Feedback on MATH3/4/62051 Hyperbolic Geometry, January 2016

A1 (i) A straightforward definition. Some of you forgot to state that the coefficients a, b, c, d must be real and that $ad - bc > 0$.

(ii) Most of you answered this question correctly. The point is to show that a Möbius transformation of $\mathbb{H}$, say γ, does map $\mathbb{H}$ to $\mathbb{H}$. That is, you want to show that if $z \in \mathbb{H}$ then $\gamma(z) \in \mathbb{H}$. For this you need to calculate $\mathrm{Im}(\gamma(z))$ and show that if $\mathrm{Im}(z) > 0$ then $\mathrm{Im}(\gamma(z))$. Note that

$$\gamma(z) = \frac{az+b}{cz+d} = \frac{az+b}{cz+d} \times \frac{c\bar{z}+d}{c\bar{z}+d} = \frac{ac|z|^2 + bd + adz + bc\bar{z}}{|cz+d|^2}.$$

The denominator in the final expression is real, as is $ac|z|^2 + bd$. Note, writing $z = x+iy$, that $adz+bc\bar{z} = (ad+bc)x+i(ad-bc)y$. Hence

$$\mathrm{Im}(\gamma(z)) = \frac{ad-bc}{|cz+d|^2}\mathrm{Im}(z) > 0$$

provided $y > 0$, as $ad - bc > 0$.

(iii) Again, this is straightforward. If $\gamma(z) = (az+b)/(cz+d)$ then $\gamma^{-1}(z) = (dz-b)/(-cz+a)$ (and note that this is a Möbius transformation as $da - (-b)(-c) = ad - bc > 0$.

A2 (i) A straightforward definition from the course: $z_0 \in \mathbb{H} \cup \partial\mathbb{H}$ is a fixed point if $\gamma(z_0) = z_0$.

(ii) Recall $\tau(\gamma) = (a+d)^2$ *when γ is in normalised form!* Of the 60 Level 3 scripts, 24 of you didn't remember this (and also forgot to normalised the transformation in part (iii) of the question). That's 40%. Seriously, I don't know what I need to do to get you to remember this. (Those doing the Level 4/6 version were slightly better but some of you also need to hang your heads in shame.)

(iii) Of the remaining 60% who did remember to normalise γ in this question, quite a few didn't normalise it correctly. The trace is $(\frac{1}{\sqrt{3}} - 0)^2 = 1/3$. Explicitly determining the fixed points requires solving the quadratic equation $z_0^2 - z_0 + 3 = 0$, for example by using the quadratic formula (which several of you appear to have forgotten!), noting that the solutions are complex conjugates, and so there is only one fixed point in $\mathbb{H}$.

A3 (i) Most of you answered this correctly and gave good explanations.

(ii) This was one of the examples of generators and relations that I did in class. I was pleased by how many of you remembered how to use the given relation to show that a, b commute.

(iii) Most you tried this question but not that many (maybe a quarter of you, I didn't count) got it right. Most of you recognised that you needed 3 generators, say a, b, c. However, many went on to say that just one relation is needed, namely $abca^{-1}b^{-1}c^{-1} = e$ (or similar). This isn't right. Your intuition should be that a, b, c correspond to $(1, 0, 0), (0, 1, 0), (0, 0, 1)$, respectively. We need all of these to pairwise commute. To get a, b to commute we need the relation $aba^{-1}b^{-1} = e$, and similarly for b, c and a, c. Hence if we set

$$\Gamma = \langle a, b, c \mid aba^{-1}b^{-1} = bcb^{-1}c^{-1} = aca^{-1}c^{-1} = e\rangle$$

then we will obtain a group that is isomorphic to $\mathbb{Z}^3$.

A4 (i) This is a standard definition from the course. Most of you got it right.

(ii) This is a straight-forward (and, to be honest, slightly tedious) exercise in chasing (vertex,side)-pairs around the diagram. There is one elliptic cycle and one elliptic cycle transformation (the exact form of which will depend on your choice of labelling the diagram and the point at which you start calculating the cycle, but will be something like $\gamma_4^{-1}\gamma_3^{-1}\gamma_4\gamma_3\gamma_2^{-1}\gamma_1^{-1}\gamma_2\gamma_1$). The angle sum is $8 \times \pi/16 = \pi/2$ and so the elliptic cycle condition holds with $m = 4$ (most of you got this). Thus Poincaré's Theorem applies and the transformations generate a Fuchsian group Γ. We can write this in terms of generators and relations as

$$\Gamma = \langle a, b, c, d \mid (d^{-1}c^{-1}dcb^{-1}a^{-1}ba)^4 = e\rangle,$$

the 4 coming from the order of the elliptic cycle. Many of you missed out the power 4 here.

B5(a)(i) This is Exercise 6.3 from the notes (which in turn is very similar to Proposition 3.5.2 which we did in class). Not that many of you attempted this.

(b)(i) First you need to write down a parametrisation of the path σ_1: $\sigma_1(t) = t$, $0 \le t \le R$ will do (there are others). You then need to calculate $\int_0^R \frac{1}{1-t^2}\,dt$, which is best done by means of partial fractions to obtain $\log(1 + R)/(1 - R)$.

(b)(ii) Again, the first thing to do is to write down a parametrisation of the path σ_2. All kinds of weird things got written down here: did you learn nothing in Complex Analysis about how to parametrise a circle?! One parametrisation is $\sigma_2(t) = Re^{it}$, $0 \le t \le 2\pi$. Then $\sigma'(t) = Rie^{it}$ so that $|\sigma'(t)| = R$. Also note that $|\sigma(t)| = R$. Hence you need to calculate $\int_0^{2\pi} \frac{2R}{1-R^2}\,dt$. One or two of you made *very* heavy weather of this: there's no need, the integrand is a constant so the integral is simply $4\pi R/(1-R^2)$.

(b)(iii) Note that many of you tried this part of the question (perhaps because it involves remembering trig and hyperbolic trig identities?). See the solution to Exercise 6.4.

(b)(iv) See the solution to Exercise 6.4 for this too.

(b)(v) Not that many of you attempted this, despite it being very straightforward. Note that

$$\frac{\text{circumference } C_r}{\text{area } C_r} = \frac{2\pi \sinh r}{4\pi \sinh^2 r/2} \sim \frac{2\pi e^r/2}{4\pi (e^{r/2}/2)^2} \sim 1$$

as $r \to \infty$. A Euclidean analogue would be that

$$\frac{\text{circumference } C_r}{\text{area } C_r} = \frac{2\pi r}{\pi r^2} = \frac{2}{r} \to 0$$

as $r \to \infty$. (Should you be interested: geometrically this means that in hyperbolic geometry, and in contrast to the Euclidean world, 'most' of the area of a hyperbolic circle is concentrated near the edge of the circle.)

B6(a)(i) This is a standard definition from the course that most of you got right.

(a)(ii) Most of you got this mostly right, but there was a lot of carelessness in the definition of $D(p)$. Recall that $D(p) = \bigcap_{\gamma \in \Gamma \setminus \mathrm{Id}} H_p(\gamma)$. Many of you forgot to exclude $\gamma = \mathrm{Id}$ (note that $H_p(\gamma)$ doesn't make sense when $\gamma = \mathrm{Id}$); several of you wrote $\bigcup$ rather than $\bigcap$.

(b)(i) Pretty much everybody got this right. It's Exercise 14.2(i) in the notes.

(b)(ii) To calculate $H_p(\gamma_1)$ and $H_p(\gamma_1^{-1})$ use the same argument that we used in lectures when considering integer translations (see Proposition 15.2.1 in the notes). It's easy then to see that $H_p(\gamma_1) = \{z \in \mathbb{H} \mid \mathrm{Re}(z) < 2\}$ and $H_{\gamma_1^{-1}} = \{z \in \mathbb{H} \mid \mathrm{Re}(z) > 0\}$.

The remainder of this part of the question was not well answered. Few of you who chose to do B6 even bothered to attempt this; few of those who did got it right. The calculations are very similar to Exercise 14.2(ii), which we also covered in one of the support classes. As a general rule of thumb: there's a good chance that I chose numbers so that the question works out nicely, so if you get a complicated-looking answer then there's a good chance you've gone wrong.

For $p = 1 + 2i$ we have $\gamma_2(p) = (6 + 8i)/5$ (a straightforward calculation). Putting p and $\gamma_2(p)$ into equation (1) gives (after some simplication) $L_p(\gamma_2) = \{z = x+iy \in \mathbb{H} \mid (x-2)^2+y^2 = 2^2\}$, i.e. the semi-circle with centre 2 and radius 2. As $|p-2| = \sqrt{5} > 2$, we see that $H_p(\gamma_2) = \{z = x + iy \in \mathbb{H} \mid (x-2)^2 + y^2 > 2^2\}$.

Similarly, we have $\gamma_2^{-1}(p) = (4 + 8i)/5$ and a similar calculation as in the previous paragraph shows that $L_p(\gamma_2^{-1}) = \{z = x+iy \in \mathbb{H} \mid x^2 + y^2 = 2^2\}$, the semi-circle with centre 0 and radius 2. Hence $H_p(\gamma_2^{-1}) = \{z = x + iy \in \mathbb{H} \mid x^2 + y^2 > 2^2\}$.

(b)(iii) Once you've calculated the half-planes in (b)(ii) this is an easy exercise in drawing pictures in the complex plane! It's a hyperbolic quadrilateral with one vertex at ∞.

(b)(iv) Let s be a side of $D(p)$. If $s \subset L_p(\gamma)$ then the side-pairing transformation associated to s is γ^{-1}.

As $D(p)$ has finite hyperbolic area (by Gauss-Bonnet), it follows from Theorem 14.4.2 in the lecture notes that the side-pairing transformations generate Γ.

B7 (i) This is a standard definition from the course: an elliptic cycle is accidental if $\text{sum}(\mathcal{E}) \neq 2\pi$.

(ii) Almost everybody gave a good explanation of how to construct $\mathbb{H}/\Gamma$.

(iii) This is Exercise 21.3(ii) from the lecture notes (which we also did in one of the support classes).

(iv) Almost everybody did well in this question. There is a certain amount of (vertex,side)-pair chasing to see that there are three parabolic cycles. Each parabolic cycle can be seen to satisfy the parabolic cycle by checking that the trace of the associated parabolic cycle transformation is 4. Hence γ_1, γ_2 generate a Fuchsian group Γ (actually, Γ is the free group on 2 generators).

The signature of Γ is $(0; -; 3)$ ($\mathbb{H}/\Gamma$ is, topologically, a sphere with 3 points removed and so has genus 0). D has hyperbolic area 2π.

C8 (i) These are standard definitions from the course.

(ii) This is the content of one implication in Lemma 23.3.1.

(iii) Γ_1 acts properly discontinuously. It's easy to see this by noting that every orbit is discrete and that the stabiliser of every point is trivial.

$\Gamma_2 = \text{PSL}(2, \mathbb{R})$ does not act properly discontinuously on $\mathbb{H}$. There are lots of ways of seeing this. One way is to note that $\Gamma_2 \supset \{\gamma_a \mid \gamma_a(z) = z + a, a \in \mathbb{R}\}$ and that $\{\gamma_(a)(i) \mid a \in \mathbb{R}\} = \{a + i \mid a \in \mathbb{R}\}$. Hence $\Gamma_2(i) \supset \{a + i \mid a \in \mathbb{R}\}$, which is not discrete, hence $\Gamma_2(i)$ is not discrete.

Several of you proved that Γ_2 acting on $\partial\mathbb{H}$ does not act properly discontinuously. This isn't sufficient: a group may act properly discontinuously on one space but not properly discontinuously on another.

C9 (i) This is a standard definition from the course.

(ii) This is Proposition 24.4.5 in the notes.

(iii) Again, this is a proof from the course: Proposition 24.4.6 in this case.

(iv) Once you can show that the limit set of Γ contains four points then you are done, by (iii) above. One way to do this is to find enough transformations in Γ and look at their fixed points.

γ_1, γ_2 are both parabolic and have fixed points at $\infty, 0$, respectively. Hence $0, \infty \in \Lambda(\Gamma)$.

Consider $\gamma_1\gamma_2(z) = (5z+4)/(z+1)$. This is hyperbolic and has fixed points at $2 \pm 2\sqrt{2}$ (use the quadratic formula to explicitly calculate the fixed points). Hence $\Lambda(\Gamma) \supset \{0, \infty, 2\pm 2\sqrt{2}\}$. Hence $\Lambda(\Gamma)$ is infinite.