Ch. 1 Intro to Pert Thy & Asy. Expans.

Ex 1.0.1 \[ x - 2 = e \cosh x \]

\( E = 0, x = 2, \) otherwise hard to find \( x(E) \)

Try expansion of form \( x = x_0 + E x_1 + E^2 x_2 + \ldots \)

\( E = 0 \Rightarrow x_0 = 2 \), substitution gives \( x_1 = \cosh 2 \)

\( 2 + c \cosh 2 \) is good for small \( E \) (e.g. \( E = 10^{-2} \) accurate to 3 d.p.)

§1. Landau or order notation aka \( \bigO \) little \( \bigO \).

Let \( f \) be f is defined on \( \mathbb{D} \), \( o \) and \( \bigO \). We say

a) \( f(x) = O(g(x)) \) as \( x \to 0 \) if there is some \( K > 0 \), \( \epsilon > 0 \) st

\( (\epsilon, \epsilon) \subset D \) and \( \forall \ x \in (-\epsilon, \epsilon) \)

\[ |f(x)| < K |g(x)| \]

"goes to zero about the same rate"

b) \( f(x) \sim g(x) \) as \( x \to 0 \) if \( \lim_{x \to 0} |f(x)/g(x)| = 0 \)

goes to zero faster than"

c) \( f(x) \sim g(x) \) as \( x \to 0 \)

\[ \lim_{x \to 0} f(x)/g(x) = 1 \]

Lemma 1.1 If \( \frac{f(x)}{g(x)} \to M \) as \( x \to 0 \) then \( f(x) = O(g(x)) \).

Proof \[ \left| \frac{f(x)}{g(x)} - M \right| < \delta \] for \( x \in \delta \), \( \Rightarrow \)

\[ \frac{|f(x)|}{|g(x)|} < M + \delta \]

Take \( \delta = M + \epsilon \) as \( \epsilon \) is defin of \( O \).

Ex 1.1.1 i) \( x^3 = o(2x) \)

ii) \( 3x^2 = O(5x^3) \)

\( x = o(x/\epsilon) \), [i] \( 2x = o(1) \)

We need to remember L'Hôpital's rule to work out limit of ratio

(\( \text{\"zero/zero\"} \)).

vi) \( \sin x \approx x \)

vii) \( \sin x = x + O(x^3) \)

viii) \( \sin x = x + O(x^3) \)

Defn of derivative \( f(x + h) = f(x) + hf'(x) + o(h) \).

Taylor's Thm \( f(x + h) = f(x) + hf'(x) + \ldots + h^n f^{(n)}(x) + o(h^n) \)

Convergent power series \[ \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^n + O(x^{n+1}) \]

i) \( \sqrt{1 + x} = 1 + \frac{1}{2} (-x) + \frac{1}{8} (-2x) + O(x^3) = 1 - x - \frac{x^2}{2} + O(x^3) \)

xi) \( \log(1 + x) = x - \frac{x^2}{2} + O(x^3) \) as \( x \to 0 \).

§1.2 "Fundamental Theorem" of Perturb Thy "FTPT":

If \( A_0 + A_1 \epsilon + \ldots + A_n \epsilon^n = O(\epsilon^{n+1}) \) then \( A_i = A_1 = \ldots = A_n = 0. \)
Remark. This means intuitively we can "equate coefficients of \( E^n \), \( 0 \leq n \leq N \), when working modulo \( O(\varepsilon^{N+1}) \).

Remark for the algebraically inclined. \( O(\varepsilon^{N+1}) \) is an ideal, modulo in the sense of a factor ring.

Proof. Suppose but not all \( A_n \neq 0 \). Let \( A_n \) be first non-zero such that \( \frac{A_n + \varepsilon A_{n+1} + \ldots + E^{N-m} A_{N}}{E^{N+1-m}} \to 0 \) so this contradicts the hypothesis.

§1.3. Perturbation Theory of Algebraic Equations.

Ex. 1.3.1. "Regular perturbation"

\[ x^2 - 3x + 2 + \varepsilon = 0. \]
Assume solutions have expansion

\[ x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3) \]

\[ (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3))^2 = 3(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3)) + 2 + \varepsilon = O(\varepsilon^3) \]

\[ (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3))^2 - 3(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3)) + 2 + \varepsilon = O(\varepsilon^3) \]

Use FTPT. All coeffs of \( \varepsilon^k \), \( k=0,1,2 \), are zero.

\[ E_0^2 = x_0^2 - 2x_0 + 2 = 0 \Rightarrow x_0 = 1, 2 \]

\[ E_1 = 2x_0 x_1 - 3x_1 + 1 = 0 \Rightarrow x_1 = 1 \text{ or } -1 \text{ respectively} \]

\[ E_2^2 = x_1^2 + 2x_0 x_2 - 3x_2 = 0 \Rightarrow x_2 = 1 \text{ or } -1 \]

So \( x = 1 + \varepsilon + \varepsilon^2 \) or \( x = 2 - \varepsilon - \varepsilon^2 + O(\varepsilon^3) \) as \( \varepsilon \to 0 \)

We can solve algebraically \( x = \frac{3 \pm \sqrt{9 - 4(2 + \varepsilon + \varepsilon^2 + O(\varepsilon^3))}}{2} \)

(we behavior series).

Ex. 1.3.2. "Singular perturbation"

\[ x^2 - 2x + 1 = 0 \]

\[ \varepsilon = 0 \Rightarrow x = \frac{1}{2}. \] Assume \( x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3) \)

\[ E_0 = -x_0 + 1 = 0 \Rightarrow x_0 = \frac{1}{2} \]

\[ E_1 = x_0^2 - 2x_1 = 0 \Rightarrow x_1 = \frac{1}{8} \]

\[ E_2 = 2x_0 x_1 - 2x_2 = 0 \Rightarrow x_2 = -\frac{1}{16} \] (note: I said \( \varepsilon \) negative).

Where is second root of \( x = \frac{1 \pm \sqrt{1 - 4 \varepsilon}}{2} \)?

\[ x = \frac{1 \pm \sqrt{1 - 4 \varepsilon}}{2} = \frac{1}{2} \pm \frac{1}{4} \varepsilon + O(\varepsilon^2) \text{ or } \frac{2}{\varepsilon} \frac{-1}{\sqrt{4 - \varepsilon^2}} + O(\varepsilon^2) \]

This is a "singular" problem. Expansion in non-negative integer powers did not work. We can use other series, e.g. fractional and negative powers.

There is a version of FTPT to cover this.