

7. The Gauss-Bonnet theorem

§7.1 Hyperbolic polygons

In Euclidean geometry, an n -sided polygon is a subset of the Euclidean plane bounded by straight lines. Thus the edges of a Euclidean polygon are formed by segments of Euclidean geodesics. A hyperbolic polygon is defined in an analogous manner.

Definition. Let $z, w \in \mathbb{H} \cup \partial\mathbb{H}$. Then there exists a unique geodesic that passes through both z and w . We denote by $[z, w]$ the part of this geodesic that connects z and w . We call $[z, w]$ the *segment* or *arc* of geodesic between z and w .

Definition. Let $z_1, \dots, z_n \in \mathbb{H} \cup \partial\mathbb{H}$. Then the *hyperbolic n -gon* P with vertices at z_1, \dots, z_n is the region of \mathbb{H} bounded by the geodesic segments

$$[z_1, z_2], \dots, [z_{n-1}, z_n], [z_n, z_1].$$

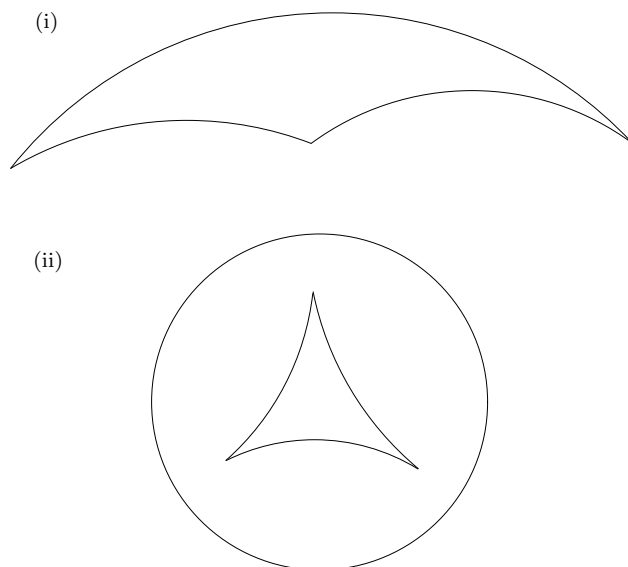


Figure 7.1: A hyperbolic triangle (i) in the upper half-plane model, (ii) in the Poincaré disc

Remark. Notice that we allow some of the vertices to lie on the circle at infinity. Such a vertex is called an *ideal vertex*. If all the vertices lie on $\partial\mathbb{H}$ then we call P an *ideal polygon*. Notice that the angle at an ideal vertex is zero; this is because all geodesics meet $\partial\mathbb{H}$ at right-angles and so the angle between any two such geodesics is zero.

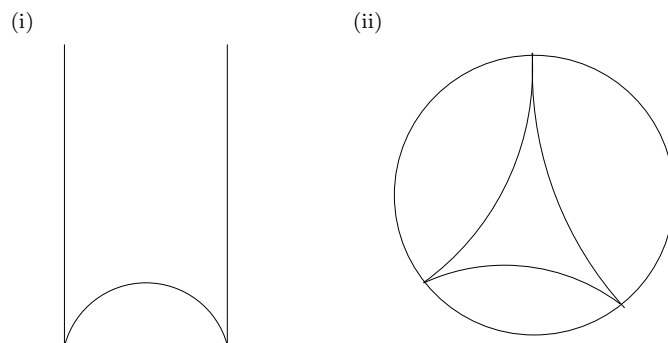


Figure 7.2: An ideal triangle (i) in the upper half-plane model, (ii) in the Poincaré disc

§7.2 The Gauss-Bonnet theorem for a triangle

The Gauss-Bonnet theorem can be stated in a wide range of contexts. In hyperbolic geometry, the Gauss-Bonnet theorem gives a formula for the area of a hyperbolic polygon in terms of its angles—a result that has no analogue in Euclidean geometry. We will use the Gauss-Bonnet theorem to study tessellations of the hyperbolic plane by regular polygons, and we will see that there are infinitely many distinct tessellations using regular polygons (whereas in Euclidean geometry there are only finitely many: equilateral triangles, squares, and hexagons).

Theorem 7.1 (Gauss-Bonnet theorem for a hyperbolic triangle)

Let Δ be a hyperbolic triangle with internal angles α, β and γ . Then

$$\text{Area}_{\mathbb{H}}(\Delta) = \pi - (\alpha + \beta + \gamma). \quad (7.1)$$

Remarks.

1. In Euclidean geometry it is well-known that the sum of the internal angles of a Euclidean triangle is equal to π (indeed, this is equivalent to the parallel postulate). In hyperbolic geometry, (7.1) implies that the sum of the internal angles of a hyperbolic triangle is strictly less than π .

2. The Gauss–Bonnet formula (7.1) implies that the area of a hyperbolic triangle is at most π . The only way that the area of a hyperbolic triangle can be equal to π is if all the internal angles are equal to zero. This means that all of the vertices of the triangle lie on the circle at infinity.
3. In Euclidean geometry, the angles of a triangle do not determine the triangle’s area (this is clear: scaling a triangle changes its area but not its angles). This is not the case in hyperbolic geometry.
4. There is an interactive java applet illustrating the Gauss-Bonnet theorem at <http://www.geom.umn.edu/java/triangle-area/>.

Proof. Let Δ be a hyperbolic triangle with internal angles α, β and γ .

We first study the case when at least one of the vertices of Δ belongs to $\partial\mathbb{H}$, and hence the angle at this vertex is zero. By applying a Möbius transformation, we can map this vertex to ∞ without altering the area or the angles. By applying the Möbius transformation $z \mapsto z + b$ for a suitable b we can assume that the circle joining the other two vertices is centred at the origin in \mathbb{C} . By applying the Möbius transformation $z \mapsto kz$ we can assume it has radius 1. Hence (see Figure 7.3)

$$\begin{aligned}
 \text{Area}_{\mathbb{H}}(\Delta) &= \int \int_{\Delta} \frac{1}{y^2} dx dy \\
 &= \int_a^b dx \int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2} dy \\
 &= \int_a^b dx \left. \frac{-1}{y} \right|_{\sqrt{1-x^2}}^{\infty} \\
 &= \int_a^b \frac{1}{\sqrt{1-x^2}} dx \\
 &= \int_{\pi-\alpha}^{\beta} -1 d\theta \text{ substituting } x = \cos \theta \\
 &= \pi - (\alpha + \beta).
 \end{aligned}$$

This proves (7.1) when one of the vertices of Δ lies on $\partial\mathbb{H}$.

Now suppose that Δ has no vertices in $\partial\mathbb{H}$. Let the vertices of Δ be A , B and C , with internal angles α , β and γ , respectively. Apply a Möbius transformation so that the side of Δ between vertices A and C lies on a vertical geodesic. Let δ be the angle at B between the side CB and the vertical. This allows us to construct two triangles, each with one vertex at ∞ : triangle $AB\infty$ and triangle $CB\infty$. See Figure 7.4.

$$\text{Area}_{\mathbb{H}}(\Delta) = \text{Area}_{\mathbb{H}}(ABC) = \text{Area}_{\mathbb{H}}(AB\infty) - \text{Area}_{\mathbb{H}}(BC\infty).$$

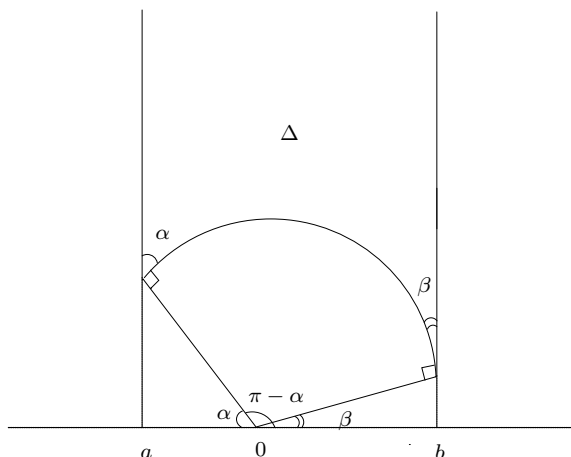


Figure 7.3: The Gauss–Bonnet theorem with one vertex of Δ at ∞

Now

$$\begin{aligned} \text{Area}_{\mathbb{H}}(AB\infty) &= \pi - (\alpha + (\beta + \delta)) \\ \text{Area}_{\mathbb{H}}(BC\infty) &= \pi - ((\pi - \gamma) + \delta). \end{aligned}$$

Hence

$$\begin{aligned} \text{Area}_{\mathbb{H}}(ABC) &= \pi - (\alpha + (\beta + \delta)) - (\pi - ((\pi - \gamma) + \delta)) \\ &= \pi - (\alpha + \beta + \gamma). \end{aligned}$$

□

Exercise 7.1

(The point of this exercise is to use the Gauss-Bonnet theorem to calculate the area of a given triangle.)

Let Δ be the hyperbolic triangle with vertices at $v_1 = i$, $v_2 = 2 + 2i$ and $v_3 = 4 + i$.

- (i) Calculate the equations of the sides of Δ .
- (ii) Let C_1 and C_2 be two circles in \mathbb{R}^2 with centres c_1, c_2 and radii r_1, r_2 , respectively. Suppose C_1 and C_2 intersect. Let θ denote the internal angle at the point of intersection (see figure). Show that

$$\cos \theta = \frac{|c_1 - c_2|^2 - r_1^2 - r_2^2}{2r_1r_2}.$$

- (iii) Use the Gauss–Bonnet theorem to show that the area of Δ is approximately 0.1377.

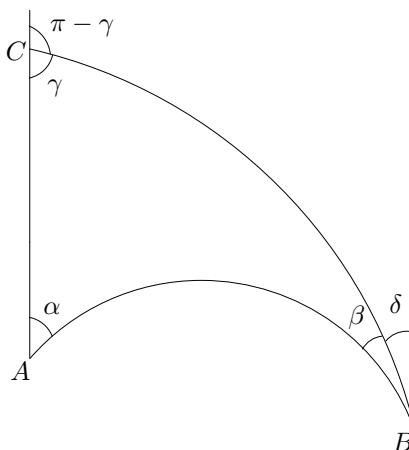


Figure 7.4: The Gauss–Bonnet theorem for the triangle ABC with no vertices on $\partial\mathbb{H}$

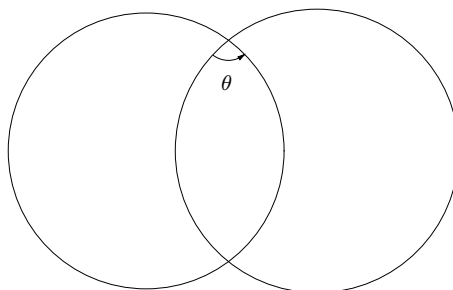


Figure 7.5: The internal angle between two circles

We can generalise the above theorem to give a formula for the area of an n -sided polygon.

Theorem 7.2 (Gauss-Bonnet theorem for a hyperbolic polygon)

Let P be an n -sided hyperbolic polygon with vertices v_1, \dots, v_n and internal angles $\alpha_1, \dots, \alpha_n$. Then

$$\text{Area}_{\mathbb{H}}(P) = (n - 2)\pi - (\alpha_1 + \dots + \alpha_n). \tag{7.2}$$

Proof (sketch). Cut up P into triangles. Apply Theorem 7.1 to each triangle and then sum the areas. \square

Exercise 7.2

Prove that the area of a hyperbolic quadrilateral with internal angles $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ is given by

$$2\pi - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4).$$

§7.3 Tessellations of the hyperbolic plane by regular polygons

Recall that a regular n -gon is an n -gon where all n sides have the same length and all internal angles are equal. We are interested in the following problem: when can we tile the plane using regular n -gons with k polygons meet at each vertex?

In Lecture 1 we remarked that the only possible tessellations of \mathbb{R}^2 are given by: equilateral triangles (with 6 triangles meeting at each vertex), squares (with 4 squares meeting at each vertex), and by hexagons (with 3 hexagons meeting at each vertex).

In hyperbolic geometry, the situation is far more interesting: there are infinitely many different tessellations by regular polygons!

Theorem 7.3

There exists a tessellation of the hyperbolic plane by regular hyperbolic n -gons with k polygons meeting at each vertex if and only if

$$\frac{1}{n} + \frac{1}{k} < \frac{1}{2}. \quad (7.3)$$

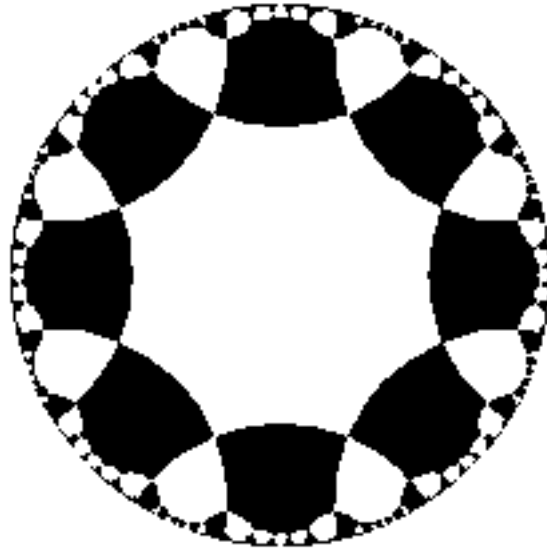
Proof. We only prove that if there is a tessellation then n, k satisfy (7.3), the converse is harder. Let α denote the internal angle of a regular n -gon P . Then as k such polygons meet at each vertex, we must have that $\alpha = 2\pi/k$. As the area of the polygon P must be positive, substituting $\alpha = 2\pi/k$ into (7.2) and re-arranging we have:

$$\frac{1}{n} + \frac{1}{k} < \frac{1}{2},$$

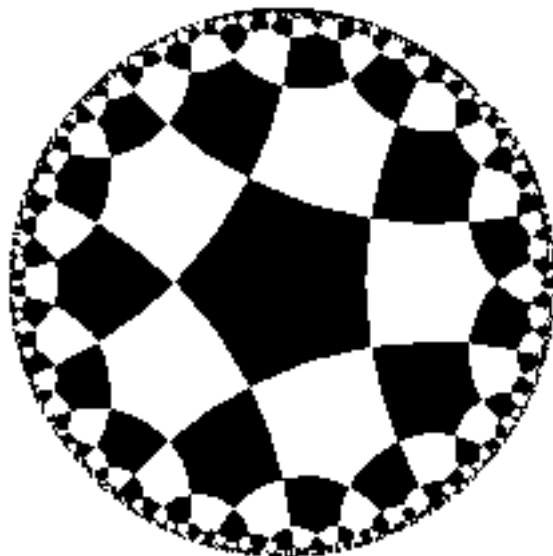
as required. □

Examples.

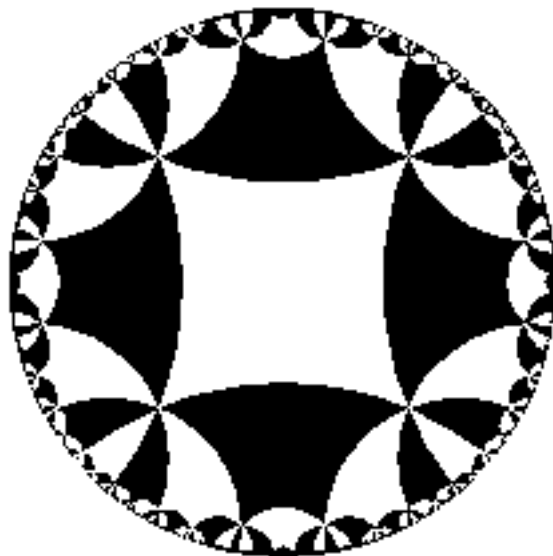
1. A tessellation of the Poincaré disc with $n = 8, k = 4$:



2. A tessellation of the Poincaré disc with $n = 5$, $k = 4$:



3. A tessellation of the Poincaré disc with $n = 4$, $k = 8$:



One technical point that we have glossed over is the existence of regular n -gons in hyperbolic geometry. To see that such polygons exist we quote the following result.

Proposition 7.4

Let $\alpha_1, \dots, \alpha_n$ be such that

$$(n-2)\pi - \sum_{k=1}^n \alpha_k > 0.$$

Then there exists a polygon with internal angles α_k .

Proof. See Beardon Theorem 7.16.2. □

One can show that if the internal angles are all equal then the lengths of the sides are all equal.

Exercise 7.3

Let $n \geq 3$. By explicit construction, show that there exists a regular n -gon with internal angle equal to α if and only if $\alpha \in [0, (n-2)\pi/n)$.

(Hint: Work in the Poincaré disc \mathbb{D} . Let $\omega = e^{2\pi i/n}$ be an n^{th} root of unity. Fix $r \in (0, 1)$ and consider the polygon $D(r)$ with vertices at $r, r\omega, r\omega^2, \dots, r\omega^{n-1}$. This is a regular n -gon (why?). Let $\alpha(r)$ denote the internal angle of $D(r)$. Use the Gauss-Bonnet Theorem to express the area of $D(r)$ in terms of $\alpha(r)$. Examine what happens as $r \rightarrow 0$ and as $r \rightarrow 1$. (To examine $\lim_{r \rightarrow 0} \text{Area}_{\mathbb{H}} D(r)$, note that $D(r)$ is contained in a hyperbolic circle $C(r)$, and use Exercise 6.3 to calculate $\lim_{r \rightarrow 0} \text{Area}_{\mathbb{H}} C(r)$.) You may use without proof the fact that $\alpha(r)$ depends continuously on r .)

In particular, conclude that there there exists a regular n -gon with each internal angle equal to a right angle whenever $n \geq 5$. This is in contrast with the Euclidean case where, of course, the only regular polygon with each internal angle equal to a right angle is the square.

Exercise 7.4

(This exercise is outside the scope of the course (and therefore not examinable!). However, anybody remotely interested in pure mathematics should get to see what is below at least once!)

A polyhedron in \mathbb{R}^3 is formed by joining together polygons along their edges. A platonic solid is a convex polyhedra where each constituent polygon is a regular n -gon, with k polygons meeting at each vertex.

By mimicking the discussions above, show that there are precisely five platonic solids: the tetrahedron, cube, octahedron, dodecahedron and icosahedron (corresponding to $(n, k) = (3, 3), (4, 3), (3, 4), (5, 3)$ and $(3, 5)$, respectively).