Notes on metric spaces

§1 Introduction

The purpose of these notes is to quickly review some of the basic concepts from Real Analysis, Metric Spaces and some related results that will be used in this course.

§2 Convergence of real numbers

§2.1 Limits

Let $x_n \in \mathbb{R}$. We say that $x_n \to x$ if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geq \mathbb{N}$ we have $|x_n - x| < \varepsilon$ for all $n > N$.

§2.2 Lim sup and lim inf

When we come to study ergodic theory, we will often have to check that various sequences converge. It will often be most convenient to do this by using lim sups and lim infs.

Definition. Let $x_n \in \mathbb{R}$ be a sequence. Define

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \left(\sup_{k\geq n} \{x_k\} \right).$$

Remark. The lim sup always exists (although it may be $\pm\infty$). This is because $\sup_{k\geq n}\{x_k\}$ is a decreasing sequence (in n), and decreasing sequences always converge.

Definition. Let $x_n \in \mathbb{R}$ be a sequence. Define

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \left(\inf_{k\geq n} \{x_k\} \right).$$

Remark. For the same reason as above, the lim inf always exists but may be equal to $\pm\infty$.

Proposition 2.1
Let x_n be a sequence. Then $x_n \to x$ if and only if $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = x$.

Remark. Thus in order to show that a sequence converges, we need only show that both the lim sup and the lim inf agree. In fact, it is clear from the definitions that $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$. Hence to show that x_n converges, we need only show that $\limsup_{n\to\infty} x_n \leq \liminf_{n\to\infty} x_n$.

Example. Take

$$x_n = (-1)^n + (-1)^n \left(\frac{1}{2^n}\right).$$

Then $\limsup_{n\to\infty} x_n = +1$ and $\liminf_{n\to\infty} x_n = -1$.

§3 Metric spaces

§3.1 Metric spaces

Let X be a set. A function $d : X \times X \to [0, \infty)$ is called a *metric* if

(i) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ (the triangle inequality).

We call (X, d) a *metric space*.

Examples.

1. Take $X = \mathbb{R}$ and define $d(x, y) = |x - y|$.
2. Take $X = \mathbb{R}^2$ and define
$$d((x_1, x_2), (y_1, y_2)) = |y_1 - x_1| + |y_2 - x_2|.$$
3. Take $X = \mathbb{R}^2$ and define
$$d((x_1, x_2), (y_1, y_2)) = \sqrt{|y_1 - x_1|^2 + |y_2 - x_2|^2}.$$
4. Consider the space
$$C([0, 1], \mathbb{R}) = \{f : [0, 1] \to \mathbb{R} \mid f \text{ is continuous}\}.$$
Define
$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|.$$
(a standard result says that this supremum is finite.) Then we can define a metric by
$$d(f, g) = \|f - g\|_\infty.$$
5. More generally, if X is a compact (see below) metric space then the space $C(X, \mathbb{R})$ of continuous real-valued functions defined on X is a metric space with metric $d(f, g) = \sup_{x\in X} |f(x) - g(x)|$.

§3.2 Convergence

Let (X, d) be a metric space. Let $x_n \in X$ be a sequence of points in X. We say that x_n *converges* to $x \in X$ (and write $x_n \to x$) if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $d(x_n, x) < \varepsilon$.

A sequence x_n is *Cauchy* if: for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $d(x_n, x_m) \leq \varepsilon$.

A metric space (X, d) is said to be *complete* if every Cauchy sequence converges. Examples of complete metric spaces include: $\mathbb{R}, \mathbb{R}^2$ (with either of the metrics above), $C(X, \mathbb{R})$ where X is a compact metric space,.... The set of all rationals $\mathbb{Q}$ is *not* complete: the sequence $x_1 = 1, x_2 = 1.4, x_3 = 1.41, x_4 = 1.414, \ldots, x_n =$ decimal expansion of $\sqrt{2}$ to n decimal places is Cauchy but does not converge in $\mathbb{Q}$ (because $\sqrt{2}$ is irrational).

§3.3 Open and closed sets

Let (X, d) be a metric space. Let $x \in X$ and let $\varepsilon > 0$. The set

$$B(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$$

of all points y that are distance at most ε from x is called the *open ball of radius ε and centre x*. We can think of $B(x, \varepsilon)$ as being a small neighbourhood around the point x.

A subset $U \subset X$ is called *open* if for all $x \in U$ there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$, i.e. every point x in U has a small neighbourhood that is also contained in U.

One can easily show that open balls $B(x, \varepsilon)$ are open subsets.

A set F is said to be *closed* if its complement $X \setminus F$ is open. There are other ways of defining closed sets:

Proposition 3.1
Let (X, d) be a metric space and let $F \subset X$. Then the following are equivalent:

(i) F is closed (i.e. $X \setminus F$ is open);

(ii) if $x_n \in F$ is a sequence of points in F such that $x_n \to x$ for some $x \in X$ then $x \in F$ (i.e. any convergent sequence of points in F has its limit in F).

§4 Compactness

We will usually study compact metric spaces. Roughly speaking, these are spaces where sequences of points cannot 'escape'.

§4.1 Sequential compactness

Let (X, d) be a metric space. We say that X is *sequentially compact* if every sequence $x_n \in X$ has a convergent subsequence, i.e. there exist $n_j \to \infty$ such that $x_{n_j} \to x$ for some $x \in X$.

Examples.

1. $[0, 1]$ is compact.
2. $(0, 1)$ is not compact (because $x_n = 1/n$ does not have a convergent subsequence in $(0, 1)$).
3. $C(X, \mathbb{R})$ is not compact.

§4.2 Compactness by open covers

Let (X, d) be a metric space. A collection of open sets $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ is called an open cover if

$$X = \bigcup_{\alpha \in A} U_\alpha.$$

We say that X is *compact* if every open cover of X has a finite subcover, i.e. if $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ is an open cover of X then there exists $\alpha_1, \ldots, \alpha_n$ such that

$$X = U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}.$$

Example. $(0, 1)$ is not compact: the open cover $\{(1/(n + 2), 1/n)\}$ does not have a finite subcover.

The two notions of compactness are the same:

Proposition 4.1
Let (X, d) be a metric space. Then X is sequentially compact if and only if it is compact.

§4.3 Properties of compact spaces

Proposition 4.2
Let X be compact. Then X is closed.

Proposition 4.3 (Heine-Borel)
A subset $X \subset \mathbb{R}^d$ is compact if and only if it is closed and bounded.

Remark. For subspaces of spaces other than $\mathbb{R}^d$, this characterisation of compactness fails. For example, there are closed and bounded subsets of $C([0, 1], \mathbb{R})$ that are not compact.

§5 Continuity

§5.1 (ε, δ)-continuity

Definition. Let (X, d) be a metric space and let $x \in X$. A function $f : X \to \mathbb{R}$ is said to be *continuous at* x if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x, y) < \delta$ then $|f(x) - f(y)|$. We say that f is *continuous* if it is continuous at every point $x \in X$.

Remark. More generally, we can easily modify this definition to define what it means for a map $f : X \to Y$ between two metric spaces X and Y to be continuous.

§5.2 Continuity by convergence of sequences

Let (X, d) be a metric space. We can give an alternative characterisation of continuity in terms of how the images of convergent subsequences behave.

Proposition 5.1
The following are equivalent:

(i) $f : X \to \mathbb{R}$ *is continuous at* $x \in X$,

(ii) for all convergent sequences $x_n \in X$ *such that* $x_n \to x$ *then* $f(x_n) \to f(x)$.

Remark. Again, this definition can easily be extended to the case of a continuous function $f : X \to Y$ between two arbitrary metric spaces.

§5.3 Continuity by preimages of open sets

Here is another characterisation of continuity in terms of open sets.

Proposition 5.2
Let X *and* Y *be two metric spaces and let* $f : X \to Y$. *The following are equivalent:*

(i) f *is continuous*

(ii) if $U \subset Y$ *is an open subset of* Y *then the set*

$$f^{-1}(U) = \{x \in X \mid f(x) \in U\}$$

is an open subset of X.

§5.4 Continuity and compactness

Continuous functions defined on compact metric spaces enjoy various nice properties and we describe some of them here.

Proposition 5.3 (Uniform continuity)
Let (X, d_X) be a compact metric space and let (Y, d_Y) be a metric space (not necessarily compact). Let $f : X \to Y$ be continuous. Then f is uniformly continuous: i.e. $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x, y \in X$, if $d_X(x, y) < \varepsilon$ then $d_Y(f(x), f(y)) < \delta$.

Remarks.

1. Recall that in the (ε, δ)-definition of continuity, given $x \in X$ and $\varepsilon > 0$ we choose $\delta > 0$ such that $d_X(x, y) < \varepsilon$ implies $d_Y(f(x), f(y)) < \delta$. The crucial point is that a priori δ depends on both ε and x. With uniform continuity, the same δ will work for *any* x.

2. In particular, if $T : X \to X$ is a continuous transformation of a compact metric space then T is uniformly continuous.

3. If $f : X \to \mathbb{R}$ is a continuous real-valued function defined on a compact metric space then it is uniformly continuous.

Proposition 5.4 (Continuous image of a compact set is compact)
Let (X, d) be a compact metric space and let $f : X \to Y$ be continuous. Then $f(X) \subset Y$ is a compact subset of Y.

Proposition 5.5 (Continuous functions are separable)
Let (X, d) be a compact metric space. Then the space

$$C(X, \mathbb{R}) = \{f : X \to \mathbb{R} \mid f \text{ is continuous}\}$$

is separable, i.e. there exists a countable dense set $\{f_n\}$, i.e. $\forall f \in C(X, \mathbb{R})$ and $\forall \varepsilon > 0$, $\exists n > 0$ such that $\|f - f_n\|_\infty < \varepsilon$

Proposition 5.6 (Stone-Weierstrass theorem)
Let (X, d) be a compact metric space and let $\mathcal{A} \subset C(X, \mathbb{R})$ be an algebra of continuous functions defined on X, i.e.

(i) the zero function $f(x) \equiv 0 \in \mathcal{A}$,

(ii) if $f, g \in \mathcal{A}$ and $\lambda \in \mathbb{R}$ then $\lambda f + g \in \mathcal{A}$,

(iii) if $f, g \in \mathcal{A}$ then the product function $fg \in \mathcal{A}$.

Suppose that $\mathcal{A}$ separates the points of X and does not vanish at any point of X, i.e.

(i) for all $x, y \in X$ with $x \neq y$ there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$,

(ii) for all $x \in X$ there is some $f \in \mathcal{A}$ for which $f(x) \neq 0$.

Then $\mathcal{A}$ is dense in $C(X, \mathbb{R})$.

Remark. A similar result holds for $C(X, \mathbb{C})$, provided one assumes that if $f \in \mathcal{A}$ then $\bar{f} \in \mathcal{A}$ (where the bar denotes complex conjugation).

Examples.

1. Take $X = [0, 1]$ and take $\mathcal{A}$ to be the algebra of all polynomials. Clearly polynomials separate points and do not vanish at any point. The Stone-Weierstrass theorem implies that any continuous function can be arbitrarily well approximated by a polynomial.

2. Take $X = \mathbb{R}/\mathbb{Z}$ and take $\mathcal{A}$ to be the algebra of all trigonometric polynomials, i.e. $\mathcal{A} = \{\sum_j \alpha_j e^{2\pi i \ell_j x} \mid \alpha_j \in \mathbb{C},\ \ell_j \in \mathbb{Z}\}$. Then $\mathcal{A}$ is an algebra that separates points and does not vanish at any point. Hence any continuous function $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ defined on the circle can be arbitrarily well approximated by a trigonometric polynomial.