MATH3/4/62051 Hyperbolic Geometry coursework test: solutions and feedback

Q1 (i) A transformation of the form $\gamma(z) = (az+b)/(cz+d)$ where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$ is called a *Möbius transformation*. [2 marks]

Feedback: You must remember to state the conditions on a, b, c, d.

(ii) Clearly the identity is in Γ: take $a = d = 1, b = c = 0$. [2 marks]
Let $\gamma_i(z) = (a_i z + b_i)/(c_i z + d_i)$ where $a_i, b_i, c_i, d_i \in \mathbb{R}$, $a_i d_i - b_i c_i = 1$ and b_i, c_i are even, a_i, d_i are odd, $i = 1, 2$. Then

$$\begin{aligned} \gamma_1\gamma_2(z) &= \frac{a_1\left(\frac{a_2 z+b_2}{c_2 z+d_2}\right)+b_1}{c_1\left(\frac{a_2 z+b_2}{c_2 z+d_2}\right)+d_1} \\ &= \frac{(a_1a_2+b_1c_2)z+(a_1b_2+b_1d_2)}{(c_1a_2+d_1c_2)z+(c_1b_2+d_1d_2)} \end{aligned}$$

Then

$$(a_1a_2+b_1c_2)(c_1b_2+d_1d_2)-(a_1b_2+b_1d_2)(c_1a_1+d_1c_2) = (a_1d_1-b_1c_1)(a_2d_2-b_2c_2) = 1.$$

Also,

$$\begin{aligned} a_1a_2 + b_1c_2 &= \text{odd} \times \text{odd} + \text{even} \times \text{even} = \text{odd} \\ a_1b_2 + b_1d_2 &= \text{odd} \times \text{even} + \text{even} \times \text{odd} = \text{even} \\ c_1a_2 + d_1c_2 &= \text{even} \times \text{odd} + \text{odd} \times \text{even} = \text{even} \\ c_1b_2 + d_1d_2 &= \text{even} \times \text{even} + \text{odd} \times \text{odd} = \text{odd}. \end{aligned}$$

Hence $\gamma_1\gamma_2 \in \Gamma$. [6 marks]
Let $\gamma(z) = (az+b)/(cz+d) \in \Gamma$. Then $\gamma^{-1}(z) = (dz-b)/(-cz+a)$. As $da-(-b)(-c) = ad-bc = 1$ and d, a are odd, $-b, -c$ are even, we have $\gamma^{-1} \in \Gamma$. [2 marks]

Feedback: You must check $\gamma_1\gamma_2 \in \Gamma$ and $\gamma^{-1} \in \Gamma$; i.e. you must check that your 'new' a, b, c, d in the composition and in the inverse satisfy the determinant condition and the odd/even condition. One can simplify the above a bit by using matrices (this is fine and I gave full credit for this).

Q2 (i) Suppose $\gamma(z_0) = z_0$. Then $(az_0+b)/(cz_0+d) = z_0$ if and only if $cz_0^2+(d-a)z_0-b = 0$. By the quadratic formula,

$$z_0 = \frac{a-d \pm \sqrt{(a-d)^2+4bc}}{2c}$$

[2 marks]

Feedback: I think almost everybody got this right.

(ii) Let $\gamma(z) = (az+b)/(cz+d) \in \text{Möb}(\mathbb{H})$ and assume that γ is in normalised form. Then $\tau(\gamma) = (a+d)^2$.

Feedback: Many of you forgot to normalise here...

From (i), the number and location of the fixed points of γ is determined by the sign of the discriminant $(a-d)^2+4bc$. As $ad-bc=1$, we have $bc=ad-1$ so that

$$(a-d)^2+4bc = a^2-2ad+d^2+4(ad-1) = a^2+2ad+d^+4 = (a+d)^2+4 = \tau(\gamma)+4.$$

[3 marks]

Feedback: Very few of you did this, instead just writing out the connection between $\tau(\gamma)$ and classifying Möbius transformations below. The question explicitly says '*By using (*), relate $\tau(\gamma)$ to the number and location of fixed points'*.

Hence

- if $\tau(\gamma)>4$ then there are two real solutions to (*), so γ has two fixed points on $\partial\mathbb{H}$ and so is hyperbolic;
- if $\tau(\gamma)=4$ then there is one real solutions to (*), so γ has one fixed point on $\partial\mathbb{H}$ and so is parabolic;
- if $\tau(\gamma)\in[0,4)$ then there are two complex conjugate solutions to (*), so γ has one fixed point in $\mathbb{H}$ and so is elliptic.

[3 marks]

Feedback: I think everybody knows this classification.

(iii) Note that $ad-bc=7\times(-1)-(-4)\times 4=-7+16=9$ so that γ is not normalised. We can write γ in normalised form as

$$\gamma(z)=\frac{\frac{7}{3}z+\frac{-4}{3}}{\frac{4}{3}z+\frac{-1}{3}}.$$

This has $\tau(\gamma)=(7/3-1/3)^2=(6/3)^2=2^2=4$ and so is parabolic. [2 marks]

Feedback: Firstly, anybody who wrote $\tau(\gamma)=(7-1)^2=36$ and said that γ is hyperbolic didn't normalise (and made a small part of me die inside). Please normalise things; it's not hard. There were a lot of arithmetic mistakes here with minus signs. Many of you wrote '$ad-bc$'=23 or $\tau(\gamma)=(7/3+1/3)^2=64/9$, etc, or other slips.

Note that $\gamma(z_0)=z_0$ iff $7z_0-4=4z_0-1$ iff $4(z_0-1)^2=0$ iff $z_0=1$. Hence γ has a unique fixed point on $\partial\mathbb{H}$ and so is parabolic. [2 marks]

Feedback: Again, quite a few of you made mathematical slips and didn't factorise the quadratic correctly, claiming that the fixed point is in $\mathbb{H}$.

(iv) If γ_1,γ_2 are parabolic then $\gamma_1\gamma_2$ need not be parabolic.

Let $\gamma_1(z)$ have matrix $\begin{pmatrix}1&2\\0&1\end{pmatrix}$ (so $\gamma_1(z)=z+2$). Let $\gamma_2(z)$ have matrix $\begin{pmatrix}1&0\\2&1\end{pmatrix}$ (so $\gamma_1(z)=z/(2z+1)$). Both are parabolic as they have trace 4 (note that both of these are normalised). However $\gamma_1\gamma_2$ has matrix $\begin{pmatrix}5&2\\2&1\end{pmatrix}$ which is normalised and has trace $36>4$ and so is hyperbolic. [4 marks]

Feedback: Several of you said: let $\gamma_1(z) = z+1$ and let $\gamma_2 = \gamma_1^{-1}$. Then γ_1, γ_2 are parabolic (with unique fixed point at ∞) but $\gamma_1\gamma_2 =$ identity which is not parabolic. This is fine, though unintended on my part (I should have said in the question that $\gamma_2 \neq \gamma_1^{-1}$).
Some of you also said: Let γ_1, γ_2 be parabolic. Choose a conjugacy g so that $g^{-1}\gamma_1 g$ is a translation. Then $g^{-1}\gamma_2 g$ is also a translation. Hence $g^{-1}\gamma_1\gamma_2 g$ is also a translation, so $\gamma_1\gamma_2$ is parabolic. This doesn't work: the map g that conjugates a parabolic Möbius transformation γ to a translation depends on γ (specifically: it depends on the fixed point of γ). If γ_1, γ_2 have different fixed points then the conjugacy map g will be different.