FOURIER SERIES AND TRANSFORMS

Fourier series

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \{a_n \cos n\omega t + b_n \sin n\omega t\} \qquad \text{(period } T = 2\pi/\omega)$$

where

$$\begin{aligned} a_n &= \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos n\omega t\, dt \\ b_n &= \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega t\, dt \end{aligned}$$

Half range Fourier series

$$\textit{sine series} \qquad a_n = 0,\ b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega t\,dt$$

$$\textit{cosine series} \qquad b_n = 0,\ a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega t\,dt$$

Finite Fourier transforms

sine transform

$$\begin{aligned}\tilde{f}_s(n) &= \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega t\,dt\\ f(t) &= \sum_{n=1}^{\infty}\tilde{f}_s(n)\sin n\omega t\end{aligned}$$

cosine transform

$$\begin{aligned}\tilde{f}_c(n) &= \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega t\,dt\\ f(t) &= \frac{1}{2}\tilde{f}_c(0) + \sum_{n=1}^{\infty}\tilde{f}_c(n)\cos n\omega t\end{aligned}$$

Fourier integral

$$\frac{1}{2}\left(\lim_{t\nearrow 0} f(t) + \lim_{t\searrow 0} f(t)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\int_{-\infty}^{\infty} f(u)e^{-i\omega u}\,du\,d\omega$$

Fourier integral transform

$$\begin{aligned}\tilde{f}(\omega) &= F\{f(t)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega u} f(u)\,du\\ f(t) &= F^{-1}\left\{\tilde{f}(\omega)\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega t}\tilde{f}(\omega)\,d\omega\end{aligned}$$