NUMERICAL FORMULAE

Iteration

Newton Raphson method for refining an approximate root x_0 of $f(x) = 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Particular case to find $\sqrt{N}$ use $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$.

Secant Method

$$x_{n+1} = x_n - f(x_n) / \left(\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}\right)$$

Interpolation

$$\begin{aligned} \Delta f_n &= f_{n+1} - f_n \quad , \quad \delta f_n = f_{n+\frac{1}{2}} - f_{n-\frac{1}{2}} \\ \nabla f_n &= f_n - f_{n-1} \quad , \quad \mu f_n = \frac{1}{2}\left(f_{n+\frac{1}{2}} + f_{n-\frac{1}{2}}\right) \end{aligned}$$

Gregory Newton Formula

$$f_p = f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + ... + \binom{p}{r}\Delta^r f_0$$

$$\text{where } p = \frac{x - x_0}{h}$$

Lagrange's Formula for n points

$$y = \sum_{i=1}^{n} y_i \ell_i(x)$$

where

$$\ell_i(x) = \frac{\Pi_{j=1, j\neq i}^{n}(x - x_j)}{\Pi_{j=1, j\neq i}^{n}(x_i - x_j)}$$

Numerical differentiation

Derivatives at a tabular point

$$\begin{aligned} hf_0' &= \mu\,\delta f_0 - \frac{1}{6}\mu\,\delta^3 f_0 + \frac{1}{30}\mu\,\delta^5 f_0 - ... \\ h^2 f_0'' &= \delta^2 f_0 - \frac{1}{12}\delta^4 f_0 + \frac{1}{90}\delta^6 f_0 - ... \\ hf_0' &= \Delta f_0 - \frac{1}{2}\Delta^2 f_0 + \frac{1}{3}\Delta^3 f_0 - \frac{1}{4}\Delta^4 f_0 + \frac{1}{5}\Delta^5 f_0 - ... \\ h^2 f_0'' &= \Delta^2 f_0 - \Delta^3 f_0 + \frac{11}{12}\Delta^4 f_0 - \frac{5}{6}\Delta^5 f_0 + ... \end{aligned}$$

Numerical Integration

Trapezium Rule

$$\int_{x_0}^{x_0+h} f(x)dx \simeq \frac{h}{2}(f_0 + f_1) + E$$

where $f_i = f(x_0 + ih),\ E = -\frac{h^3}{12}f''(a),\ x_0 < a < x_0 + h$

Composite Trapezium Rule

$$\int_{x_0}^{x_0+nh} f(x)dx \simeq \frac{h}{2}\{f_0 + 2f_1 + 2f_2 + ...2f_{n-1} + f_n\} - \frac{h^2}{12}(f_n' - f_0') + \frac{h^4}{720}(f_n''' - f_0''')...$$

where $f_0' = f'(x_0),\ f_n' = f'(x_0 + nh)$, etc

Simpson's Rule

$$\int_{x_0}^{x_0+2h} f(x)dx \simeq \frac{h}{3}(f_0 + 4f_1 + f_2) + E$$

where $E = -\frac{h^5}{90}f^{(4)}(a) \qquad x_0 < a < x_0 + 2h.$

Composite Simpson's Rule (n even)

$$\int_{x_0}^{x_0+nh} f(x)dx \simeq \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + ... + 2f_{n-2} + 4f_{n-1} + f_n) + E$$

where $E = -\frac{nh^5}{180}f^{(4)}(a). \qquad x_0 < a < x_0 + nh$

Gauss order 1. (Midpoint)

$$\int_{-1}^{1} f(x)dx = 2f\left(0\right) + E$$

where $$E = \frac{2}{3} f^{''}(a). \qquad -1 < a < 1$$

Gauss order 2.

$$\int_{-1}^{1} f(x)dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) + E$$

where $$E = \frac{1}{135} f^{\prime v}(a). \qquad -1 < a < 1$$

Differential Equations

To solve $y' = f(x, y)$ given initial condition y_0 at $x_0, x_n = x_0 + nh$.

Euler's forward method

$$y_{n+1} = y_n + hf(x_n, y_n) \qquad n = 0, 1, 2, ...$$

Euler's backward method

$$y_{n+1} = y_n + hf(x_{n+1}, y_{n+1}) \qquad n = 0, 1, 2, ...$$

Heun's method (Runge Kutta order 2)

$$y_{n+1} = y_n + \frac{h}{2}(f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))).$$

Runge Kutta order 4.

$$y_{n+1} = y_n + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

where

$$\begin{aligned} K_1 &= f(x_n, y_n) \\ K_2 &= f\left(x_n + \frac{h}{2}, y_n + \frac{hK_1}{2}\right) \\ K_3 &= f\left(x_n + \frac{h}{2}, y_n + \frac{hK_2}{2}\right) \\ K_4 &= f(x_n + h, y_n + hK_3) \end{aligned}$$

Chebyshev Polynomials

$$T_n(x) = \cos\left[n(\cos^{-1} x)\right]$$

$$T_o(x) = 1 \qquad\qquad T_1(x) = x$$

$$U_{n-1}(x) = \frac{T_n'(x)}{n} = \frac{\sin\left[n(\cos^{-1} x)\right]}{\sqrt{1-x^2}}$$

$$\begin{aligned}
T_m(T_n(x)) &= T_{mn}(x). \\
T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x) \\
U_{n+1}(x) &= 2xU_n(x) - U_{n-1}(x) \\
\int T_n(x)dx &= \frac{1}{2}\left\{\frac{T_{n+1}(x)}{n+1} - \frac{T_{n-1}(x)}{n-1}\right\} + \text{constant}, \qquad\qquad n \geq 2
\end{aligned}$$

$$f(x) = \frac{1}{2}a_0T_0(x) + a_1T_1(x)...a_jT_j(x) + ...$$

where $$a_j = \frac{2}{\pi}\int_0^\pi f(\cos\theta)\cos j\theta d\theta \qquad\qquad j \geq 0$$

and $\int f(x)dx = \text{constant } + A_1T_1(x) + A_2T_2(x) + ...A_jT_j(x) + ...$
where $A_j = (a_{j-1} - a_{j+1})/2j \qquad j \geq 1$