W}}{\partial y^2} + D_y \frac{\partial^4 \hat{W}}{\partial y^4} = 0; \tag{3.3}$$

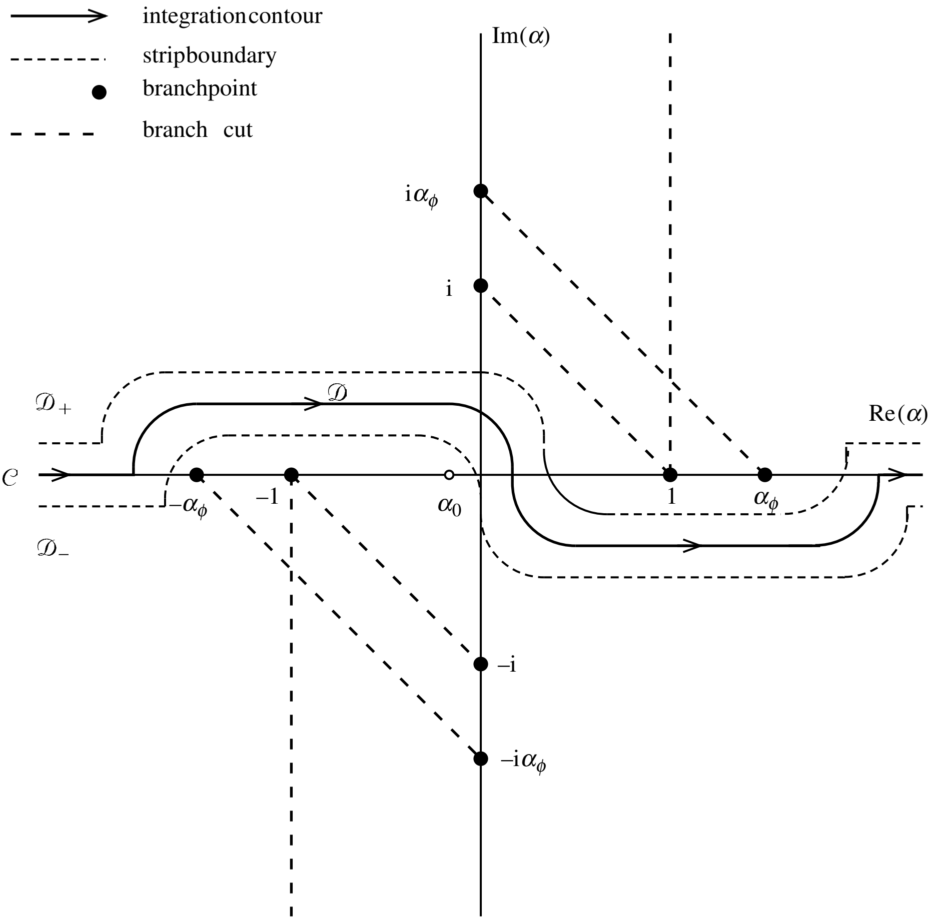

Figure 2. The contour $\mathcal{C}$, running inside the strip $\mathcal{D}$. The upper half-space, $\mathcal{D}_+$, is the region above the lower boundary of $\mathcal{D}$, and the lower half-space, $\mathcal{D}_-$, lies below the upper boundary. Observe that $\mathcal{D}$ is defined so that the point $\alpha=\alpha_0$ lies in the lower half plane (i.e. $\notin\mathcal{D}_+$). Also shown are the branch cuts, for the case $H^2<D_xD_y$. In cases where $H^2>D_xD_y$, the only change to this figure is the rotation about the origin of the branch points of the function ϕ through an anticlockwise angle $\pi/4$.

this is the governing equation in Fourier space. Seeking solutions of the form $\hat{W}=A_m(\alpha)\mathrm{e}^{-\lambda_m(\alpha)y}$ leads to

$$D_x(\alpha^4-1)-2H\alpha^2\lambda_m^2+D_y\lambda_m^4=0, \tag{3.4}$$

and so

$$\lambda_m(\alpha)=\frac{1}{\sqrt{D_y}}[H\alpha^2+(-1)^mD_x\phi]^{1/2}, \tag{3.5}$$

in which

$$\phi(\alpha)=\frac{1}{D_x}[(H^2-D_xD_y)\alpha^4+D_xD_y]^{1/2}, \tag{3.6}$$

$m\in\{1,2\}$, and the factor $1/D_x$ has been added in the last expression for later convenience. The unknown functions $A_m(\alpha)$ must be algebraic in nature, since an essential singularity at infinity would prevent $\hat{W}$ from satisfying the boundary conditions. The integral (3.2) must not contain contributions that are incoming, or that grow as y is increased. It is obvious that at most, one branch of each function λ_m can satisfy these restrictions, so the most general solution to equation (3.3) possesses two distinct exponents. Now, for the function ϕ, we position finite branch cuts on the line sections $[\alpha_\phi, \mathrm{i}\alpha_\phi]$ and $[-\alpha_\phi, -\mathrm{i}\alpha_\phi]$, wherein

$$\alpha_\phi = \left(\frac{D_x D_y}{D_x D_y - H^2}\right)^{1/4} \tag{3.7}$$

(see, for example, figure 2), and specify that $\phi(0) = \sqrt{D_y/D_x}$. This implies that the function λ_1 (λ_2) has branch points located at $\alpha = \pm 1$ ($\alpha = \pm\mathrm{i}$), and changes sign if either of these points is encircled. Some care is required with the interpretation of the exponent functions λ_1 and λ_2, since they are doubly multivalued. Despite this, their values at regular points on the real line can easily be obtained, and this turns out to be sufficient for the means of completing the formal solution. Thus, it is not difficult to show that $\mathrm{Re}[\lambda_2(\alpha)] \neq 0$ for $\alpha \in \mathbb{R}$, and we must select the branch for which $\mathrm{Re}[\lambda_2(\alpha)] > 0$ and discard the other. Furthermore, regardless of the values of the material parameters, we always have

$$\lambda_1 \lambda_2 = \gamma(\alpha), \tag{3.8}$$

wherein

$$\gamma(\alpha) = \sqrt{D_x/D_y}(\alpha^4 - 1)^{1/2}. \tag{3.9}$$

The implicit time harmonic factor is $\mathrm{e}^{-\mathrm{i}t}$, therefore, negative imaginary values of λ_1 are permissible for $\alpha \in (-1,1)$, whereas positive imaginary values are not. Hence, we must have $\gamma(0) = -\mathrm{i}\sqrt{D_x/D_y}$, since λ_2 is positive real here. For later convenience, we use finite branch cuts, along the lines $[-\mathrm{i}, -1]$ and $[1, \mathrm{i}]$. Given the behaviour of γ for large $|\alpha|$, it follows from equation (3.8) that

$$\lambda_m \sim \frac{|\alpha|}{\sqrt{D_y}}[H + (-1)^m (H^2 - D_x D_y)^{1/2}]^{1/2}, \tag{3.10}$$

as $\alpha \to \infty \in \mathcal{D}$. Finally, we observe that an additional pair of branch cuts is required in order to construct a plane in which both λ_1 and λ_2 are analytic; we place these along the lines $\alpha = -1 - \mathrm{i}u$ and $\alpha = 1 + \mathrm{i}u$, for $u \geq 0$. These cuts are also used for the function $\eta(\alpha)$, defined as

$$\eta(\alpha) = (\alpha^2 - 1)^{1/2}, \tag{3.11}$$

with $\eta(0) = -\mathrm{i}$, which is required later. Note that the quotient of η and λ_1 is analytic at $\alpha = \pm 1$, since both functions change sign if either of these points is encircled. The cut plane in the case $H^2 < D_x D_y$ is shown in figure 2.

For later use, we now evaluate the functions ϕ, λ_1, λ_2 and γ at the point $\alpha = \alpha_0$ in equation (3.1). We obtain

$$\gamma^0 = -\frac{\mathrm{i}k^2 \sin\Theta}{\sqrt{D_y}}\sqrt{2H\cos^2\Theta + D_y \sin^2\Theta}, \tag{3.12}$$

$$\phi^0 = \frac{k^2}{D_x}(H\cos^2\Theta + D_y\sin^2\Theta), \tag{3.13}$$

$$\lambda_1^0 = -\mathrm{i}k\sin\Theta \tag{3.14}$$

and

$$\lambda_2^0 = \frac{k}{\sqrt{D_y}}\sqrt{2H\cos^2\Theta + D_y\sin^2\Theta}. \tag{3.15}$$

The Fourier representation of W has the form

$$W(x,y) = \frac{1}{2\pi}\int_C (A_1 e^{-\lambda_1 y} + A_2 e^{-\lambda_2 y}) e^{-i\alpha x}\, d\alpha, \tag{3.16}$$

and we shall also make use of its y derivative

$$Y(x,y) = -\frac{1}{2\pi}\int_C (\lambda_1 A_1 e^{-\lambda_1 y} + \lambda_2 A_2 e^{-\lambda_2 y}) e^{-i\alpha x}\, d\alpha. \tag{3.17}$$

(c) *Kirchhoff edge conditions*

We now obtain transformed expressions for M and V. First, commuting the operator $\mathcal{M}$ in equation (2.19) with the integral (3.2), we find that

$$M(x,y) = -\frac{1}{2\pi D_x}\int_C [(D_y\lambda_1^2 - D_1\alpha^2)A_1 e^{-\lambda_1 y} + (D_y\lambda_2^2 - D_1\alpha^2)A_2 e^{-\lambda_2 y}] e^{-i\alpha x}\, d\alpha.$$

Defining

$$L_m(\alpha) = \frac{1}{D_x}[2D_{xy}\alpha^2 + (-1)^m D_x\phi], \tag{3.18}$$

the expression for the bending moment simplifies to

$$M(x,y) = -\frac{1}{2\pi}\int_C [L_1 A_1 e^{-\lambda_1 y} + L_2 A_2 e^{-\lambda_2 y}] e^{-i\alpha x} d\alpha. \tag{3.19}$$

More care is required when obtaining a representation of the second edge condition, and we decompose V into two terms in order to pre-empt a mathematical technicality that arises owing to the strength of the singularity at the crack tip. Thus, we write

$$Y(x,y) = \frac{-1}{2\pi}\int_C \left[\lambda_1 A_1 e^{-\lambda_1 y} + \lambda_2 A_2 e^{-\lambda_2 y} + Q\hat{T}_- e^{-\eta y\sqrt{H/D_y}}\right] e^{-i\alpha x}\, d\alpha + QT(x,y), \tag{3.20}$$

where Q is a constant at our disposal, and

$$T(x,y) = \frac{1}{2\pi}\int_C \hat{T}_- e^{-\eta y\sqrt{H/D_y}} e^{-i\alpha x} d\alpha. \tag{3.21}$$

The multifunction $\hat{T}_-(\alpha)$ is given by

$$\hat{T}_- = (\alpha - 1)^{-3/2}, \tag{3.22}$$

with a branch cut placed along the line $\alpha = 1 + iu$, $u \geq 0$ and $T\_(0) = -i$. If we now commute the operator $\mathcal{V}$, defined in equation (2.17), with the integral in equation (3.20), we obtain

$$\begin{aligned} V(x,y) = -\frac{1}{2\pi D_x}\int_C \Big[& [(D_1 + 4D_{xy})\alpha^2 - D_y\lambda_1^2]\lambda_1 A_1 e^{-\lambda_1 y} \\ & + [(D_1 + 4D_{xy})\alpha^2 - D_y\lambda_2^2]\lambda_2 A_2 e^{-\lambda_2 y} + (2D_{xy}\alpha^2 + H)Q\hat{T}_- e^{-\eta y\sqrt{H/D_y}}\Big] \\ & \times e^{-i\alpha x}\, d\alpha + Q\mathcal{V}T(x,y), \end{aligned}$$

which simplifies to

$$V(x,y) = -\frac{1}{2\pi}\int_{\mathcal{C}}\left[L_2\lambda_1 A_1 \mathrm{e}^{-\lambda_1 y} + L_1\lambda_2 A_2 \mathrm{e}^{-\lambda_2 y} + \frac{1}{D_x}(2D_{xy}\alpha^2 + H)Q\hat{T}_-\mathrm{e}^{-\eta y\sqrt{H/D_y}}\right] \times \mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha + Q\mathcal{V}T(x,y), \tag{3.23}$$

in view of the definition (3.18). At a later stage, we will choose the constant Q to ensure that the integral in equation (3.23) is convergent. This has the effect of removing the (non-integrable) crack tip singularity from the transform, and placing it in the final term, which can be evaluated explicitly using methods outlined in Noble (1988). This final step will, however, prove unnecessary, since we shall only require the value of $\mathcal{V}T(x,y)$ on the line $y=0$. Thus, we observe from equation (3.21) that

$$\frac{H}{D_x}\left[\frac{D_y}{H}\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} + 1\right]T = 0,$$

and combining this with equation (2.17) leads us to

$$\mathcal{V}T(x,y) = \frac{1}{D_x}\left[-2D_{xy}\frac{\partial^2}{\partial x^2} + H\right]T.$$

Since only the x derivative remains present, we may set $y=0$ *before* evaluating the integral, which appears in the definition of T (3.21). Consequently, we find that

$$\mathcal{V}T(x,0) = \frac{1}{D_x}\left[-2D_{xy}\frac{\partial^2}{\partial x^2} + H\right]\frac{\mathrm{e}^{3\mathrm{i}\pi/4}}{\sqrt{\pi}}\mathrm{He}(-x)\mathrm{e}^{-\mathrm{i}x}|x|^{1/2},$$

in which He is Heaviside's unit function, defined as $\mathrm{He}(x)=1$, $x>0$, and $\mathrm{He}(x)=0$, $x\leq 0$, and is so denoted here to avoid confusion with the material parameter H. We will not evaluate V at the crack tip, so we may take $\mathrm{dHe}(x)/\mathrm{d}x\equiv 0$, and therefore

$$\mathcal{V}T(x,0) = \frac{D_{xy}}{D_x}\frac{\mathrm{e}^{3\mathrm{i}\pi/4}}{2\sqrt{\pi}}\mathrm{He}(-x)\mathrm{e}^{-\mathrm{i}x}\left[2\frac{D_1+4D_{xy}}{D_{xy}}|x|^{1/2} - 4\mathrm{i}|x|^{-1/2} + |x|^{-3/2}\right]. \tag{3.24}$$

(*d*) *The tip condition*

Finally, we must determine the constraints that the tip condition imposes. First, consider the symmetric field. Recalling equation (2.27S), we observe that the Fourier representation of $Y(x,0)$ is analytic in $\mathcal{D}^+$. Therefore, in this case we may write

$$Y(x,0) = \frac{1}{2\pi}\int_{\mathcal{C}}\hat{Y}_+\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha,$$

where

$$\hat{Y}_+ \sim b_1\alpha^{-\zeta}, \tag{3.25}$$

for some constant b_1 and $\zeta>0$ as $\alpha\to\infty\in\mathcal{D}^+$. Now, let $x>0$, and make the substitution $s=\alpha x$. The contour of integration need not be altered, so we obtain

$$Y(x,0) = \frac{1}{2\pi x}\int_{\mathcal{C}}\hat{Y}_+\left(\frac{s}{x}\right)\mathrm{e}^{-\mathrm{i}s}\,\mathrm{d}s.$$

Since the integrand is analytic in $\mathscr{D}^+$, we may deform $\mathscr{C}$ into a new contour $\mathscr{C}'$, on which $|s|>M$, for some fixed, positive quantity M. Allowing x to approach zero, we obtain

$$Y(x,0) \sim b_1 x^{\zeta-1} \frac{1}{2\pi} \int_{\mathscr{C}'} s^{-\zeta} \mathrm{e}^{-\mathrm{i}s}\, \mathrm{d}s. \tag{3.26}$$

This integral can be evaluated by a relatively straightforward process (Osborne 1999), but this is unnecessary here. We need only compare equation (3.26) with (2.24) to deduce that

$$\zeta \geq 1, \tag{3.27}$$

as $\alpha \to \infty \in \mathscr{D}^+$. In the antisymmetric case, we note that the constant C in equation (2.23) must vanish, in view of the boundary condition (2.27A), since first spatial derivatives of displacement are finite in the region of the crack tip (see §2*c*). Consequently, we find that

$$\hat{W}_+^A(\alpha) \sim b_2 \alpha^{-(\zeta+1)}, \tag{3.28}$$

for some constant b_2. Since the functions A_1 and A_2 are known to be algebraic in nature, this behaviour is independent of arg α, therefore equations (3.25) and (3.28) apply as $\alpha \to \infty$ *in any direction.*

4. Solution via the Wiener–Hopf technique

(*a*) *Wiener–Hopf equation for the symmetric case*

We now employ the edge conditions (2.26S–2.28S), along with the Fourier representations (3.17), (3.19) and (3.23), to obtain a Wiener–Hopf equation. From equation (2.26S), we observe that $V(x,0)=0$ for all x, so in this case there is no singularity at the crack tip, and we may set $Q=0$ in equation (3.23), to obtain

$$\int_{\mathscr{C}} (L_2 \lambda_1 A_1 + L_1 \lambda_2 A_2) \mathrm{e}^{-\mathrm{i}\alpha x}\, \mathrm{d}\alpha = 0, \tag{4.1}$$

for all x. For negative x, we have from equations (2.27S) and (3.17)

$$\int_{\mathscr{C}} (\lambda_1 A_1 + \lambda_2 A_2) \mathrm{e}^{-\mathrm{i}\alpha x}\, \mathrm{d}\alpha = 0. \tag{4.2}$$

Finally, from equations (2.28S) and (3.19) we obtain, for $x \geq 0$,

$$-\frac{1}{2\pi} \int_{\mathscr{C}} (L_1 A_1 + L_2 A_2) \mathrm{e}^{-\mathrm{i}\alpha x}\, \mathrm{d}\alpha = -\frac{k^2}{D_x} (D_1 \cos^2 \Theta + D_y \sin^2 \Theta) \mathrm{e}^{\mathrm{i}kx \cos \Theta}. \tag{4.3}$$

Appropriate deformations of the contour $\mathscr{C}$ (i.e. into the lower half plane for $x \geq 0$, and upper half plane for $x<0$) in equations (4.1–4.3) now yield

$$L_2 \lambda_1 A_1 + L_1 \lambda_2 A_2 = 0, \tag{4.4}$$

$$-(\lambda_1 A_1 + \lambda_2 A_2) = \hat{Y}_+ \tag{4.5}$$

and

$$-(L_1 A_1 + L_2 A_2) = \hat{M}_+ + \hat{M}_-. \tag{4.6}$$

Here, the functions $\hat{Y}_+$ and $\hat{M}_-$ are unknown, but $\hat{M}_+$ is easily determined by writing equation (4.3) in the form

$$\frac{1}{2\pi}\int_{\mathcal{C}}\left[\hat{M}_+ + \hat{M}_- + \frac{\mathrm{i}k^2(D_1\cos^2\Theta + D_y\sin^2\Theta)}{D_x(\alpha-\alpha_0)}\right]\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha = 0,$$

where α_0 is given by equation (3.1). Since this must hold for $x \ge 0$, it follows that there are no singularities in the lower half plane, and hence

$$\hat{M}_+ = -\frac{\mathrm{i}k^2(D_1\cos^2\Theta + D_y\sin^2\Theta)}{D_x(\alpha-\alpha_0)}.$$

In view of equations (3.13) and (3.18), this can be written in the compact form

$$\hat{M}_+ = \frac{\mathrm{i}L_1^0}{\alpha-\alpha_0}. \tag{4.7}$$

Next, we make use of equations (4.4) and (4.5), to find that

$$A_1 = \frac{L_1\hat{Y}_+}{2\phi\lambda_1} \tag{4.8}$$

and

$$A_2 = -\frac{L_2\hat{Y}_+}{2\phi\lambda_2}. \tag{4.9}$$

Substituting these into equation (4.6), we obtain the Wiener–Hopf equation for the symmetric case,

$$-\frac{\hat{Y}_+}{\gamma K} = \hat{M}_+ + \hat{M}_-, \tag{4.10}$$

in which the kernel function, $K(\alpha)$, is given by

$$K(\alpha) = \frac{2\phi}{L_1^2\lambda_2 - L_2^2\lambda_1}. \tag{4.11}$$

(*b*) *Wiener–Hopf equation for the antisymmetric case*

Proceeding as in §4*a*, we now make use of the edge conditions (2.26A–2.27A), and the Fourier representations (3.2) and (3.19) to obtain

$$L_1A_1 + L_2A_2 = 0 \tag{4.12}$$

and

$$A_1 + A_2 = \hat{W}_+. \tag{4.13}$$

Now, we can define the quantity

$$U(x,0) = V(x,0) - Q\mathcal{V}T(x,0), \tag{4.14}$$

which is equivalent to $V(x,0)$ for $x \ge 0$, and so from equation (3.23), its Fourier representation is

$$\hat{V}_+ + \hat{U}_- = -\left[L_2\lambda_1A_1 + L_1\lambda_2A_2 + \frac{1}{D_x}(2D_{xy}\alpha^2 + H)Q\hat{T}_-\right]. \tag{4.15}$$

Deforming the contour $\mathcal{C}$ of the integral in equation (3.23) into the lower half plane when $x \ge 0$, and assuming that there is no contribution from the arc at infinity, we deduce from equation (2.28A) that the only contribution comes from a simple pole at the point $\alpha=\alpha_0$, as in the symmetric case. Therefore,

$$\hat{V}_+ = \frac{\mathrm{i}\lambda_1^0 L_2^0}{\alpha - \alpha_0}. \tag{4.16}$$

Manipulation of equations (4.12) and (4.13) yields

$$A_1 = \frac{L_2 \hat{W}_+}{2\phi} \tag{4.17}$$

and

$$A_2 = -\frac{L_1 \hat{W}_+}{2\phi}, \tag{4.18}$$

and using these in equation (4.15), we obtain

$$\frac{\hat{W}_+}{K} = \hat{V}_+ + \hat{U}_- + \frac{1}{D_x}(2D_{xy}\alpha^2 + H)Q\hat{T}_-. \tag{4.19}$$

This is the Wiener–Hopf equation for the antisymmetric problem, in which the kernel function $K(\alpha)$ is again given by equation (4.11).

(c) The kernel factorization

The analytic product factorizations of the functions $\gamma(\alpha)$ and $K(\alpha)$, as defined by equations (3.9) and (4.11), respectively, are now required. The former is trivial; we have

$$\gamma_+ = -\mathrm{i}\sqrt[4]{D_x/D_y}[(\alpha+1)(\alpha+\mathrm{i})]^{1/2} \tag{4.20}$$

and

$$\gamma_- = \mathrm{i}\sqrt[4]{D_x/D_y}[(\alpha-1)(\alpha-\mathrm{i})]^{1/2}, \tag{4.21}$$

where $\arg[\gamma_\pm(0)]=-\pi/4$. Note that the product factors as defined have the relationship

$$\gamma_+(\alpha)=\gamma_-(-\alpha).$$

In the case of $K(\alpha)$, we expand equation (4.11), and then by making use of equations (3.5), (3.8) and (3.18), we find that

$$\begin{aligned} K(\alpha) &= \frac{2D_x^2\phi}{(D_y\lambda_1^2 - D_1\alpha^2)^2\lambda_2 - (D_y\lambda_2^2 - D_1\alpha^2)^2\lambda_1} \\ &= \frac{2D_x^2\phi}{[D_y^2\gamma^2 + 4D_yD_{xy}\alpha^2\gamma - D_1^2\alpha^4](\lambda_1-\lambda_2)}. \end{aligned}$$

Now, to perform a product factorization, it is convenient to separate $K(\alpha)$ into three functions,

$$K(\alpha) = K_1(\alpha)K_2(\alpha)K_3(\alpha), \tag{4.22}$$

in which

$$K_1(\alpha) = \frac{D_x^2(\alpha^4 - \alpha_{\mathrm{e}}^4)/K_1^\infty}{D_xD_y(\alpha^4-1) + 4D_yD_{xy}\alpha^2\gamma - D_1^2\alpha^4}, \tag{4.23}$$

$$K_2(\alpha) = \frac{2\phi/K_2^\infty}{\gamma - \lambda_1^2}\frac{\lambda_1}{\eta} \tag{4.24}$$

and

$$K_3(\alpha) = -K_1^\infty K_2^\infty \frac{\eta}{\alpha^4 - \alpha_e^4}. \tag{4.25}$$

Here, K_1 and K_2 are chosen so as to possess only finite-branch cuts, and a factorization of K_3 is available by inspection. The function η is given by equation (3.11), and the constants K_m^∞ are chosen so that the functions $K_m \to 1$ as $\alpha \to \infty$, for $m \in \{1,2\}$. Hence, it is easy to show that

$$K_1^\infty = \frac{D_x^2}{D_x D_y + 4\sqrt{D_x D_y} D_{xy} - D_1^2} \tag{4.26}$$

and

$$K_2^\infty = \frac{1}{D_x}\sqrt{2D_y\left(H + \sqrt{D_x D_y}\right)}. \tag{4.27}$$

This simple form of K_2^∞ is obtained from equation (4.24) by cancelling a factor of λ_1 and squaring the resulting expression. The signum is then determined by considering the behaviour of the functions ϕ, λ_1, λ_2 and η as $\alpha \to \infty$ on the real line (cf. equations (3.6), (3.10) and (3.11)).

The value of α_e is chosen so as to remove the poles of K_1, thus, setting the denominator of equation (4.23) to zero and using equation (3.9), we obtain

$$D_y^2 \gamma^2 + 4D_y D_{xy} \alpha^2 \gamma - D_1^2 \alpha^4 = 0, \tag{4.28}$$

which leads directly to

$$\frac{\gamma}{\alpha^2} = \frac{1}{D_y}\left[-2D_{xy} \pm \sqrt{4D_{xy}^2 + D_1^2}\right].$$

The values of α obtained by squaring and rearranging this equation can be substituted back, and, given the location of branch cuts of γ (figure 2) and the fact that $\gamma(0) = -\mathrm{i}\sqrt{D_x/D_y}$ on the chosen sheet of the Riemann surface, it is easy to determine those which are actually roots. Thus, four poles are located at the points $\alpha = \pm\alpha_e$, $\alpha = \pm \mathrm{i}\alpha_e$, wherein

$$\alpha_e = \left\{ D_x D_y \Big/ \left[D_x D_y - \left(\sqrt{4D_{xy}^2 + D_1^2} - 2D_{xy} \right)^2 \right] \right\}^{1/4}. \tag{4.29}$$

These singularities are associated with the possibility of flexural edge wave propagation (hence the notation α_e), and in fact α_e is the edge wave wavenumber discussed by Norris (1994). This is to be expected, since equation (4.28) is a form of the edge wave dispersion relation studied by the aforementioned author. We note that the points $\alpha = \pm\alpha_e$, $\alpha = \pm \mathrm{i}\alpha_e$ are simple poles, since equation (4.28) can be manipulated to yield an octic polynomial, with distinct roots, of which these are four. Under certain (physically unusual) circumstances, specifically when

$$5D_x D_y \le \left[2D_{xy} + \sqrt{D_1^2 + 4D_{xy}^2}\right]^2, \tag{4.30}$$

two additional poles are present at the points $\alpha = \pm\alpha_p$, where

$$\alpha_p = \left\{ D_x D_y \Big/ \left[D_x D_y - \left(\sqrt{4D_{xy}^2 + D_1^2} + 2D_{xy} \right)^2 \right] \right\}^{1/4}.$$

These lie on the line $\mathrm{Re}(\alpha)=\mathrm{Im}(\alpha)$, between the origin and the branch cuts of the function γ. An appropriate indentation of the branch cuts of γ, prior to carrying out the kernel factorization, will banish these to the other sheet of the Riemann surface, without affecting the factorized functions in their regions of analyticity. Thus, in this case, the branch cut on the line [i,1] should be replaced by cuts along $[\mathrm{i},\alpha_\mathrm{p}/M]$ and $[\alpha_\mathrm{p}/M,1]$ say, where $M>1$, and a similar indentation must be applied to the cut along $[-1,-\mathrm{i}]$. This situation will not arise for most physically realistic parameter sets, and thus, in most cases, the integrals below can be taken along straight lines. Should the need to study a material whose parameters satisfy the inequality (4.30) arise, however, then the appropriate integration paths must be adjusted accordingly.

Having applied the adjustments described above to the branch cuts of the function γ if necessary, we now prove that K (as defined by equation (4.11)) is analytic at the branch points of the function ϕ (3.6). Thus, let α_Φ be any one of these points, and let $\mathcal{L}$ be a small circular contour enclosing α_Φ (but not enclosing any other branch points). Consider the effect on K if α traverses such a contour once. Clearly, the function ϕ changes sign, and L_1 is interchanged with L_2, but the effect on λ_1 and λ_2 is slightly less obvious. To see that λ_1 is interchanged with λ_2 (and not λ_1 with $-\lambda_2$, etc.), we observe from equation (3.8) that

$$\gamma(\alpha_\Phi) = \lim_{\alpha\to\alpha_\Phi} \lambda_1(\alpha)\lambda_2(\alpha) = \pm\alpha_\Phi^2 H/D_y,$$

wherein the upper sign must be chosen, since $\arg[\gamma(\alpha_\Phi)]=2\arg[\alpha_\Phi]$. Hence $\lambda_1\approx\lambda_2$ for $\alpha\in\mathcal{L}$, and consequently K is unchanged, since both its numerator and denominator switch sign.

Factorizations of K_1 and K_2 can now be obtained in terms of contour integrals, by means of the standard method (Noble 1988), while that of K_3 is available by inspection. Thus, let the upper and lower boundaries of the strip $\mathcal{D}$ be denoted $\mathcal{C}_+$ and $\mathcal{C}_-$, respectively. If $f(\alpha)$ is a function which is analytic and zero-free in $\mathcal{D}$, then

$$f = f_+ f_-, \tag{4.31}$$

where f_+ and f_- are non-zero in their respective regions of analyticity ($\mathcal{D}_+$ and $\mathcal{D}_-$),

$$f_\pm = \exp\left[\frac{I_\pm}{2\pi\mathrm{i}}\right] \tag{4.32}$$

and

$$I_\pm = \pm\int_{\mathcal{C}_\mp} \log[f(z)]\frac{\mathrm{d}z}{z-\alpha}. \tag{4.33}$$

Substituting K_1 for f in equation (4.33), and taking the upper sign throughout, we write

$$I_{1_+} = \int_{\mathcal{C}_-} \log[K_1(z)]\frac{\mathrm{d}z}{z-\alpha}.$$

Note that since $K_1\to1$ as $\alpha\to\infty$ the process of taking logarithms does not create a branch point at infinity. Nor, from the above arguments, are there any logarithmic branch points in the finite part of the cut plane, because $K_1(\alpha)$ remains bounded and non-zero. Therefore, by deforming $\mathcal{C}_-$ into the lower half plane, we can reduce this integral to

$$I_{1_+} = \int_{-1}^{-\mathrm{i}} \log\left[\frac{K_1^{\mathrm{U}}(z)}{K_1^{\mathrm{L}}(z)}\right] \frac{\mathrm{d}z}{z-\alpha},$$

where the superscripts 'U' and 'L' refer to evaluation on the upper-right and lower-left faces of the branch cuts, respectively. Following a similar process for K_{1_-}, and then using equation (4.23), we find that

$$I_{1_\pm} = \int_1^{\mathrm{i}} \log\left[\frac{D_xD_y(z^4-1)+4D_yD_{xy}z^2\gamma^{\mathrm{U}}(z)-D_1^2z^4}{D_xD_y(z^4-1)-4D_yD_{xy}z^2\gamma^{\mathrm{U}}(z)-D_1^2z^4}\right] \frac{\mathrm{d}z}{z\pm\alpha}. \tag{4.34}$$

Note that the substitution $z\to -z$, which has been made in the integral for I_{1_+}, has the effect of changing $\gamma^{\mathrm{L}}(z)$ to $\gamma^{\mathrm{U}}(z)$ and vice versa. Similarly, the function K_2 does not possess a branch point at infinity, and since K, K_1 and K_3 are analytic at the branch points of the function ϕ, it follows that there are only finite-branch cuts from 1 to i, and from -1 to $-\mathrm{i}$, due to γ. Consequently, from equation (4.33), we find that

$$I_{2_+} = \int_{-1}^{-\mathrm{i}} \log\left[\frac{\lambda_1^2(z)-\gamma^{\mathrm{L}}(z)}{\lambda_1^2(z)-\gamma^{\mathrm{U}}(z)}\right] \frac{\mathrm{d}z}{z-\alpha}.$$

This integral possesses an endpoint removable singularity, which may be eliminated by cancelling a factor of $\lambda_1^{\mathrm{L}}(z)$, for example. We can then simplify matters by multiplying the numerator and denominator of the term in square brackets by a factor of $\lambda_1^{\mathrm{L}}(z)+\lambda_2(z)$. This enables us to remove both λ_1 and $\lambda_2(z)$ from the integrand, which is desirable, since it is somewhat difficult to define computable expressions for these functions for $z\notin\mathbb{R}$. After a little algebra, we are left with

$$I_{2_\pm} = \int_1^{\mathrm{i}} \log\left[\frac{-D_x\phi(z)}{Hz^2+D_y\gamma^U(z)}\right] \frac{\mathrm{d}z}{z\pm\alpha}, \tag{4.35}$$

where a similar calculation has been carried out for I_{2_-}, and a substitution $z\to -z$ performed on the integral for I_{2_+}. Finally, for $K_3(\alpha)$, we have by inspection

$$K_{3_\pm} = \sqrt{K_1^\infty K_2^\infty}\,\frac{\mathrm{i}\eta_\pm}{(\alpha\pm\alpha_{\mathrm{e}})(\alpha\pm\mathrm{i}\alpha_{\mathrm{e}})}, \tag{4.36}$$

wherein

$$\eta_\pm(\alpha) = \mathrm{e}^{\mp\mathrm{i}\pi/4}(\alpha\pm 1)^{1/2} \tag{4.37}$$

and $\eta_\pm(0)=\mathrm{e}^{-\mathrm{i}\pi/4}$. The entire factorization process is now complete; we have

$$K_\pm = K_{3_\pm}\exp\left[\frac{1}{2\pi\mathrm{i}}(I_{1_\pm}+I_{2_\pm})\right], \tag{4.38}$$

and for $\alpha\in\mathscr{D}_+$ it is clear that

$$K_+(\alpha) = K_-(-\alpha).$$

Crucially, the integrals involved in equation (4.38), i.e. those given in equations (4.34) and (4.35), are definite and non-singular; hence $(I_{1_\pm}+I_{2_\pm})\to 0$ as $\alpha\to\infty$.

Thus, from equations (4.36) and (4.37), it is evident that

$$K_{\pm} \to \sqrt{K_1^{\infty} K_2^{\infty}} \frac{\mathrm{i}}{\alpha^2} \eta_{\pm}(\alpha), \tag{4.39}$$

as $\alpha \to \infty \in \mathscr{D}^{\pm}$.

(d) Formal solution in the symmetric case

We now obtain explicit expressions for the functions A_1 and A_2, thereby completing the solution. Since the only singularity of the function $\hat{M}_+$ is a simple pole at the point $\alpha = \alpha_0$, we rewrite equation (4.10) in the form

$$\frac{\hat{Y}_+}{K_+\gamma_+} + K_-^0\gamma_-^0\hat{M}_+ = [K_-^0\gamma_-^0 - K_-\gamma_-]\hat{M}_+ - K_-\gamma_-\hat{M}_-. \tag{4.40}$$

The right-hand side of this equation is analytic in $\mathscr{D}_-$, since it has no pole at the point $\alpha = \alpha_0$, and the left-hand side is analytic in $\mathscr{D}_+$, since the functions K_+ and γ_+ are non-zero in this latter region. They are therefore analytic continuations of each other, as both sides are analytic in the strip $\mathscr{D}$, and so equal an entire function $J^{\mathrm{S}}(\alpha)$, say. The right-hand side of equation (4.40) is now superfluous (except possibly to bound the behaviour of the entire function at infinity), and from the left-hand side we obtain

$$\hat{Y}_+ = K_+\gamma_+[J^{\mathrm{S}} - K_-^0\gamma_-^0\hat{M}_+].$$

The unknown entire function J^{S} is uniquely determined by the physically required condition at the crack tip. Thus, taking into account the asymptotic behaviour of all the known functions in the above expression ((3.5), (3.6), (3.18), (4.7), (4.20) and (4.39)), it is clear that the restriction (3.25) can only be satisfied if $J^{\mathrm{S}}(\alpha) \to 0$ as $|\alpha| \to \infty$. Hence, by Liouville's theorem, we may conclude that

$$J^{\mathrm{S}} \equiv 0. \tag{4.41}$$

Note that in the situation $H^2 = D_x D_y$ the leading order term of the factor $(L_1/\lambda_1 - L_2/\lambda_2)$ disappears, and the function ϕ reduces to a constant, leaving the overall asymptotic behaviour unchanged. The formal solution for the symmetric case is now complete: we have

$$W(x, y) = -K_-^0\gamma_-^0 L_1^0 \frac{\mathrm{i}}{4\pi} \int_{\mathscr{C}} \frac{K_+}{\phi\gamma_-}(\lambda_2 L_1 \mathrm{e}^{-\lambda_1 y} - \lambda_1 L_2 \mathrm{e}^{-\lambda_2 y}) \frac{\mathrm{e}^{-\mathrm{i}\alpha x}}{\alpha - \alpha_0} \mathrm{d}\alpha, \tag{4.42}$$

in which the expression for $\hat{M}_+$, given in equation (4.7), has been used.

(e) Formal solution in the antisymmetric case

Proceeding as before, we express equation (4.19) in the form

$$\frac{\hat{W}_+}{K_+} - K_-^0\hat{V}_+ = K_-[\hat{U}_- + Q(2D_{xy}\alpha^2 + H)\hat{T}_-] + [K_- - K_-^0]\hat{V}_+. \tag{4.43}$$

Again, the left-hand side is analytic in $\mathscr{D}_+$, and the right-hand side in $\mathscr{D}_-$, so by analytic continuation both must be equal to an entire function which we denote $J^{\mathrm{A}}(\alpha)$. Equating the left-hand side of equation (4.43) to J^{A} and considering the asymptotic behaviour of the known functions in this last expression (i.e. equations (4.16) and (4.39)) it is observed from equation (3.28) that we must have

$$J^{\mathrm{A}} \equiv 0. \tag{4.44}$$

Making use once more of equation (4.16), the Fourier representation of W in the antisymmetric case now becomes

$$W(x,y) = K_-^0 \lambda_1^0 L_2^0 \frac{\mathrm{i}}{4\pi} \int_{\mathcal{C}} \frac{K_+}{\phi} \left(L_2 \mathrm{e}^{-\lambda_1 y} - L_1 \mathrm{e}^{-\lambda_2 y} \right) \frac{\mathrm{e}^{-\mathrm{i}\alpha x}}{\alpha - \alpha_0} \mathrm{d}\alpha. \tag{4.45}$$

(*f*) *The complete solution*

Applying symmetry conditions to equations (4.42) and (4.45), we now obtain the full solution for the scattered plate displacement,

$$W(x,y) = -\frac{\mathrm{i}K_-^0}{4\pi} \int_{\mathcal{C}} \frac{K_+}{\phi} \left[\frac{\gamma_-^0}{\gamma_-} L_1^0 \left(\lambda_2 L_1 \mathrm{e}^{-\lambda_1 |y|} - \lambda_1 L_2 \mathrm{e}^{-\lambda_2 |y|} \right) \right.$$
$$\left. - \operatorname{sgn}(y) \lambda_1^0 L_2^0 \left(L_2 \mathrm{e}^{-\lambda_1 |y|} - L_1 \mathrm{e}^{-\lambda_2 |y|} \right) \right] \frac{\mathrm{e}^{-\mathrm{i}\alpha x}}{\alpha - \alpha_0} \mathrm{d}\alpha, \tag{4.46}$$

which is valid for all x and y (aside from the point $x=y=0$). The contour $\mathcal{C}$ traverses the real line but is indented above all singularities lying along $\alpha \le \alpha_0$, and above those for which $\alpha > \alpha_0$. Note that the integrand in equation (4.46) is single valued at the branch points of the function ϕ; the same arguments used in §4*c* also apply here.

Our choice of integration path may now be justified by considering the behaviour of the functions λ_1 and λ_2 on this indented real line contour. Note that integrals around 'small arc indentations' can be evaluated using standard results (Osborne 1999). Indentations around branch points do not contribute to the solution, while the contribution from a pole can be expressed in terms of its residue. At every regular point on the real line, $\mathrm{Re}[\lambda_2] > 0$ and $\mathrm{Re}[\lambda_1] \ge 0$. Where equality holds, $\mathrm{Im}[\lambda_1] < 0$, therefore the solution clearly represents only outgoing or decaying waves, since the implicit time harmonic factor is $\mathrm{e}^{-\mathrm{i}t}$. An integration path that permits a sufficiently general solution to equation (3.3), and whose orientation with respect to any of the four branch points $\alpha = \pm 1$, $\alpha = \pm \mathrm{i}$ is different to that of $\mathcal{C}$, cannot satisfy this requirement. Furthermore, in order to satisfy the tip condition, it is necessary for the function $K_+(\alpha)$ to possess at least two poles (see §4). The function $K(\alpha)$ has four poles located at the points $\alpha = \pm \alpha_{\mathrm{e}}$ and $\alpha = \pm \mathrm{i}\alpha_{\mathrm{e}}$. The definition of the contour $\mathcal{C}$ implies that $\alpha = -\alpha_{\mathrm{e}}$ and $\alpha = -\mathrm{i}\alpha_{\mathrm{e}}$ are located in the region $\mathcal{D}_-$, while $\alpha = \alpha_{\mathrm{e}}$ and $\alpha = \mathrm{i}\alpha_{\mathrm{e}}$ lie within $\mathcal{D}_+$. Had we chosen a path of integration that distributed these singularities in a different manner, an unphysical wave (either exponentially growing or incoming) would be present on the upper face of the crack. To demonstrate this, one must simply set $y=0^+$ and $x \ge 0$ in equation (4.46), deform the contour $\mathcal{C}$ into the lower half plane, and calculate the appropriate residues.

5. Discussion of the solution

(*a*) *Verification*

We now confirm that the integral (4.46) is indeed the solution of the boundary-value problem outlined in §2. In doing so, we shall also confirm the validity of the interchanges performed in §3 between differential operators and infinite contour integrals. That the governing equation (2.7) is satisfied is immediately clear, since on applying the operator $\mathcal{G}$ (2.8) under the integral sign in equation (4.46), each exponential term generates a multiplicative factor of $[D_x(\alpha^4-1)-2H\lambda_j^2+D_y\lambda_j^4]$, the very polynomial of which both λ_1 and λ_2 are by definition roots. The resulting expression is, therefore, identically zero. When dealing with the edge conditions, it is consistent to consider the symmetric and antisymmetric components of the solution separately. Thus, in place of equations (2.21) and (2.22), we verify the equivalent conditions (2.26S), (2.28S), (2.26A) and (2.28A) along the half line $y=0$, $x\geq 0$. Note that the conditions for $x<0$ do not require verification, since these are merely consequences of symmetry. Some care must be taken regarding the convergence of the integral expressions involved in this process, in particular in the neighbourhood of the crack tip. Thus, fix $\varepsilon>0$, and for the remainder of this section, we impose the restriction

$$\sqrt{x^2+y^2}>\varepsilon. \tag{5.1}$$

Since ε may be arbitrarily small, the resulting expressions may be used to determine the behaviour of physical quantities as the point $x=y=0$ is approached. Now, it is not difficult to see that the application of the operator $\mathcal{M}$ of equation (2.16) to (4.42) leads to an integral which is uniformly convergent, and by setting $y=0$ we obtain

$$\begin{aligned}M^{\mathrm{S}}(x,0) &= -K_-^0\gamma_-^0L_1^0\frac{\mathrm{i}}{4\pi D_x}\\ &\quad\times\int_{\mathcal{C}}\frac{K_+}{\phi\gamma_-}[(D_1\alpha^2-D_y\lambda_1^2)\lambda_2L_1-(D_1\alpha^2-D_y\lambda_2^2)\lambda_1L_2]\frac{\mathrm{e}^{-\mathrm{i}\alpha x}}{\alpha-\alpha_0}\mathrm{d}\alpha\\ &= K_-^0\gamma_-^0L_1^0\frac{\mathrm{i}}{2\pi}\int_{\mathcal{C}}\frac{1}{K_-\gamma_-}\frac{\mathrm{e}^{-\mathrm{i}\alpha x}}{\alpha-\alpha_0}\mathrm{d}\alpha.\end{aligned}$$

The only singularity in the lower half plane is the simple pole at $\alpha=\alpha_0$, hence if $x\geq 0$, we deform the contour downwards and the integral reduces to a clockwise circle around this point, since Jordan's lemma (Osborne 1999) shows that the arc at infinity gives no contribution. Making use of equations (3.13) and (3.18), it is immediately clear that

$$M^{\mathrm{S}}(x,0) = -\frac{k^2}{D_x}(D_1\cos^2\Theta+D_y\sin^2\Theta)\mathrm{e}^{\mathrm{i}kx\cos\Theta},$$

which shows that condition (2.28S) is indeed satisfied. Similarly, for the antisymmetric solution (4.45), we obtain

$$M^{\mathrm{A}}(x,0)=K_-^0\lambda_1^0L_2^0\frac{\mathrm{i}}{4\pi D_x}\int_{\mathcal{C}}\frac{K_+}{\phi}[(D_1\alpha^2-D_y\lambda_1^2)L_2-(D_1\alpha^2-D_y\lambda_2^2)L_1]\frac{\mathrm{e}^{-\mathrm{i}\alpha x}}{\alpha-\alpha_0}\mathrm{d}\alpha=0;$$

the term in square brackets vanishing in view of equations (3.5) and (3.18). Thus, condition (2.26A) is also satisfied. For the second Kirchhoff edge condition, application of the operator $\mathcal{V}\partial/\partial y$ under the integral sign in equation (4.42) leads to

$$\begin{aligned}V^{\mathrm{S}}(x,0) &= K_-^0\gamma_-^0L_1^0\frac{\mathrm{i}}{4\pi D_x}\\ &\quad\times\int_{\mathcal{C}}\frac{K_+\gamma_+}{\phi}[(D_1+4D_{xy})\alpha^2(L_2-L_1)+D_y(\lambda_1^2L_1-\lambda_2^2L_2)]\mathrm{e}^{-\mathrm{i}\alpha x}\mathrm{d}\alpha=0,\end{aligned}$$

as required by condition (2.26S); the term in square brackets vanishes, due to equations (3.5) and (3.18) as above. In the antisymmetric case, the procedure is slightly different, since we cannot apply the operator $\mathcal{V}\partial/\partial y$ to equation (4.45) directly. Instead, we use the results of §4 in equation (3.23) to obtain

$$V^{\mathrm{A}}(x,0)=\frac{1}{2\pi}\int_{C}\left[\frac{\mathrm{i}\lambda_1^0 L_2^0}{\alpha-\alpha_0}\,\frac{K_-^0}{K_-}-\frac{Q\hat{T}_-}{D_x}(2D_{xy}\alpha^2+H)\right]\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha+Q\mathcal{V}T(x,0), \tag{5.2}$$

in which the final term is written explicitly in equation (3.24). Given the asymptotic behaviour of the functions K_- and $\hat{T}_-$ as $|\alpha|\to\infty$ (equations (4.39) and (3.22)), it is easy to see that the integral in equation (5.2) converges if we choose

$$Q=\mathrm{e}^{-\mathrm{i}\pi/4}\frac{D_x}{2D_{xy}}\frac{K_-^0}{\sqrt{K_1^\infty K_2^\infty}}\lambda_1^0 L_2^0. \tag{5.3}$$

If we now set $x\geq 0$, and deform the contour C into the lower half plane, we obtain

$$V^{\mathrm{A}}(x,0)=-\frac{\mathrm{i}k^3}{D_x}\sin\Theta[(D_1+4D_{xy})\cos^2\Theta+D_y\sin^2\Theta]\mathrm{e}^{\mathrm{i}kx\cos\Theta},$$

in agreement with equation (2.28A). On the other hand, if $x<0$, then we can employ the uniformly valid expression for $V^{\mathrm{A}}(x,0)$, equation (5.2), to obtain its behaviour as $x\to 0^-$; from equations (3.24) and (5.3), it can immediately be shown that

$$V^{\mathrm{A}}(x,0)\sim\frac{K_-^0}{\sqrt{K_1^\infty K_2^\infty}}\frac{k^3\sin\Theta}{4\sqrt{\pi}D_x}[(D_1+4D_{xy})\cos^2\Theta+D_y\sin^2\Theta]|x|^{-3/2}.$$

(*b*) *Concluding remarks*

In this article we have obtained and verified an exact integral representation of the flexural wave field W, (4.46), scattered by a semi-infinite straight crack aligned with a principal axis of an orthotropic thin elastic plate. A detailed asymptotic evaluation of this solution via the method of steepest descents (Jeffreys & Jeffreys 1956) will be presented in the second part of this work; however, several comments are in order at this point. The integrand in equation (4.46) possesses three poles, $\alpha=\alpha_0$, $\alpha=-\alpha_{\mathrm{e}}$ and $\alpha=\mathrm{i}\alpha_{\mathrm{e}}$. Respectively, these are associated with the reflected field, flexural edge waves, and evanescent edge waves. The last of these decay exponentially in all directions away from the crack tip, and consequently, are of little interest. In their analysis, Norris & Wang (1994) preferred to apply the method of stationary phase (Jeffreys & Jeffreys 1956) rather than steepest descents. While the two are, in principle, equivalent to each other, in practice, stationary phase is difficult to employ correctly and efficiently in this rather more complicated scattering problem. Norris & Wang also noted that the edge wave that could propagate in the isotropic case was none other than that apparently first discussed by Thurston *et al.* (1974). (In fact it was originally discovered by Konenkov (1960) but published in a Russian journal.) Analogously, the wave associated with the pole $\alpha=-\alpha_{\mathrm{e}}$ in this article is precisely that examined by Norris (1994). This edge wave has interesting properties, and has recently been shown by the authors (Thompson *et al.* 2002) to persist when the free edge is inclined at an arbitrary angle to the principal axes of orthotropy.

A final point concerns the reduction of equation (4.46) to the isotropic case. The required parameters for isotropy are $D_x = D_y = D$, $D_1 = \nu D$ and $D_{xy} = (1-\nu)D/2$, where D is the isotropic bending stiffness and ν is the Poisson ratio. (Note that this leads to $H = D$.) Using these relations in equation (4.46), the solution can be compared with (4.13) of Norris & Wang (1994). All but one of the apparent differences between these expressions are accounted for by the definitions of the Fourier transform. The remaining conflict is the factor of 1/2, which Norris & Wang (1994) chose to absorb into the kernel. Thus, the two are found to be entirely equivalent.

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