Chapter 7. The WKB method. See eg Simmonds & Man Ch VI.

In this chapter we focus on eigenvalue problems for 2nd order ODEs, specifically

\[ \frac{d^2y}{dx^2} + \lambda^2 r(x) y = 0 \quad y(0) = y(1) = 0 \]  \[ \text{[1]} \]

(note y as fn of x rather than x as fn of t as comes from stationary waves.)

7.1 Motivation

Stationary waves eg a string satisfy the wave equation

\[ T \frac{d^2y}{dx^2} = \rho(x) \frac{1}{u^2} \frac{d^2y}{dt^2} \quad T = \text{tension} \]
\[ \rho = \text{density of string}. \]
\[ y(0, t) = y(1, t) = 0 \]
\[ \text{Pinned at both ends} \]

Standing wave solutions have form \( y(x, t) = y(x) \cos\omega t \)

Reducing wave equation to

\[ \frac{d^2y}{dx^2} + \frac{\rho(x)}{T} \omega^2 y = 0 \]

which is of the form \( \text{[1]} \).

7.1.1 Density constant \( \rho(x) = 1 \)

\[ \frac{d^2y}{dx^2} + \left( \frac{\omega^2}{c^2} \right) y = 0 \quad c^2 = \frac{\rho}{T} \]

Solve \( y(x) = A \cos \lambda x + B \sin \lambda x \) \quad \( \lambda^2 = \frac{\omega^2}{c^2} \).

\( y(0) = 0 \Rightarrow A = 0 \quad y(1) = 0 \Rightarrow B \sin \lambda = 0 \)
\[ \Rightarrow \lambda = n\pi \quad n = \pm 1, \pm 2, \ldots \]

So solve have form \( y = B \sin(n\pi x) \)

but \#0 solns only exist for \( n \in \mathbb{N} \), ie special or "eigen" values.

In general we only get closed form solutions for very simple \( r(x) \).

Note: It is using the "boundary conditions" on \( y(0) \) and \( y(1) \) rather than \( y(0), y'(0) \) that means solutions do not exist for all \( n \).

And like eigenvectors, "eigenvectors" can be multiplied by any constant \( B \).
7.2 The WKB Approximation

WKB stands for Wentzel, Kramers, Brillouin, maybe Jefferys' name should be added along with a few others but name has stuck.

If \( w \) (or \( \lambda \)) is very large then the zeros of the solutions are close together. We are planning on doing an asymptotic expansion for large \( \lambda \).

If \( r \) was constant in (17) we would have solutions

\[
y(x) = A e^{\pm i\pi x R}
\]

We therefore try \( y(x, A) = e^{\lambda g(x, A)} \)

Substitute in (17) gives

\[
\lambda^{-1} g'' + g'^2 + r = 0
\]

(1st order eqn in \( h = g' \)).

\[
\lambda^{-1} h' + h^2 + r = 0
\]

We then seek a regular expansion of the form

\[
h(x) = h_0(x) + \frac{1}{\lambda} h_1(x) + \frac{1}{\lambda^2} h_2(x) + O(\frac{1}{\lambda^3})
\]

(\( \frac{1}{\lambda} \) is like \( e \)).

Following this back to the original problem this is looking for a solution of the form

\[
y = \exp \left( \lambda g_0(x) + g_1(x) + \frac{1}{\lambda} g_2(x) + O(\frac{1}{\lambda^2}) \right)
\]
7.3 Example Airy's Equation \( y'' + \lambda^2 y = 0 \)

Consider \( y = \exp(\lambda g_0 + g_1 + \frac{1}{3} g_2 + ...) \)
\[
y' = \exp[\lambda g_0 + g_1 + \frac{1}{3} g_2 + ..] (\lambda g_0' + g_1' + ..)
\]
\[
y'' = \exp[\lambda g_0 + g_1 + \frac{1}{3} g_2 + ..] (\lambda g_0' + g_1' + ..)^2 + \exp[\lambda g_0 + g_1 + ..] (\lambda g_0'' + g_1'' + \frac{1}{3} g_2'' + ..)
\]
Substitute in ODE and cancel common \( \exp \).

\[
(\lambda g_0' + g_1' + ..)^2 + \lambda g_0'' + g_1'' + .. + \lambda^2 x = 0
\]

**Terms of order \( \lambda^2 \)**

\[
g_0'' + 2g_0'g_1' = 0
\]

\[
\Rightarrow \quad \frac{1}{2}i x^{-1/2} + 2i x^{1/2} g_1' = 0
\]

\[
\Rightarrow \quad g_1' = \frac{-i}{4x} \Rightarrow g_1 = -\frac{i}{4} \log x + B
\]

[Note: disregard \( A \) as multiplying \( \exp(\lambda x) \) ]

Terms order \( \lambda \), with \( g_0 \)

\[
g_0 = \pm i \frac{2}{3} x^{3/2}
\]

( + term first)

now \( g_0 = -i \frac{2}{3} x^{3/2} \) gives same result [check]

So general solution is of form

\[
y = A \exp \left[ i \lambda \frac{2}{3} x^{3/2} - \frac{1}{4} \log x + .. \right] + B \exp \left[ -i \lambda \frac{2}{3} x^{3/2} - \frac{1}{4} \log x + .. \right]
\]

Notice \( \exp(-\frac{1}{4} \log x) = x^{-1/4} \) so
\[ y(x) = \frac{1}{x^{3/4}} \left( C \cos\left(\frac{2}{3}\lambda x^{3/2}\right) + D \sin\left(\frac{2}{3}\lambda x^{3/2}\right) \right. \\
\left. + O\left(\frac{1}{x^2}\right) \right) \]

If we repeat this for the general case \(r(x)\) then following Simmons & Mann p.77-79 we obtain
\[ y(x,\lambda) = \left( \frac{1}{r(x)} \right)^{1/4} + O\left(\frac{1}{\lambda}\right) \right) \left\{ A \cos \lambda \int_a^x \sqrt{r(t)} \, dt + O\left(\frac{1}{\lambda}\right) \\
+ B \sin \lambda \int_a^x \sqrt{r(t)} \, dt + O\left(\frac{1}{\lambda^2}\right) \right\} \]
where \(A, B\) are arbitrary real constants, \(a\) can be any convenient value.

Note: as in MNS any expression \(A e^{i\xi t} + B e^{-i\xi t}\) is real if it equals its conjugate and hence \(A = \overline{B}\) in that case \(A e^{i\xi t} + \overline{B} e^{-i\xi t} = 2 \Re(A \cos \xi t) \neq -2i \Im(A \sin \xi t)\).

But the point is we can just say
\[ C \cos \xi t + D \sin \xi t \]
Some real \(C\) and \(D\).
7.3a Example with more complicated $\lambda$ dependence.

$$y'' + (\lambda^2(1+x)^2 + \frac{\lambda}{1+x})y = 0$$

is not exactly $\lambda^2 r(x)$.

Again try $y = \exp(\lambda g_0 + g_1 + O(\lambda^2))$

giving

$$\lambda g_0'' + g_1'' + \cdots + (\lambda g_0' + g_1' + \lambda r)$$

$$+ \lambda^2(1+x)^2 + \frac{\lambda}{1+x} = 0$$

Terms in $\lambda^2$

$$g_0'' + (1+x)^2 = 0$$

$$g_0' = \pm i (1+x)$$

$$g_0 = \pm i (x + \frac{1}{2}x^2)$$

Terms in $\lambda$

$$g_0'' + 2g_0'g_1' + \frac{1}{1+x} = 0$$

$$g_0 = + i (x + \frac{1}{2}x^2) \quad (+\text{case})$$

$$g_0' = i (1+x)$$

$$g_0'' = i$$

$$g_1' = \frac{-i}{2(1+x)} + \frac{i}{2(1+x)^2}$$

$$g_1 = -\frac{i}{2} \log(1+x) - \frac{i}{2} \frac{1}{(1+x)}$$

(-case) $g_0 = - i (x + \frac{1}{2}x^2) \quad g_1 = \frac{i}{2} \log(1+x) + \frac{i}{2} \frac{1}{(1+x)}$

Hence general soln is

$$y = A \exp \left\{ i \lambda (x + \frac{x^2}{2}) - \frac{i}{2} \log(1+x) - \frac{i}{2} \frac{1}{1+x} \right\}$$

$$+ B \exp \left\{ -i \lambda (x + \frac{x^2}{2}) - \frac{i}{2} \log(1+x) + \frac{i}{2} \frac{1}{1+x} \right\}$$
Usual simplification

\[ y \approx \frac{1}{(1 + x)^{1/2}} \left\{ \cos \left( \frac{\lambda}{2} x^{1/2} \right) - \frac{1}{2} \frac{\lambda}{(1 + x)^{1/2}} \right\} \\
+ D \sin \left( \frac{\lambda}{2} x^{1/2} - \frac{3}{4} \frac{\lambda}{(1 + x)^{1/2}} \right) \]

We can solve more general eqs of form \( y'' + f(x, \lambda) y = 0 \).

7.4 Estimating eigenvalues

Often eigenvalues are more important than eigenfunctions.

We get these estimates by applying the boundary conditions \( y(a) = y(b) = 0 \) to the WKB approximation.

(some \( a, b \))

The Airy example is easy for \( y(a) = y(b) = 0 \). For \( a = 0 \) we see the cosine term vanishes (otherwise soln is unbounded at \( 0 \)). The sine term (note this is second order despite the \( \lambda^{1/4} \)) gives us simply \( \frac{2}{3} \lambda = n \pi \), \( n \in \mathbb{N} \).

So \( \lambda \approx \frac{3n\pi}{2} \), \( n \in \mathbb{N} \), but for \( n \) large, \( (a) \) we neglected \( O(\lambda^2) \) terms in the WKB approximation.

Let's do a more interesting example.

\[ y'' + \lambda^2 (1 + 3 \sin^2 x)^2 y = 0 \quad y(0) = y(1) = 0. \]

\[ y \approx \exp \left( \gamma g_0 + g_1 + \ldots \right) \]

\[ g_0^2 + (1 + 3 \sin^2 x)^2 = 0 \]

\[ g_0 = \pm i \left( 1 + 3 \sin^2 x \right) \]

\[ g_0 = \pm i \left( 1 + 3 \sin^2 \left( \frac{5\pi}{2} - \frac{3}{4} \sin 2x \right) \right) = \pm i \left( \frac{5\pi}{2} - \frac{3}{4} \sin 2x \right) \]

\[ g_1 = - \log \left( 1 + 3 \sin^2 x \right) \text{ (for both } g_0 \text{)} \]

Hence, \( y \approx \frac{1}{(1 + 3 \sin^2 x)^{1/2}} \left\{ \cos \left( \frac{\lambda}{2} x - \frac{3}{4} \sin 2x \right) + D \sin \lambda \left( \frac{5\pi}{2} - \frac{3}{4} \sin 2x \right) \right\} \)

(Exactly the general result for \( r(x) \) on page)
Now we apply the boundary conditions to obtain the approximate eigenvalues.

\[ y(0) = 0 \Rightarrow C = 0 \]
\[ y(1) = 0 \Rightarrow D \sin \left( \frac{\pi}{2} x - \frac{3}{4} \sin 2x \right) = 0 \]

Hence

\[ \lambda \left( \frac{\pi}{2} - \frac{3}{4} \sin 2 \right) = n \pi \quad n \in \mathbb{N} \]

\[ \lambda_n \approx \frac{n \pi}{\pi/2 - 3/4 \sin 2} \]

E.g. for \( n = 5 \), \( \lambda_5 \approx 13.82 \) is the exact value and our approx gives 13.824.

7.4.2 One further example.

\[ y'' + \left( \lambda^2 e^{-2x} - 1 \right) y = 0 \quad y(1) = 0, y(\infty) = 0. \]

As before
\[ g_0 = -i e^{-x} \]
\[ g_1 = \frac{x}{2} \quad \text{for both} + \text{and} - \]

\[ y = A \exp \left( \frac{x}{2} \right) + i \lambda e^{-x} + \frac{x}{2} e^{x} + B \exp \left( -i \lambda e^{x/2} \right) \]

\[ = e^{x/2} \left[ C \cos(\lambda e^{-x}) + D \sin(\lambda e^{-x}) \right] \]

As \( x \to \infty \), \( \cos(\lambda e^{-x}) \to 1 \) so \( C = 0 \).

\[ y(1) = 0 \Rightarrow \sin(\lambda e^{-1}) = 0 \quad \lambda_n \approx n \pi e \]

\[ \lambda_0 \approx 31.45 e \quad \text{exact} \quad 31.2196 e \]