3 hours

THE UNIVERSITY OF MANCHESTER

HYPERBOLIC GEOMETRY

29 January 2016

9:45 – 12:45

Answer **all** four questions in **Section A** (40 marks in all) and
two of the three questions in **Section B** (30 marks each) and
both questions in **Section C** (50 marks in all).
If all three questions from Section B are attempted then credit will be given for the two best answers.

Electronic calculators may be used, provided that they cannot store text/transmit or receive information/display graphics.

Notation: Throughout, $\mathbb{H}$ denotes the upper half-plane, $\partial\mathbb{H}$ denotes the boundary of $\mathbb{H}$, $\mathbb{D}$ denotes the Poincaré disc, and $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$.

SECTION A

Answer **ALL** four questions

A1.

(i) What does it mean to say that a transformation γ is a Möbius transformation of $\mathbb{H}$?

[2 marks]

(ii) Let γ be a Möbius transformation of $\mathbb{H}$. Show that if $z \in \mathbb{H}$ then $\gamma(z) \in \mathbb{H}$.

[4 marks]

(iii) Let γ be a Möbius transformation of $\mathbb{H}$. Show that γ^{-1} is also a Möbius transformation of $\mathbb{H}$.

[2 marks]

A2.

(i) Let γ be a Möbius transformation of $\mathbb{H}$. What does it mean to say that $z_0 \in \mathbb{H} \cup \partial\mathbb{H}$ is a fixed point of γ?

[2 marks]

(ii) Define the trace $\tau(\gamma)$ of a Möbius transformation γ. Suppose that γ is not the identity. State (without proof) how the trace $\tau(\gamma)$ can be used to classify γ as either a hyperbolic, parabolic or elliptic Möbius transformation.

[4 marks]

(iii) Consider the Möbius transformation of $\mathbb{H}$ given by

$$\gamma(z) = \frac{z-3}{z}.$$

Calculate the trace of γ and hence decide if γ is hyperbolic, parabolic or elliptic. Check your answer by explicitly calculating the fixed points of γ.

[4 marks]

A3.

(i) Let $\mathcal{S} = \{a_1, \dots, a_k\}$ be a finite set of symbols and let $w_1, \dots, w_m$ be a finite set of words in symbols chosen from $\mathcal{S} \cup \mathcal{S}^{-1}$. Briefly explain how to construct the group

$$\Gamma = \langle a_1, \dots, a_k \mid w_1 = \cdots = w_m = e \rangle.$$

(Your answer should include: a description of the elements of Γ, a description of the group operation, a description of the group identity, and a brief explanation of how to find the inverse of an element in Γ. You do not need to prove that the group operation is associative.)

[6 marks]

(ii) Explain why the group Γ written in terms of generators and relations as

$$\Gamma = \langle a, b \mid aba^{-1}b^{-1} = e \rangle$$

is isomorphic to the additive group

$$\mathbb{Z}^2 = \{(n, m) \mid n, m \in \mathbb{Z}\}.$$

[4 marks]

(iii) How could you write the additive group $\mathbb{Z}^3 = \{(n, m, p) \mid n, m, p \in \mathbb{Z}\}$ in terms of generators and relations?

[2 marks]

A4.

(i) Let D be a convex hyperbolic polygon with no boundary vertices. Suppose that D is equipped with a set of side-pairing transformations so that no side is paired with itself. Let $\mathcal{E}$ be an elliptic cycle. What does it mean to say that $\mathcal{E}$ satisfies the elliptic cycle condition?

[2 marks]

(ii) Consider the regular hyperbolic octagon in Figure 1 below with all internal angles equal to $\pi/16$ and the sides paired as indicated.

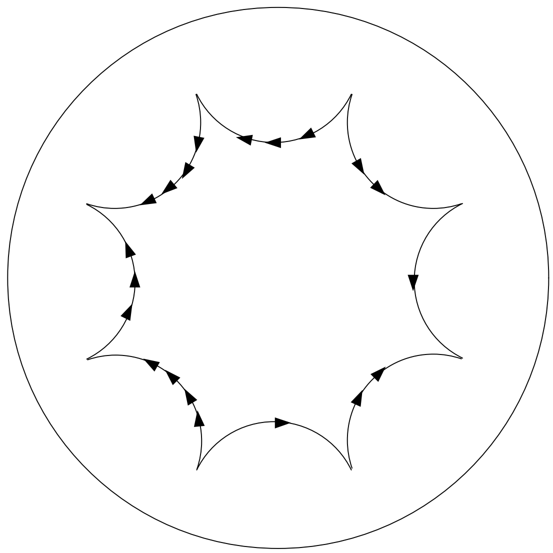

Figure 1: A regular hyperbolic octagon with all internal angles equal to $\pi/16$ and sides paired as indicated. See Question A4(ii).

Show that there is just one elliptic cycle and show that the elliptic cycle condition is satisfied. Hence use Poincaré's Theorem to show that the side pairing transformations generate a Fuchsian group. Give a presentation of Γ in terms of generators and relations.

[8 marks]

SECTION B

Answer **TWO** of the three questions

B5.

(a) (i) Recall that vertical straight lines and semi-circles with real centres in $\mathbb{H}$ have equations of the form

$$\alpha z\bar{z} + \beta z + \beta\bar{z} + \gamma = 0$$

where $\alpha, \beta, \gamma \in \mathbb{R}$. Recall the map $h : \mathbb{H} \to \mathbb{D}$ given by

$$h(z) = \frac{z-i}{iz-1}.$$

Show that h maps vertical straight lines and semi-circles with real centres in $\mathbb{H}$ to arcs of circles and lines in $\mathbb{D}$ with equations of the form

$$\alpha' z\bar{z} + \beta' z + \bar{\beta}'\bar{z} + \alpha' = 0$$

where $\alpha' \in \mathbb{R}$ and $\beta' \in \mathbb{C}$.

[6 marks]

(b) (i) Let $\sigma : [a,b] \to \mathbb{D}$ be a smooth path in the Poincaré disc $\mathbb{D}$. Recall that the hyperbolic length of σ is defined to be

$$\mathrm{length}_{\mathbb{D}}(\sigma) = \int_a^b \frac{2}{1-|\sigma(t)|^2}|\sigma'(t)|\,dt.$$

Let $R \in (0,1)$. Let σ_1 be the path along the real axis in $\mathbb{D}$ from 0 to R. Calculate $\mathrm{length}_{\mathbb{D}}(\sigma_1)$.

[4 marks]

(ii) Let $R \in (0,1)$. Let σ_2 denote the path that travels once, anticlockwise, around the (Euclidean) circle with centre 0 and radius R. Write down a parametrisation of σ_2. Calculate $\mathrm{length}_{\mathbb{D}}(\sigma_2)$.

[4 marks]

(iii) Let $C_r = \{z \in \mathbb{D} \mid d_{\mathbb{D}}(z,0) = r\}$ denote the hyperbolic circle in $\mathbb{D}$ with centre 0 and hyperbolic radius r. It can be shown that C_r is also a Euclidean circle with centre 0 and radius, say, R (you do not need to prove this). Use the fact that $d_{\mathbb{D}}(0,r) = \log\left(\frac{1+r}{1-r}\right)$ to show that $R = \tanh r/2$. Hence show that the hyperbolic circumference of C_r is

$$\mathrm{circumference}_{\mathbb{D}}(C_r) = 2\pi \sinh r.$$

[6 marks]

(iv) Recall that the hyperbolic area of a subset $D \subset \mathbb{D}$ is given by

$$\mathrm{Area}_{\mathbb{D}}(D) = \int\int_D \frac{4}{(1-(x^2+y^2))^2}\,dx\,dy.$$

Also recall that, when integrating using polar co-ordinates (ρ, θ), the area element is $\rho\,d\rho\,d\theta$.

Show that the hyperbolic area of the region enclosed by C_r is given by

$$\mathrm{Area}_{\mathbb{D}}(C_r) = 4\pi \sinh^2 r/2.$$

[6 marks]

(v) Show that

$$\lim_{r\to\infty} \frac{\mathrm{circumference}_{\mathbb{D}}(C_r)}{\mathrm{Area}_{\mathbb{D}}(C_r)} = 1.$$

(You may use that, for large r, $\sinh r \sim \frac{e^r}{2}$.)

What is the Euclidean analogue of this result?

[4 marks]

B6.

(a) (i) Let Γ be a Fuchsian group acting on $\mathbb{H}$. What does it mean to say that an open set $F \subset \mathbb{H}$ is a fundamental domain for Γ?

[2 marks]

(ii) Briefly explain an algorithm for calculating a Dirichlet polygon for a Fuchsian group Γ. (Your answer should include a definition of the terms $[p, \gamma(p)]$, $L_p(\gamma)$, $H_p(\gamma)$, $D(p)$ that were defined in the lectures.)

[4 marks]

(b) (i) It was proved in the lectures that the perpendicular bisector of the arc of geodesic $[z_1, z_2]$ between two points $z_1, z_2 \in \mathbb{H}$ is given by

$$\{z \in \mathbb{H} \mid d_{\mathbb{H}}(z, z_1) = d_{\mathbb{H}}(z, z_2)\}. \tag{1}$$

Write $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Show that (1) can be written in the form

$$\{z \in \mathbb{H} \mid y_2|z - z_1|^2 = y_1|z - z_2|^2\}.$$

(You may use the fact that

$$\cosh d_{\mathbb{H}}(z, w) = 1 + \frac{|z - w|^2}{2\,\mathrm{Im}(z)\,\mathrm{Im}(w)}.)$$

[4 marks]

(ii) Let

$$\Gamma = \left\{\gamma(z) \mid \gamma(z) = \frac{az + 2b}{\frac{c}{2}z + d}, a, b, c, d \in \mathbb{Z}, ad - bc = 1\right\}.$$

Let

$$\gamma_1(z) = z + 2, \quad \gamma_1^{-1}(z) = z - 2, \quad \gamma_2(z) = \frac{2z - 4}{z}, \quad \gamma_2^{-1}(z) = \frac{4}{-z + 2}.$$

(Note that $\gamma_1, \gamma_2 \in \Gamma$. You may assume that Γ is a Fuchsian group.)
Let $p = 1 + 2i$. Calculate $H_p(\gamma_1), H_p(\gamma_1^{-1}), H_p(\gamma_2), H_p(\gamma_2^{-1})$.
(Hint: use geometric intuition for $H_p(\gamma_1^{\pm 1})$ and the result in B6(b)(i) for $H_p(\gamma_2^{\pm 1})$.)
Conclude that

$$D(p) \subset H_p(\gamma_1) \cap H_p(\gamma_1^{-1}) \cap H_p(\gamma_2) \cap H_p(\gamma_2^{-1}). \tag{2}$$

[14 marks]

(iii) One can prove that the inclusion in (2) is an equality (you do not need to do this). Sketch a picture of $D(p)$.

[2 marks]

(iv) For each side of $D(p)$, write down the associated side-pairing transformation.
Briefly explain why Γ is generated by γ_1 and γ_2.

[4 marks]

B7. Let Γ be a Fuchsian group and let D be a Dirichlet polygon for Γ. Suppose that D has some vertices on the boundary but that there are no free edges. Equip D with a set of side-pairing transformations and assume that no side of D is paired with itself.

(i) What does it mean to say that an elliptic cycle is a non-accidental elliptic cycle?

[2 marks]

(ii) Briefly explain how to construct the space $\mathbb{H}/\Gamma$. (As well as explaining how to construct $\mathbb{H}/\Gamma$, your account should include a brief description of the roles played by elliptic cycles and parabolic cycles, how marked points are formed and how cusps are formed.)

[6 marks]

(iii) In the course the signature $\mathrm{sig}(\Gamma)$ of a cocompact Fuchsian group was defined. This definition can be extended to the case when Γ is not cocompact and $\mathbb{H}/\Gamma$ has cusps as follows: Suppose that Γ and D are as above. Let $m_1, \ldots, m_r$ denotes the orders of the non-accidental elliptic cycles. (There may also be some accidental elliptic cycles.) Suppose that $\mathbb{H}/\Gamma$ has genus $g \geq 0$ and $c \geq 1$ cusps. Then the signature of Γ is defined to be

$$\mathrm{sig}(\Gamma) = (g; m_1, \ldots, m_r; c).$$

We write $\mathrm{sig}(\Gamma) = (g; —; c)$ if there are no non-accidental elliptic cycles.

Recall that the Gauss-Bonnet Theorem gives the following formula for the hyperbolic area of an n-sided hyperbolic polygon P with internal angles $\alpha_1, \ldots, \alpha_n$:

$$\mathrm{Area}_{\mathbb{H}}(P) = (n-2)\pi - (\alpha_1 + \cdots + \alpha_n).$$

Also recall Euler's formula: if a space X has a triangulation with V vertices, E edges and F faces then the genus g of X is given by

$$2 - 2g = V - E + F.$$

Prove that if $\mathrm{sig}(\Gamma) = (g; m_1, \ldots, m_r; c)$ then

$$\mathrm{Area}_{\mathbb{H}}(D) = 2\pi\left((2g-2) + \sum_{j=1}^{r}\left(1 - \frac{1}{m_j}\right) + c\right). \tag{3}$$

[12 marks]

(iv) Consider the hyperbolic quadrilateral D illustrated in Figure 2 on the next page. Here

$$\gamma_1(z) = z + 2, \qquad \gamma_2(z) = \frac{z}{2z+1}.$$

Use Poincaré's Theorem to show that γ_1, γ_2 generate a Fuchsian group. Write down the signature $\mathrm{sig}(\Gamma)$ and use (3) to calculate the hyperbolic area $\mathrm{Area}_{\mathbb{H}}(D)$.

Check your answer by using the Gauss-Bonnet Theorem to calculate the hyperbolic area of D.

[10 marks]

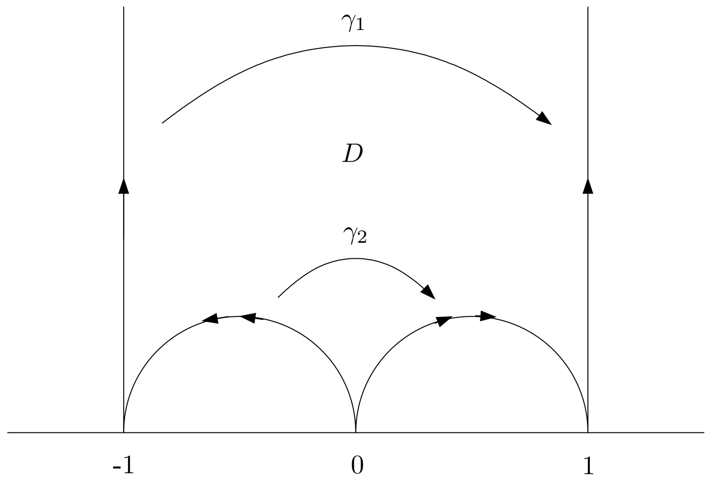

Figure 2: See Question B7(iv).

SECTION C

Answer **BOTH** of the questions

C8.

(i) Let (X, d) be a metric space and let Γ be a group of homeomorphisms that acts on X.

Define what it means to say that Γ acts properly discontinuously on X.

Let $x \in X$. Define the orbit $\Gamma(x)$ and the stabiliser $\mathrm{Stab}_\Gamma(x)$ of x.

[6 marks]

(ii) Suppose that

- the orbit $\Gamma(x)$ is a discrete subset of X for all $x \in X$, and
- the stabiliser $\mathrm{Stab}_\Gamma(x)$ is a finite subgroup of Γ for all $x \in X$.

Prove that Γ acts properly discontinuously on X.

[14 marks]

(iii) Consider the following spaces and groups acting on them:

- $X_1 = \mathbb{R}^3$, $\Gamma_1 = \{\gamma_{n,m} \mid \gamma_{n,m}(x, y, z) = (x + n, y + m, z), n \in \mathbb{Z}, m \in \mathbb{Z}\}$.
- $X_2 = \mathbb{H}$, $\Gamma_2 = \mathrm{PSL}(2, \mathbb{R})$.

Use the results of (ii) above to show that Γ_1 acts properly discontinuously on X_1 but that Γ_2 does not act properly discontinuously on X_2.

[6 marks]

C9.

(i) Let Γ be a Fuchsian group acting on $\mathbb{D}$. What is meant by a limit point of $\Gamma(z)$?

[2 marks]

Recall that the limit set $\Lambda(\Gamma)$ is defined to be the set of all limit points of $\Gamma(z)$ and that this set is independent of z. You may use this fact without proof in the remainder of this question.

(ii) Suppose that $\gamma \in \Gamma$ is parabolic. Prove that the fixed point of γ belong to $\Lambda(\Gamma)$.

[4 marks]

(iii) Prove that if $\Lambda(\Gamma)$ contains 3 or more points then it is infinite. (You may use without proof the fact that $\Lambda(\Gamma)$ is Γ-invariant.)

[12 marks]

(iv) Let Γ be the group generated by

$$\gamma_1(z) = z + 4, \qquad \gamma_2(z) = \frac{z}{z+1}.$$

One can show that Γ is a Fuchsian group (you do not need to prove this).

Using the results from (ii) and (iii) above, together with any other standard results from the course, show that $\Lambda(\Gamma)$ is infinite.

[6 marks]

END OF EXAMINATION PAPER