4 (a) Try \( x = x_0 + \epsilon x_1 + \cdots \)

\[ e^0 : \ddot{x}_0 + 2x_0 = 0 \quad , \quad x_0(0) = 1 \quad \Rightarrow \quad x_0 = e^{-2t} \]

\[ e^1 : \dot{x}_1 + 2x_1 = x_0^3 = e^{-6t} \quad , \quad x_1(0) = 0 \quad \Rightarrow \quad x_1 = \frac{1}{4} e^{-2t} - \frac{1}{4} e^{-6t} \]

So

\[ x = e^{-2t} + \epsilon \left( \frac{1}{4} e^{-2t} - \frac{1}{4} e^{-6t} \right) + O(\epsilon^2) \]

There is no problem with this expansion and it is uniformly valid. In this context “uniformly \( O \)” means the constant in the definition of \( O \) can be chosen independently of \( t \).

(b) Try \( x = x_0 + \epsilon x_1 + \cdots \)

\[ e^0 : \ddot{x}_0 = t \quad , \quad x_0(0) = \dot{x}_0(0) = 0 \quad \Rightarrow \quad x_0 = \frac{1}{6} t^3 \]

\[ e^1 : \ddot{x}_1 = -2x_0(x_0+1) = -\frac{1}{18} t^6 - \frac{1}{3} t^3 \quad , \quad x_1(0) = 0 = \dot{x}_1(0) = 0 \quad \Rightarrow \quad x_1 = -\frac{t^8}{1008} - \frac{t^5}{60} \]

So

\[ x = \frac{1}{6} t^3 + \epsilon \left( -\frac{t^8}{1008} - \frac{t^5}{60} \right) + O(\epsilon^2) \]

which is non-uniform as \( t \to \infty \). In fact it fails when \( t = O(\epsilon^{-\frac{1}{2}}) \).

(c) Try \( x = x_0 + \epsilon x_1 + \cdots \)

\[ e^0 : \ddot{x}_0 = 0 \quad , \quad x_0(0) = 1, \dot{x}_0(0) = 0 \quad \Rightarrow \quad x_0 = 1 \]

\[ e^1 : \ddot{x}_1 = 1 - \dot{x}_0 = 1 \quad , \quad x_1(0) = 0 = \dot{x}_1(0) = 0 \quad \Rightarrow \quad x_1 = \frac{1}{2} t^2 \]

So

\[ x = 1 + \epsilon \left( \frac{1}{2} t^2 \right) + O(\epsilon^2) \]

which is non-uniform as \( t \to \infty \). Problem arises when \( t = O \left( \frac{1}{\sqrt{\epsilon}} \right) \).

5 First solve the homogeneous equation \( \ddot{x} + \epsilon \dot{x} = 0 \) gives the complementary solutions as

\[ x = A + B e^{-\epsilon t} \]

Now look for a particular integral of the form \( x_p = Ct + D \Rightarrow x_p = t \). Hence applying the boundary conditions, reveals the exact solution is

\[ x = 1 + t + \frac{1}{\epsilon} \left( e^{-\epsilon t} - 1 \right) \]

The difficulty is created by the non-uniformity of \( e^{-\epsilon t} \) as \( t \to \infty \) and \( \epsilon \to 0 \).
The idea is to order the term $f_1(\epsilon), f_2(\epsilon), f_3(\epsilon)$ in such a way that $\frac{f_{n+1}}{f_n} \rightarrow 0$ as $\epsilon \rightarrow 0$. This result is

(a) 
\[ \log \frac{1}{\epsilon}, \log \left( \log \left( \frac{1}{\epsilon} \right) \right), 1, \epsilon^{1/2} \log \left( \frac{1}{\epsilon} \right), \epsilon^{1/2}, \epsilon \log \frac{1}{\epsilon}, \epsilon^{3/2}, \epsilon^2 \log \frac{1}{\epsilon}, \epsilon^2. \]

(b) Now $\cot \epsilon \sim \frac{1}{\epsilon}$ so the order is 
\[ \cot \epsilon, \epsilon^{-0.01}, \log \left( \frac{1}{\epsilon} \right), \exp \left( -\frac{1}{\epsilon} \right). \]

It might help you to think of $\epsilon = e^{-n}$ and let $n \rightarrow \infty$, as it is probably easier to spot null sequences than functions that go to zero. If you are stuck try putting in small values for $\epsilon$, or plotting graphs of the functions. When you think you have the answer test it by working out the limits of the ratios. You probably know useful things like $x \log x \rightarrow 0$ as $x \rightarrow 0$ (see eg Math10131, Problems 3 q 6).